

Homework for Partial Differential Equations

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Exercises are from
Introduction to Partial Differential Equations with MATLAB
by Cooper. For example “Exercise 2.6.4” is exercise 4
in section 6 of chapter 2 in Cooper.
Beware: Some solutions may be incorrect!

PARTIAL DIFFERENTIAL EQUATIONS

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HOMEWORK 1

2. First-Order Equations.

2.1. Generalities.

Exercise 2.1.1. Verify that, for any C^1 function $f(x)$, $u(x, t) = f(x - ct)$ is a solution of the PDE $u_t + cu_x = 0$, c a constant.

Solution.

$$\begin{aligned}u_t + cu_x &= f'(x - ct) \frac{\partial(x - ct)}{\partial t} + c \left(f'(x - ct) \cdot \frac{\partial(x - ct)}{\partial x} \right) \\&= -cf'(x - ct) + cf'(x - ct) \\&= 0.\end{aligned}$$

□

Exercise 2.1.2. Verify that $u(x, y) = \sqrt{x^2 + y^2}$ satisfies the nonlinear PDE $u_x^2 + u_y^2 = 1$ for $(x, y) \neq (0, 0)$.

Solution. $u_x^2 + u_y^2 = \left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2 = 1$

□

Exercise 2.1.3. Show that $u(x, t) = f(x/t)$ satisfies

$$u_t + \left(\frac{x}{t} \right) u_x = 0 \quad \text{for } t > 0,$$

where f is any C^1 function.

Solution.

$$\begin{aligned}u_t + \left(\frac{x}{t} \right) u_x &= f_t(x/t) \frac{\partial(x/t)}{\partial t} + \left(\frac{x}{t} \right) f_x(x/t) \frac{\partial(x/t)}{\partial x} \\&= \left(-\frac{x}{t^2} \right) f_t(x/t) + \left(\frac{x}{t^2} \right) f_x(x/t) \\&= 0.\end{aligned}$$

□

2.2. First-Order PDE's.

In the next exercise, the PDE has the characteristic $x = x_0 + ct$. Let $v(t) = u(x_0 + ct, t)$ be the restriction of u to a characteristic line. Find the ODE solved by v . Solve it, and then restate the answer in terms of u .

Exercise 2.2.1. Solve the IVP

$$u_t + cu_x + u = 0, \quad u(x, 0) = f(x).$$

Solution. Notice that

$$\frac{d}{dt}v(t) = cu_x(x_0 + ct) + u_t(x_0 + ct) = -u(x_0 + ct) = -v(t),$$

and so $v(t) = Ce^{-t}$ for some constant C . Then

$$C = v(0) = u(x_0, 0) = f(x_0) = f(x - ct).$$

So $u(x_0 + ct, t) = v(t) = f(x_0)e^{-t}$ for all x_0 , and hence $u(x, t) = f(x - ct)e^{-t}$. \square

Exercise 2.2.4. Let l be any line in the x, t plane and consider the IVP

$$u_t + cu_x = 0, \quad u \text{ given on } l.$$

Can this problem always be solved? Pay special attention to the case when l is a characteristic $x - ct = \text{constant}$.

Solution. In the case when l is a line that is not a characteristic, the problem may be solved. Indeed, if l intersects a characteristic line $x - ct = x_0$ at a point (x, t) , then, as u is constant along a characteristic line, we have that

$$u(x_0, 0) = u(x, t).$$

Hence the initial data u given along l can be used to define a function f by the rule

$$f(x_0) = \text{value of } l \text{ at } (x, t),$$

where (x, t) is the point that l intersects the characteristic line $x - ct = x_0$. Hence our initial data becomes $u(x, 0) = f(x)$, in which case we can solve the PDE by Theorem 2.1.

However, in the case when l is itself a characteristic line, the initial data u given along l does not give enough information, because this only tells us that u is constant along l , and we cannot use this data to construct a function as before in order to explicitly solve the PDE. \square

Exercise 2.2.5. Let $u(x,t) = f(xe^{-t})$ be the sol of $u_t + xu_x = 0, u(x,0) = f(x)$.

- Use MATLAB to plot snapshots of the solution on $[0, 15]$ for $t = .4, .8, 1.2, 1.6$. Make a fine mesh with $\Delta x = .01$.
- We want to approximate the velocity of the top of the hump at two different times, say $t = .5$ and $t = 1.5$. Plot $u(x,t)$ at times $t = .5, 1.5 + \Delta t, .5, 1.5 + \Delta t$. First use $\Delta t = .1$. Use the zoom feature of MATLAB to determine how far the top of the hump moves over the time interval $[t, t + \Delta t]$. Determine the average velocity of the top of the hump over this time interval. Compute the average velocities of the hump again, this time with $\Delta t = .05$.
- Use the formula for the characteristic through $x_0 = 2$ to compute the instantaneous velocity of the top of the hump at times $t = .5$ and $t = 1.5$. How well does the average velocity as you computed it in part (b) compare with the instantaneous velocity?
- Why does the hump broaden as the wave moves to the right?

Solution.

(a)

Function definition in script, and Command lines:

```

1 function y = u(x,t)
2     p = x*exp(-t);
3     y = exp(-2*(p-2).^2);

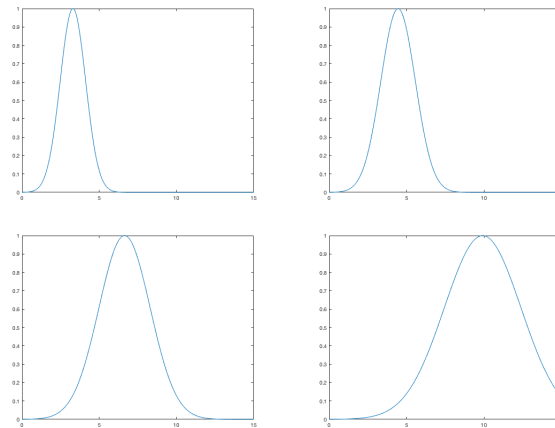
```

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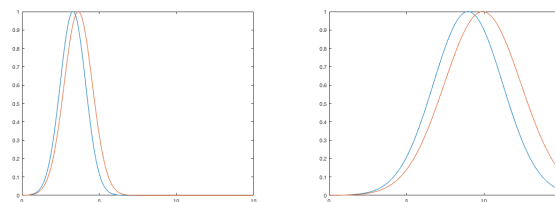
>> x = 0:.01:15;
>> plot(x,u(x,.4))
>> plot(x,u(x,.8))
>> plot(x,u(x,1.2))
>> plot(x,u(x,1.6))

```

Plots of the solution $u(x,t)$ for $t = .4, .8, 1.2, 1.6$, respectively:

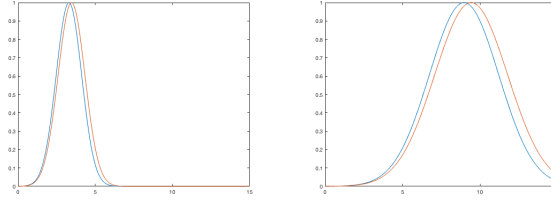


(b) Command lines for $\Delta t = .1$: `>> plot(x,u(x,.5),x,u(x,.5+.1))`
`>> plot(x,u(x,1.5),x,u(x,1.5+.1))`, which gave the figures:



On the time interval $[.5, .5 + .1]$ the hump moves $\approx 3.65 - 3.3 = .35$, giving a velocity of 3.5. On $[1.5, 1.5 + .1]$ the hump moves $\approx 9.8 - 8.9 = .9$, giving a velocity of 9.

Command lines for $\Delta t = .05$: `>> plot(x,u(x,.5),x,u(x,.5+.05))`
`>> plot(x,u(x,1.5),x,u(x,1.5+.05))`, which gave the figures:



On the time interval $[.5, .5 + .05]$ the hump moves $\approx 3.47 - 3.3 = .17$, giving a velocity of 3.4. On $[1.5, 1.5 + .05]$ the hump moves $\approx 9.4 - 8.9 = .5$, giving a velocity of 10.

- (c) Since $dx/dt = x$, then we know $x(t) = x_0 e^t$. So at $x_0 = 2$, $x(t) = 2e^t$. At times $t = .5$ and $t = 1.5$, we have instantaneous velocities $x(.5) \approx 3.3$ and $x(1.5) \approx 9$. These are close to the answers in part (b), but I suppose “close” is subjective.
- (d) The characteristic curves in the x, t plane are logarithms which diverge from each other. Since the solution u is constant on the curves, the humps broaden as t increases.

To see this more explicitly, fix a time $t = t_0$ and consider the distance in the x, t -plane between two points $u(a_0, t_0)$ and $u(b_0, t_0)$ for distinct fixed position values $x = a_0$ and $x = b_0$, say $\alpha := \text{dist}(u(a_0, t_0), u(b_0, t_0))$. (We pick a_0 and b_0 so that they lie on either side of the “hump”).

For another fixed time $t_1 > t_0$, pick two values a_1 and b_1 so that (a_0, t_0) and (a_1, t_1) lie on the same characteristic curve, and so that (b_0, t_0) and (b_1, t_1) lie on the same characteristic curve. Let $\beta := \text{dist}(u(a_1, t_1), u(b_1, t_1))$. Since the characteristic curves in the x, t plane are logarithms which are diverging from one another, then $\beta > \alpha$. This is not yet enough to say that the hump is broadening. However, if we consider the fact that u is constant on the characteristic curves, then

$$u(a_0, t_0) = u(a_1, t_1) \quad \text{and} \quad u(b_0, t_0) = u(b_1, t_1).$$

With this additional information, the inequality $\beta > \alpha$ now tells us that as time increases, points along two distinct characteristic curves remain at constant “heights” above the x, t plane, but that they are getting farther apart from one another, hence broadening the hump.

□

Exercise 2.2.8. Solve the IVP for the linear equation

$$u_t + x^2 u_x = 0, \quad u(x, 0) = f(x).$$

You will need to use the formula of Example (b) of Section 1.3. Over what region in the x, t plane does the solution exist?

Solution. The characteristic curves for the PDE are given by solutions to the ODE

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0,$$

and so our characteristic curves are

$$x(t) = \frac{x_0}{1 - tx_0}.$$

Solving explicitly for x_0 to obtain $p(x, t)$:

$$x_0 = p(x, t) = \frac{x}{1 + tx}$$

which is only defined when $x \neq -1/t$. The curve $x = -1/t$ splits the x, t plane into two regions, and the solution will only exist in the region containing the x axis, i.e. the region given by $x < -1/t$. On this region, the solution to the given IVP is $u(x, t) = f(p(x, t)) = f\left(\frac{x}{1+tx}\right)$. \square

HOMEWORK 2

2. First-Order Equations.**2.3. Nonlinear conservation laws.**

Exercise 2.3.2. Show that if the initial data $f(x)$ has $f'(x_0) < 0$ for some x_0 , then the C^1 solution of

$$u_t + uu_x = 0, \quad u(x, 0) = f(x)$$

must break down at some time $t > 0$.

Proof. Since f is C^1 , then the inequality $f'(x_0) < 0$ implies that f is decreasing on some interval $[x_0, x_1]$. The characteristic lines emanating from x_0 and x_1 are respectively given by

$$x = x_0 + f(x_0)t \quad \text{and} \quad x = x_1 + f(x_1)t.$$

In terms of t , these become

$$t = \frac{x - x_0}{f(x_0)} \quad \text{and} \quad t = \frac{x - x_1}{f(x_1)}.$$

Now $f(x_0) > f(x_1)$ implies $1/f(x_0) < 1/f(x_1)$. So the slope of the line at x_1 in the x, t plane is steeper than that at x_0 . But $x_0 < x_1$, which means that these lines must intersect at some time value $t = T$. At this value T is where the solution breaks down. \square

Exercise 2.3.3. Show that if $f'(x) \geq 0$ for all x , then the characteristic lines emanating from the x axis do not intersect in $t > 0$. What does this say about the existence for $t > 0$.

Proof. If $f'(x) = 0$ for all x , then f is constant so the characteristic lines are vertical lines, and so they do not intersect.

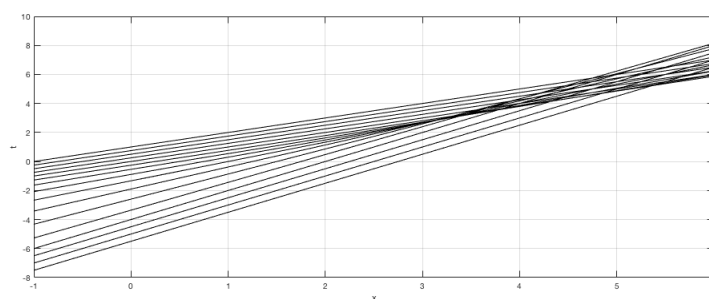
Now since $f'(x) \geq 0$ for all x , f is non-decreasing. Since the characteristic lines are given by $x = x_0 + f(x_0)t$, or, $t = \frac{x - x_0}{f(x_0)}$, then as x_0 increases, the slopes of the characteristic lines decrease. In particular, if $x_0 < x_1$ then the lines corresponding to these points either remain parallel (if $f(x_0) = f(x_1)$) or diverge from one another as time t increases, and hence never intersect. So the solution exists for $t > 0$ when $f'(x) \geq 0$ for all x . \square

Exercise 2.3.6. Next consider the IVP $u_t + uu_x = 0, u(x, 0) = f(x)$ with initial data

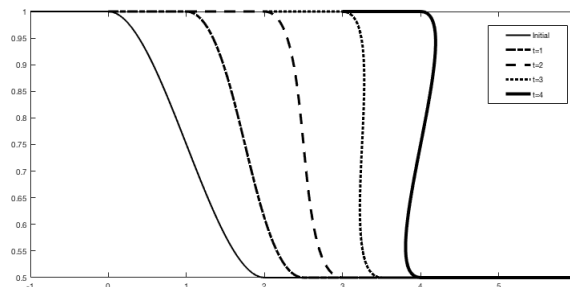
$$f(x) = \begin{cases} 1, & x \leq 0 \\ 1 - (1/8)x^2(3 - x), & 0 \leq x \leq 2 \\ 1/2, & x \geq 2. \end{cases}$$

Sketch some characteristics in the x, t plane. Do they intersect in $t > 0$? Run `mtc` with this data choice at times $t = 1, 2, 3, 4$. By choosing other times t_1, t_2, t_3, t_4 , carefully determine the time t_* when the profile first has a vertical tangent. Use the zoom feature to blow up the important parts of the graph.

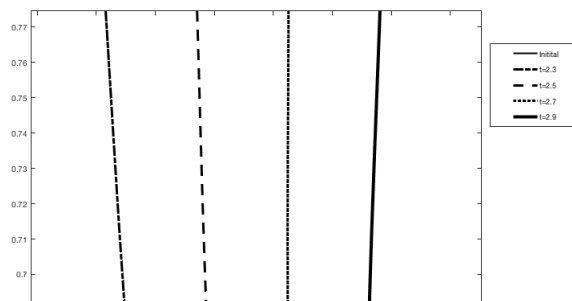
Solution. Here are some characteristics. They intersect at some time between $t = 2$ and $t = 4$:



Using `mtc` at times 1, 2, 3 and 4, we have the following profiles for $u(x, t)$:



Now, using `mtc` at times 2.3, 2.5, 2.7, and 2.9, and zooming in, we see that we have a vertical tangent at some time between $t = 2.7$ and $t = 2.9$:



□

2.4. Linearization.

Exercise 2.4.1. If u_0 is a constant, and u is a solution of $u_t + c(u)u_x = 0$, $u(x, 0) = f(x)$, show that $\max |u(x, t) - u_0| = \max |u(x, 0) - u_0|$.

Proof. Since u is constant on characteristic lines, then if (x, t) and $(x_0, 0)$ lie on the same characteristic line, we have $u(x, t) = u(x_0, 0)$. So $u(x, t) - u_0 = u(x_0, 0) - u_0$, and hence

$$\max_{(x,t)} |u(x, t) - u_0| = \max_x |u(x, 0) - u_0|.$$

□

HOMEWORK 3

2. First-Order Equations.

2.5. Weak solutions.

Exercise 2.5.1. Let $u(x, t)$ be defined for $(x, t) \in \mathbb{R}^2$ by

$$u(x, t) = \begin{cases} 1 & \text{for } x < t/2 \\ 0 & \text{for } x > t/2. \end{cases}$$

- (a) Show that u is a weak solution of $u_t + uu_x = 0$.
 (b) Show that u satisfies the integral form

$$\frac{d}{dt} \int_a^b u(x, t) dx = F(a, t) - F(b, t)$$

of the conservation law when $F(u) = u^2/2$.

Solution.

- (a) We use the change of variables $y = x - t/2$ to obtain:

$$\begin{aligned} \iint_{\mathbb{R}^2} \left(u \varphi_t(x, t) + \frac{u^2}{2} \varphi_x(x, t) \right) dx dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{t/2} \left(\varphi_t(x, t) + \frac{1}{2} \varphi_x(x, t) \right) dx dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^0 \left(\varphi_t(y + t/2, t) + \frac{1}{2} \varphi_x(y + t/2, t) \right) dy dt \\ &= \int_{-\infty}^0 \int_{-\infty}^{\infty} \left(\varphi_t(y + t/2, t) + \frac{1}{2} \varphi_x(y + t/2, t) \right) dt dy \\ &= \int_{-\infty}^0 \left(\int_{-\infty}^{\infty} \frac{d}{dt} \varphi(y + t/2, t) dt \right) dy \\ &= \int_{-\infty}^0 \left(\lim_{\tau \rightarrow \infty} [\varphi(y + \tau/2, \tau)] - \lim_{\tau \rightarrow -\infty} [\varphi(y + \tau/2, \tau)] \right) dy \\ &= 0 \end{aligned}$$

where the last equality follows from the fact that φ has finite support.

- (b) We use the hint given in the back of the book and consider three cases:
 (i) $a < b < t/2$:

$$\begin{aligned} \frac{d}{dt} \int_a^b u(x, t) dx &= \frac{d}{dt} (b - a) = 0 = \frac{1}{2} - \frac{1}{2} \\ &= \frac{u(a, t)^2}{2} - \frac{u(b, t)^2}{2} \\ &= F(u(a, t)) - F(u(b, t)) \end{aligned}$$

(ii) $a < t/2 < b$:

$$\begin{aligned} \frac{d}{dt} \int_a^b u(x,t) dx &= \frac{d}{dt} \int_a^{t/2} u(x,t) dx = \frac{d}{dt} \left(\frac{t}{2} - a \right) \\ &= \frac{1}{2} \\ &= \frac{u(a,t)^2}{2} - \frac{u(b,t)^2}{2} \\ &= F(u(a,t)) - F(u(b,t)) \end{aligned}$$

(iii) $t/2 < a < b$:

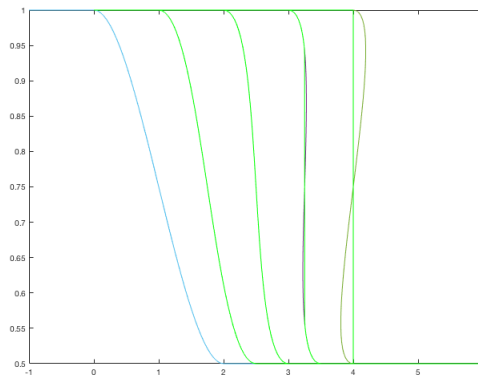
$$\frac{d}{dt} \int_a^b u(x,t) dx = 0 = \frac{u(a,t)^2}{2} - \frac{u(b,t)^2}{2} = F(u(a,t)) - F(u(b,t))$$

□

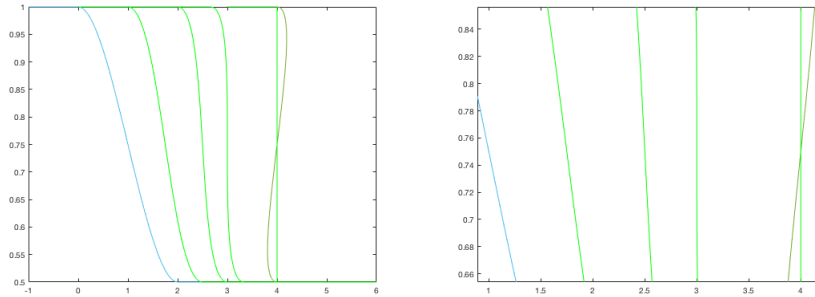
Exercise 2.5.2. Run `mtc` with data choice (1) and with $t_1 = 1, t_2 = 2, t_3 = 3$, and $t_4 = 4$. Type the MATLAB command `hold on`. Then run the program `shocks` with data choice (1) and the same times.

- Verify that $t = 8/3$ is a vertical tangent of u graphically by repeating the above process with appropriate choices of times. Use the zoom feature of MATLAB.
- Now run the program `shocks` with times $t_1 = 8/3$ and the other times at intervals of .2. The vertical segment of the profile indicates the location of the shock at that time. How fast is the shock moving? The values u_ℓ and u_r are the heights of the top and bottom of the vertical segment. Is the Rankine-Hugoniot condition satisfied?

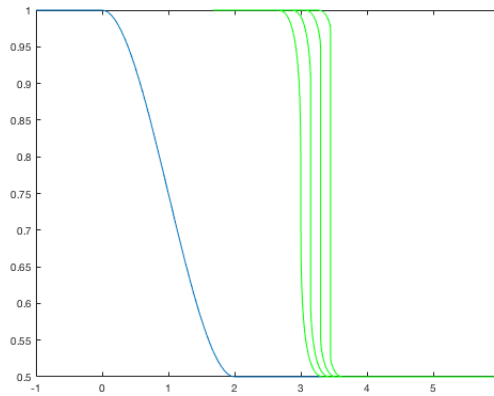
Solution. `mtc` with data choice (1) and with $t_1 = 1, t_2 = 2, t_3 = 3$, and $t_4 = 4$:



(a) **mtc** with data choice (1) and with $t_1 = 1, t_2 = 2, t_3 = 8/3$, and $t_4 = 4$



(b) **shocks** with data choice (1) and with $t_1 = 8/3, t_2 = 8/3 + .2, t_3 = 8/3 + .4$, and $t_4 = 8/3 + .6$.



The change in x is approximately $.15$, and the change in t is $.2$ so the shock is moving $.75$ units.

Using the zoom feature, we find that the tops of the vertical segments for times $t_1 = 8/3, t_2 = 8/3 + .2, t_3 = 8/3 + .4$, and $t_4 = 8/3 + .6$ are, $.945, .945, .96$, and $.98$, respectively. Similarly, the bottoms of the vertical segments are $.6, .57, .55$ and $.52$, respectively.

Checking the RH condition:

$$\frac{\frac{.945^2}{2} - \frac{.6^2}{2}}{.945 - .6} \approx .7725$$

$$\frac{\frac{.945^2}{2} - \frac{.57^2}{2}}{.945 - .57} \approx .7575$$

$$\frac{\frac{.96^2}{2} - \frac{.55^2}{2}}{.96 - .55} \approx .755$$

$$\frac{\frac{.98^2}{2} - \frac{.52^2}{2}}{.98 - .52} \approx .75$$

Ignoring user error, it looks like the RH condition is satisfied. \square

Exercise 2.5.5.

- (a) Verify that, when $u_l < u_r$ and $t > 0$,

$$v(x, t) = \begin{cases} u_l, & x \leq x_0 + tu_l \\ (x - x_0)/t, & x_0 + tu_l < x < x_0 + tu_r \\ u_r, & x \geq x_0 + tu_r \end{cases}$$

is a continuous weak solution of $v_t = vv_x = 0$. v is a rarefaction wave centered at $x = x_0$.

- (b) Sketch the characteristics of v , i.e., the lines along which v is constant.
 (c) Let $f(x)$ be a piecewise constant function

$$f(x) = \begin{cases} 1, & x < -1 \\ 1/2, & -1 < x < 1 \\ 3/2, & 1 < x < 2 \\ 1, & x > 2 \end{cases}.$$

Construct the weak solution of the IVP, $u_t + uu_x = 0$, $u(x, 0) = f(x)$ for small values of $t > 0$ using step shock waves and centered rarefaction waves. Sketch the characteristics of the solution.

Solution.

- (a) Without loss of generality, assume $x_0 = 0$. We have initial data

$$u(x, 0) = \begin{cases} u_\ell & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

We also have $F(u) = u^2/2$ and so the entropy condition

$$F'(u_\ell) = u_\ell > s > F'(u_r) = u_r$$

is not satisfied since $u_\ell < u_r$. Hence we have a rarefaction wave solution

$$u(x, t) = \begin{cases} u_\ell & x < u_\ell t \\ (F')^{-1}\left(\frac{x}{t}\right) & u_\ell t < x < u_r t \\ u_r & u_r t < x \end{cases} = \begin{cases} u_\ell & x < u_\ell t \\ \frac{x}{t} & u_\ell t < x < u_r t \\ u_r & x > u_r t, \end{cases}$$

which we know is a weak solution.

(b)

(c) We break up the constant values of the initial data into pairs and check the entropy conditions for each pair:

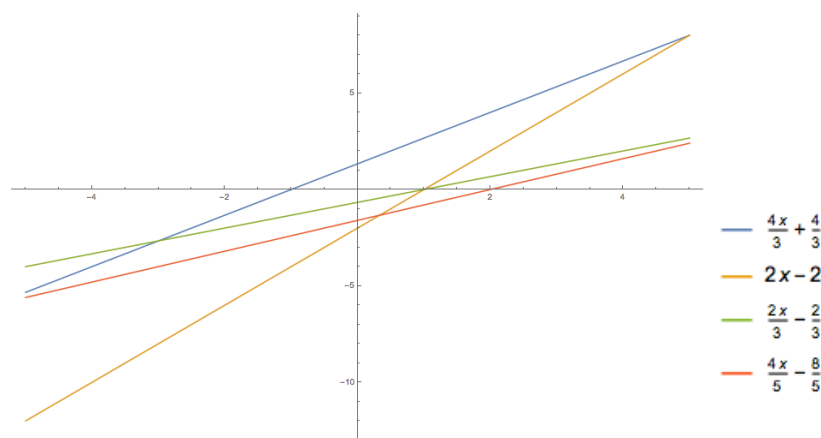
For $1, 1/2 \rightsquigarrow s = 3/4 \rightsquigarrow 1 > 3/4 > 1/2 \implies$ Shock Wave

For $1/2, 3/2 \rightsquigarrow s = 1 \rightsquigarrow 1/2 < 1 < 3/2 \implies$ Rarefaction Wave

For $3/2, 1 \rightsquigarrow s = 5/4 \rightsquigarrow 3/2 > 5/4 > 1 \implies$ Shock Wave

So our solution is

$$u(x, t) = \begin{cases} 1 & x < \frac{3}{4}t - 1 \\ \frac{1}{2} & \frac{3}{4}t - 1 < x < \frac{1}{2}t + 1 \\ \frac{x+1}{2} & \frac{1}{2}t + 1 < x < \frac{3}{2}t + 1 \\ \frac{3}{2} & \frac{3}{2}t + 1 < x < \frac{5}{4}t + 2 \\ 1 & x > \frac{5}{4}t + 2 \end{cases}$$



□

2.6. Numerical methods.

Exercise 2.6.1. Let a mesh $\Delta x, \Delta t$ be chosen, and set $\rho = \Delta x / \Delta t$. Consider the equation $u_t + cu_x = 0, u(x, 0) = f(x)$ with initial data $f(x) = \cos(\pi x / \Delta x)$.

(a) With $u_{j,0} = f(x_j) = (-1)^j$, show that the scheme

$$u_{j,n+1} = u_{j,n} - \frac{c}{\rho}(u_{j+1,n} - u_{j,n})$$

yields $u_{j,n} = (1 + 2c/\rho)^n (-1)^j$.

(b) For the same initial data, show that the scheme

$$u_{j,n+1} = (1 - \frac{c}{\rho})u_{j,n} + \frac{c}{\rho}u_{j-1,n}$$

yields $u_{j,n} = (1 - 2c/\rho)^n (-1)^j$.

Solution.

(a) By induction. The case $n = 0$ is easily verified. Suppose the claim holds for n . Then

$$\begin{aligned} u_{j,n+1} &= u_{j,n} - \frac{c}{\rho}(u_{j+1,n} - u_{j,n}) \\ &= (1 - 2c/\rho)^n (-1)^j - \frac{c}{\rho}((1 - 2c/\rho)^n (-1)^{j+1} - (1 - 2c/\rho)^n (-1)^j) \\ &= (1 - 2c/\rho)^n (-1)^j - \frac{c}{\rho}((1 - 2c/\rho)^n (-1)^{j+1} + (1 - 2c/\rho)^n (-1)^{j+1}) \\ &= (1 - 2c/\rho)^n (-1)^j - \frac{2c}{\rho}(1 - 2c/\rho)^n (-1)^{j+1} \\ &= (1 - 2c/\rho)^n (-1)^j + \frac{2c}{\rho}(1 - 2c/\rho)^n (-1)^j \\ &= (1 - 2c/\rho)(1 - 2c/\rho)^n (-1)^j \\ &= (1 - 2c/\rho)^{n+1} (-1)^j. \end{aligned}$$

(b) By induction. The case $n = 0$ is easily verified. Suppose the claim holds for n . Then

$$\begin{aligned} u_{j,n+1} &= (1 - \frac{c}{\rho})((1 - 2c/\rho)^n (-1)^j) + \frac{c}{\rho}((1 - 2c/\rho)^n (-1)^{j-1}) \\ &= ((1 - 2c/\rho)^n (-1)^j) - \frac{c}{\rho}((1 - 2c/\rho)^n (-1)^j) - \frac{c}{\rho}((1 - 2c/\rho)^n (-1)^j) \\ &= ((1 - 2c/\rho)^n (-1)^j) - \frac{2c}{\rho}((1 - 2c/\rho)^n (-1)^j) \\ &= (1 - 2c/\rho)(1 - 2c/\rho)^n (-1)^j \\ &= (1 - 2c/\rho)^{n+1} (-1)^j. \end{aligned}$$

□

Exercise 2.6.3.

- (a) Use the flux function
- $F(u) = u(\beta - u)$
- of example
- A*
- in the scheme

$$u_{j,n+1} = u_{j,n} - \frac{1}{\rho} [F(u_{j,n}) - F(u_{j-1,n})].$$

What is the resultant scheme?

- (b) What restrictions must we place on the values of the initial data so that $F'(u) \geq 0$ for all the values of u encountered in the problem?
- (c) Under this restriction on the initial data, how would you state the CFL condition?

Solution.

- (a) $u_{j,n+1} = u_{j,n} - \frac{1}{\rho} [u_{j,n}(\beta - u_{j,n}) - u_{j-1,n}(\beta - u_{j-1,n})].$
- (b) $0 \leq u \leq \beta/2.$
- (c) $\frac{\beta}{\rho} < 1.$

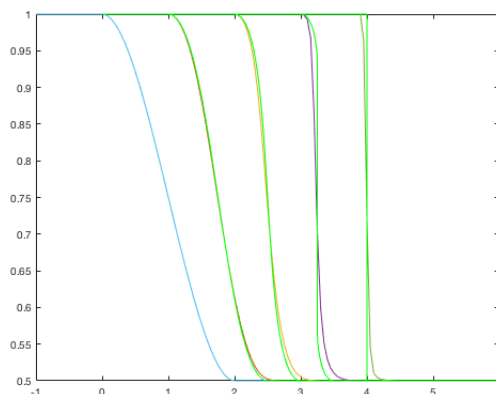
□

Exercise 2.6.4. Run `ct` with data choice (1), $\Delta x = \Delta t = .05$, with 20 time steps between snapshots. This will produce plots at times $t = 0, 1, 2, 3, 4$. Then type the command `hold on`. Next run program `shocks` with data choice (1) and the same times.

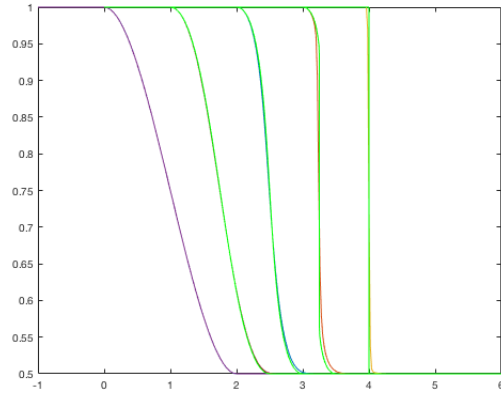
- (a) The plots are not identical. When do you begin to see significant differences?
- (b) Repeat the calculations for `shocks` and `c1`, but this time use $\Delta x = \Delta t = .02$ and 50 time steps between snapshots. How do the plots compare now?

Solution.

- (a) We begin to see differences around time $t = 3$.



(b) The plots are better now.



□

HOMEWORK 4

3. Diffusion.**3.2. The maximum principle.**

Exercise 3.2.1. Verify that each of the following functions satisfies the heat equation.

- (a) $u(x, t) = kt + \frac{1}{2}x^2 + C$.
- (b) $v(x, t) = \exp(-\gamma^2 kt) \sin(\gamma x)$, for any real γ .
- (c) $w(x, t) = \exp(-\gamma^2 kt) \cos(\gamma x)$, for any real γ .
- (d) $z(x, t) = \exp(kt \pm x)$.

Solution.

(a) $u_t(x, t) = k = ku_{xx}(x, t)$.

(b)

$$\begin{aligned} v_t(x, t) &= -\gamma^2 k \exp(-\gamma^2 kt) \sin(\gamma x) \\ v_x(x, t) &= \gamma \exp(-\gamma^2 kt) \cos(\gamma x) \\ v_{xx}(x, t) &= -\gamma^2 \exp(-\gamma^2 kt) \sin(\gamma x) \\ \implies v_t &= kv_{xx}. \end{aligned}$$

(c)

$$\begin{aligned} w_t(x, t) &= -\gamma^2 k \exp(-\gamma^2 kt) \cos(\gamma x) \\ w_x(x, t) &= -\gamma \exp(-\gamma^2 kt) \sin(\gamma x) \\ w_{xx}(x, t) &= -\gamma^2 \exp(-\gamma^2 kt) \cos(\gamma x) \\ \implies w_t &= kw_{xx}. \end{aligned}$$

(d)

$$\begin{aligned} z_t(x, t) &= k \exp(kt \pm x) \\ z_x(x, t) &= \exp(kt \pm x) \\ z_{xx}(x, t) &= \exp(kt \pm x) \\ \implies z_t &= kz_{xx}. \end{aligned}$$

□

Exercise 3.2.2.

- (a) For each of the functions in exercise 1, find the maximum and minimum over the rectangle $[-a, a] \times [0, T]$. Verify that the maximum principle of Theorem 3.1 is satisfied in each case. To which of the cases can you apply the maximum principle of Theorem 3.2?
- (b) Make 3-D plots of each of these functions using MATLAB, thereby visually verifying your results of part (a).

Solution.

(a) (i) $u(x, t) = kt + \frac{1}{2}x^2 + C$.

Since $u_t = k \neq 0$, u doesn't attain extreme values on the interior of the rectangle. So the maximum/minimum values u attains over the rectangle are that which it achieves on the boundary of the rectangle, hence Theorem 3.1 is verified. On the boundary, we have:

$$\max_{0 \leq t \leq T} u(\pm a, t) = \max_{0 \leq t \leq T} kt + \frac{a^2}{2} + C = kT + \frac{a^2}{2} + C$$

$$\min_{0 \leq t \leq T} u(\pm a, t) = \min_{0 \leq t \leq T} kt + \frac{a^2}{2} + C = \frac{a^2}{2} + C$$

$$\max_{-a \leq x \leq a} u(x, 0) = \max_{-a \leq x \leq a} \frac{1}{2}x^2 + C = \frac{a^2}{2} + C$$

$$\min_{-a \leq x \leq a} u(x, 0) = \min_{-a \leq x \leq a} \frac{1}{2}x^2 + C = C$$

$$\max_{-a \leq x \leq a} u(x, T) = \max_{-a \leq x \leq a} kT + \frac{1}{2}x^2 + C = kT + \frac{a^2}{2} + C$$

$$\min_{-a \leq x \leq a} u(x, T) = \min_{-a \leq x \leq a} kT + \frac{1}{2}x^2 + C = kT + C.$$

So on the given rectangle, we have

$$\max_{[-a, a] \times [0, T]} u(x, t) = kT + \frac{a^2}{2} + C \quad \text{and} \quad \min_{[-a, a] \times [0, T]} u(x, t) = C.$$

However, Theorem 3.2 does not apply since u is not bounded on the region $(-\infty, \infty) \times [0, T]$.

(ii) $v(x, t) = \exp(-\gamma^2 kt) \sin(\gamma x)$, for any real γ .

Setting $v_t = 0$, we must have $\gamma = 0$ or $\sin(\gamma x) = 0$. Setting $v_x = 0$, we must have $\gamma = 0$ or $\cos(\gamma x) = 0$. If $\gamma = 0$, then $v = 0$, and we are done. Suppose $\gamma \neq 0$. The condition that $\sin(\gamma x) = 0 = \cos(\gamma x)$ cannot happen, so v has no extreme values in the interior of the rectangle. So the maximum/minimum values u attains over the rectangle are that which it achieves on the boundary of the rectangle, hence Theorem 3.1 is verified. On the boundary, we have:

$$\begin{aligned} \max_{0 \leq t \leq T} v(a, t) &= \max_{0 \leq t \leq T} \exp(-\gamma^2 kt) \sin(\gamma a) = \begin{cases} \sin(\gamma a) & \text{if } \sin(\gamma a) \geq 0 \\ \exp(-\gamma^2 kT) \sin(\gamma a) & \text{if } \sin(\gamma a) < 0 \end{cases} \\ \min_{0 \leq t \leq T} v(a, t) &= \min_{0 \leq t \leq T} \exp(-\gamma^2 kt) \sin(\gamma a) = \begin{cases} \exp(-\gamma^2 kT) \sin(\gamma a) & \text{if } \sin(\gamma a) \geq 0 \\ \sin(\gamma a) & \text{if } \sin(\gamma a) < 0 \end{cases} \\ \max_{0 \leq t \leq T} v(-a, t) &= \max_{0 \leq t \leq T} \exp(-\gamma^2 kt) \sin(-\gamma a) \\ &= \max_{0 \leq t \leq T} -\exp(-\gamma^2 kt) \sin(\gamma a) = \begin{cases} \exp(-\gamma^2 kT) \sin(\gamma a) & \text{if } \sin(\gamma a) \geq 0 \\ \sin(\gamma a) & \text{if } \sin(\gamma a) < 0 \end{cases} \\ \min_{0 \leq t \leq T} v(-a, t) &= \min_{0 \leq t \leq T} \\ &= \max_{0 \leq t \leq T} -\exp(-\gamma^2 kt) \sin(\gamma a) = \begin{cases} \sin(\gamma a) & \text{if } \sin(\gamma a) \geq 0 \\ \exp(-\gamma^2 kT) \sin(\gamma a) & \text{if } \sin(\gamma a) < 0 \end{cases} \\ \max_{-a \leq x \leq a} v(x, 0) &= \max_{-a \leq x \leq a} \sin(\gamma x) = \begin{cases} 1 & \text{if } \gamma a \geq \pi \\ \sin(\gamma a) & \text{if } \gamma a < \pi \end{cases} \\ \min_{-a \leq x \leq a} v(x, 0) &= \min_{-a \leq x \leq a} \sin(\gamma x) = \begin{cases} -1 & \text{if } \gamma a \geq \pi \\ -\sin(\gamma a) & \text{if } \gamma a < \pi \end{cases} \\ \max_{-a \leq x \leq a} v(x, T) &= \max_{-a \leq x \leq a} \exp(-\gamma^2 kT) \sin(\gamma x) = \begin{cases} \exp(-\gamma^2 kT) & \text{if } \gamma a \geq \pi \\ \exp(-\gamma^2 kT) \sin(\gamma a) & \text{if } \gamma a < \pi \end{cases} \\ \min_{-a \leq x \leq a} v(x, T) &= \min_{-a \leq x \leq a} v(x, T) = \begin{cases} -\exp(-\gamma^2 kT) & \text{if } \gamma a \geq \pi \\ -\exp(-\gamma^2 kT) \sin(\gamma a) & \text{if } \gamma a < \pi \end{cases} \end{aligned}$$

So on the given rectangle, we have

$$\max_{[-a, a] \times [0, T]} v(x, t) = \begin{cases} 1 & \text{if } \gamma a \geq \pi \\ \sin(\gamma a) & \text{if } \gamma a < \pi \end{cases} \quad \text{and} \quad \min_{[-a, a] \times [0, T]} v(x, t) = \begin{cases} -1 & \text{if } \gamma a \geq \pi \\ -\sin(\gamma a) & \text{if } \gamma a < \pi \end{cases}$$

Since $|v(x, t)| \leq 1$, Theorem 3.2 applies.

(iii) $w(x, t) = \exp(-\gamma^2 kt) \cos(\gamma x)$, for any real γ . By the same reasoning as in (ii), Theorem 3.1 is satisfied. On the boundary, we have

$$\max_{0 \leq t \leq T} w(\pm a, t) = \max_{0 \leq t \leq T} \exp(-\gamma^2 kt) \cos(\gamma a) = \begin{cases} \cos(\gamma a) & \text{if } \cos(\gamma a) \geq 0 \\ \exp(-\gamma^2 kT) \cos(\gamma a) & \text{if } \cos(\gamma a) < 0 \end{cases}$$

$$\min_{0 \leq t \leq T} w(\pm a, t) = \min_{0 \leq t \leq T} \exp(-\gamma^2 kt) \cos(\gamma a) = \begin{cases} \exp(-\gamma^2 kT) \cos(\gamma a) & \text{if } \cos(\gamma a) \geq 0 \\ \cos(\gamma a) & \text{if } \cos(\gamma a) < 0 \end{cases}$$

$$\max_{-a \leq x \leq a} w(x, 0) = \max_{-a \leq x \leq a} \cos(\gamma x) = 1$$

$$\min_{-a \leq x \leq a} w(x, 0) = \min_{-a \leq x \leq a} \cos(\gamma x) = \begin{cases} -1 & \text{if } \gamma a \geq \pi \\ \cos(\gamma a) & \text{if } \gamma a < \pi \end{cases}$$

$$\max_{-a \leq x \leq a} w(x, T) = \max_{-a \leq x \leq a} \exp(-\gamma^2 kT) \cos(\gamma x) = \begin{cases} \exp(-\gamma^2 kT) & \text{if } \gamma a \geq \pi \\ \exp(-\gamma^2 kT) \cos(\gamma a) & \text{if } \gamma a < \pi \end{cases}$$

$$\min_{-a \leq x \leq a} w(x, T) = \min_{-a \leq x \leq a} w(x, T) = \begin{cases} -\exp(-\gamma^2 kT) & \text{if } \gamma a \geq \pi \\ \exp(-\gamma^2 kT) \cos(\gamma a) & \text{if } \gamma a < \pi \end{cases}$$

So on the given rectangle, we have

$$\max_{[-a, a] \times [0, T]} w(x, t) = 1$$

and

$$\min_{[-a, a] \times [0, T]} w(x, t) = \begin{cases} -1 & \text{if } \gamma a \geq \pi \\ \begin{cases} \exp(-\gamma^2 kT) \cos(\gamma a) & \text{if } \cos(\gamma a) \geq 0 \\ \cos(\gamma a) & \text{if } \cos(\gamma a) < 0 \end{cases} & \text{if } \gamma a < \pi \end{cases}$$

Since $|w(x, t)| \leq 1$, Theorem 3.2 applies.

(iv) $z(x, t) = \exp(kt \pm x)$.

We have $z_x(x, t) = \exp(kt \pm x)$ and $v_t(x, t) = k \exp(kt \pm x)$ which cannot be 0. So z has no extreme values in the interior of the rectangle. So the maximum/minimum values z attains over the rectangle are that which it achieves on the boundary of the rectangle, hence Theorem 3.1 is verified. On the boundary, we have for $z(x, t) = \exp(kt + x)$,

$$\begin{aligned} \max_{0 \leq t \leq T} z(a, t) &= \max_{0 \leq t \leq T} \exp(kt + a) = \exp(kT + a) \\ \min_{0 \leq t \leq T} z(a, t) &= \min_{0 \leq t \leq T} \exp(kt + a) = \exp(a) \\ \max_{0 \leq t \leq T} z(-a, t) &= \max_{0 \leq t \leq T} \exp(kt - a) = \exp(kT - a) \\ \min_{0 \leq t \leq T} z(-a, t) &= \min_{0 \leq t \leq T} \exp(kt - a) = \exp(-a) \\ \max_{-a \leq x \leq a} z(x, 0) &= \max_{-a \leq x \leq a} \exp(x) = \exp(a) \\ \min_{-a \leq x \leq a} z(x, 0) &= \min_{-a \leq x \leq a} \exp(x) = \exp(-a) \\ \max_{-a \leq x \leq a} z(x, T) &= \max_{-a \leq x \leq a} \exp(kT + x) = \exp(kT + a) \\ \min_{-a \leq x \leq a} z(x, T) &= \min_{-a \leq x \leq a} \exp(kT + x) = \exp(kT - a). \end{aligned}$$

Moreover, if $z(x, t) = \exp(kt - x)$, we have

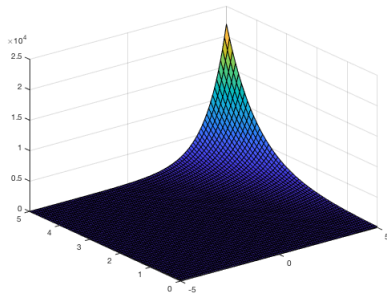
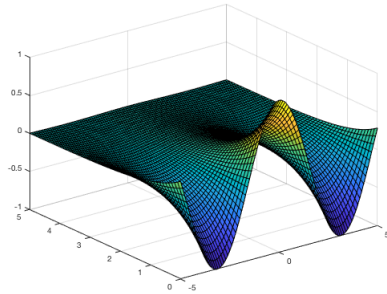
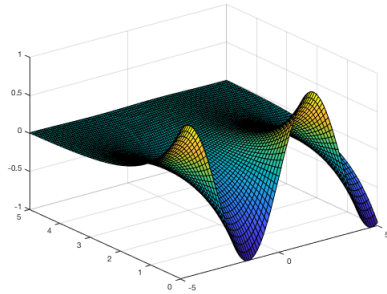
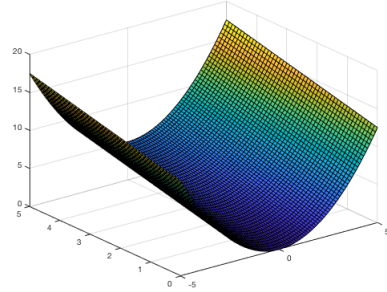
$$\begin{aligned} \max_{0 \leq t \leq T} z(a, t) &= \max_{0 \leq t \leq T} \exp(kt - a) = \exp(kT - a) \\ \min_{0 \leq t \leq T} z(a, t) &= \min_{0 \leq t \leq T} \exp(kt - a) = \exp(-a) \\ \max_{0 \leq t \leq T} z(-a, t) &= \max_{0 \leq t \leq T} \exp(kt + a) = \exp(kT + a) \\ \min_{0 \leq t \leq T} z(-a, t) &= \min_{0 \leq t \leq T} \exp(kt + a) = \exp(a) \\ \max_{-a \leq x \leq a} z(x, 0) &= \max_{-a \leq x \leq a} \exp(-x) = \exp(a) \\ \min_{-a \leq x \leq a} z(x, 0) &= \min_{-a \leq x \leq a} \exp(-x) = \exp(-a) \\ \max_{-a \leq x \leq a} z(x, T) &= \max_{-a \leq x \leq a} \exp(kT - x) = \exp(kT + a) \\ \min_{-a \leq x \leq a} z(x, T) &= \min_{-a \leq x \leq a} \exp(kT - x) = \exp(kT - a). \end{aligned}$$

So on the given rectangle, we have (for either $\exp(kt+x)$ or $\exp(kt-x)$),

$$\max_{[-a, a] \times [0, T]} z(x, t) = \exp(kT + a) \quad \text{and} \quad \min_{[-a, a] \times [0, T]} z(x, t) = \exp(-a)$$

Since \exp is not bounded on $(-\infty, \infty) \times [0, T]$, Theorem 3.2 does not apply.

(b) These plots help me visually verify the results from (a). They are given in order.



□

Exercise 3.2.3. Let $u(x, t)$ and $v(x, t)$ both be solutions of the equation

$$u_t - ku_{xx} = q.$$

Suppose that $u(x, t) \leq v(x, t)$ for $|x| \leq L$, $t = 0$, and for $x = \pm L$, $0 \leq t \leq T$. Show that $u(x, t) \leq v(x, t)$ for $|x| \leq L$, $0 \leq t \leq T$.

Solution. Notice that

$$(u - v)_t - k(u - v)_{xx} = u_t - ku_{xx} - v_t - kv_{xx} = q - q = 0.$$

So $u - v$ satisfies the heat equation. Our assumptions on u and v imply that $u(x, t) - v(x, t) \leq 0$ on the region

$$\Gamma_T := \{(x, t) : x = -L, L, 0 \leq t \leq T\} \cup \{(x, t) : t = 0, -L \leq x \leq L\}.$$

Let Q_T be

$$Q_T = \{(x, t) \in \mathbb{R}^2 : -L < x < L, 0 < t < T\}.$$

Then by Theorem 3.1, we conclude that

$$\max_{\overline{Q_T}}(u - v) = \max_{\Gamma_T}(u - v) \leq 0,$$

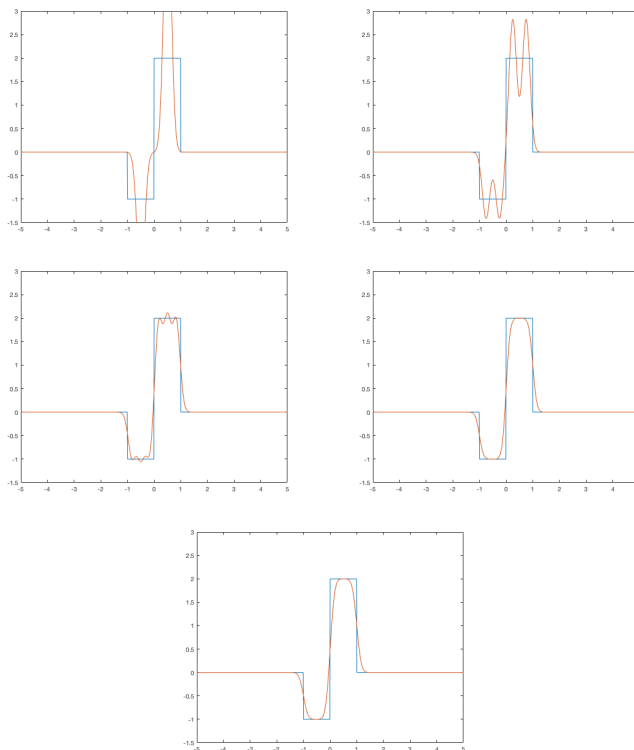
and hence $u(x, t) \leq v(x, t)$ on the region $\overline{Q_T}$, as desired. \square

HOMEWORK 5

3. Diffusion.**3.3. The heat equation with boundaries.**

Exercise 3.3.2. Run `heat1` with time $t = .01$, $k = 1$, and with $n = 2, 4, 6, 8, 12$. To get a better view, restrict the plots to the interval $[-5, 5]$ with the command `axis([-5 5 -1.5 3])`. What happens as n increases?

Solution. The plots are displayed in order for $n = 2, 4, 6, 8, 12$. As n increases, the approximation better resembles the initial data f .



□

Exercise 3.3.4. Let $f(x) = \alpha$ for $x < 0$, and $f(x) = \beta$ for $x > 0$ and let $u(x, t)$ be the solution of $u_t = ku_{xx}$, $u(x, 0) = f(x)$ for $x \in \mathbb{R}$.

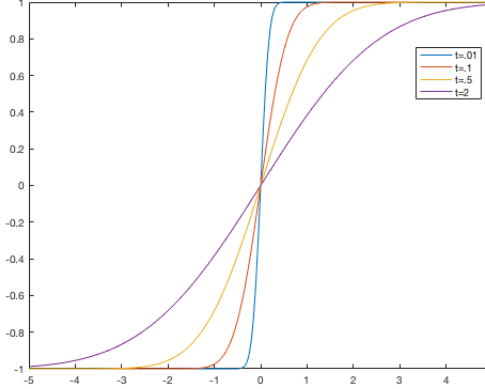
- Show that for this data, the formula $u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} f(y) dy$ can be rewritten in terms of the (extended) error function. You will need to make the change of variable $z = \frac{(y-x)}{\sqrt{4kt}}$.
- Show that for each x , $\lim_{t \rightarrow \infty} u(x, t) = (\alpha + \beta)/2$.
- Using the error function of MATLAB `erf(x)`, plot the solution on $[-5, 5]$ for $k = 1$ and $t = .01, .1, .5, 2$.

Solution.

(a) We apply the change of variable $z = \frac{(y-x)}{\sqrt{4kt}}$ at the third equality:

$$\begin{aligned}
 u(x, t) &= \frac{1}{\sqrt{4\pi kt}} \left(\alpha \int_{-\infty}^0 e^{-\frac{(x-y)^2}{4kt}} dy + \beta \int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy \right) \\
 &= \frac{1}{\sqrt{\pi}} \left(\alpha \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-z^2} dz + \beta \int_{\frac{-x}{\sqrt{4kt}}}^{\infty} e^{-z^2} dz \right) \\
 &= \frac{1}{\sqrt{\pi}} \left(\alpha \left(\int_0^{\infty} e^{-z^2} dz - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz \right) \right. \\
 &\quad \left. + \beta \left(\int_0^{\infty} e^{-z^2} dz - \int_0^{\frac{-x}{\sqrt{4kt}}} e^{-z^2} dz \right) \right) \\
 &= \frac{1}{\sqrt{\pi}} \left(\alpha \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) + \beta \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf} \left(\frac{-x}{\sqrt{4kt}} \right) \right) \right) \\
 &= \frac{\alpha}{2} \left(1 - \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) + \frac{\beta}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right)
 \end{aligned}$$

(b) This follows from our formula found in (a) and the fact that $\lim_{x \rightarrow 0} \operatorname{erf}(x) = 0$.
 (c)



□

Exercise 3.3.6. Assume that $u(x, t)$ is a solution of $u_t = ku_{xx}$, such that u and u_x tend to zero rapidly, as $x \rightarrow \pm\infty$. Let $Q = \int_{\mathbb{R}} u(x, t) dx$. We have already seen that Q is a conserved quantity. Here are two more quantities associated with solutions of the heat equation that have a special behavior. In the following, assume that $Q \neq 0$.

- (a) Show that $m = \frac{1}{Q} \int_{\mathbb{R}} xu(x, t) dx$ is independent of t .
 (b) Let

$$p(t) = \frac{1}{Q} \int_{\mathbb{R}} (x - m)^2 u(x, t) dx.$$

Show that $p(t) = p(0) + 2kt$.

(c) Show that

$$p(t) = \frac{1}{Q} \int_{\mathbb{R}} x^2 u(x, t) dx - m^2$$

(d) Find m and $p(t)$ for the fundamental solution $S(x, t)$.

(e) Evaluate the integral

$$\int_{-\sqrt{p(t)}}^{\sqrt{p(t)}} S(x, t) dx$$

using error function. The amount of heat contained in the interval $[-\sqrt{p(t)}, \sqrt{p(t)}]$ is a constant fraction of the total amount of heat.

Solution.

(a) Below, the second equality follows since u satisfies the heat equation, and the last equality follows since $u, u_x \rightarrow 0$ rapidly as $x \rightarrow \pm\infty$.

$$\begin{aligned} \frac{dm}{dt} &= \frac{1}{Q} \int_{\mathbb{R}} x u_t(x, t) dx \\ &= \frac{k}{Q} \int_{\mathbb{R}} x u_{xx}(x, t) dx \\ &= \frac{k}{Q} \left(x u_x(x, t) \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} u_x(x, t) dx \right) \\ &= 0. \end{aligned}$$

(b) First, notice that

$$\begin{aligned} \frac{dm}{dx} &= \frac{1}{Q} \int_{\mathbb{R}} [u(x, t) + x u_x(x, t)] dx \\ &= \frac{1}{Q} \left(Q + x u(x, t) \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} u(x, t) dx \right) \\ &= \frac{1}{Q} (Q + 0 - Q) \\ &= 0 \end{aligned}$$

since $u, u_x \rightarrow 0$ as $x \rightarrow \pm\infty$. Hence

$$\begin{aligned} \frac{dp}{dt} &= \frac{k}{Q} \int_{\mathbb{R}} (x - m)^2 u_{xx}(x, t) dx \\ &= \frac{k}{Q} \left((x - m)^2 u_x(x, t) \Big|_{-\infty}^{\infty} - 2 \int_{\mathbb{R}} u_x(x, t) (x - m) dx \right) \\ &= \frac{k}{Q} \left(0 - 2 \left((x - m) u(x, t) \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} u(x, t) dx \right) \right) \\ &= \frac{k}{Q} (-2(0 - Q)) \\ &= 2k. \end{aligned}$$

So $p(t) = 2kt + C$, C constant. Then $p(0) = C$, and hence $p(t) = 2kt + p(0)$.

(c)

$$\begin{aligned}
p(t) &= \frac{1}{Q} \int_{\mathbb{R}} (x^2 - 2xm + m^2)u(x, t)dx \\
&= \frac{1}{Q} \int_{\mathbb{R}} x^2 u(x, t)dx - \frac{2m}{Q} \int_{\mathbb{R}} xu(x, t)dx + \frac{m^2}{Q} \int_{\mathbb{R}} u(x, t)dx \\
&= \frac{1}{Q} \int_{\mathbb{R}} x^2 u(x, t)dx - 2m^2 + m^2 \\
&= \frac{1}{Q} \int_{\mathbb{R}} x^2 u(x, t)dx - m^2.
\end{aligned}$$

(d) Note that $g(x, t) := xe^{-\frac{x^2}{4kt}}$ is an odd function in x , so $\int_{\mathbb{R}} g(x, t)dx = 0$. Hence for $u = S$,

$$m = \frac{1}{Q} \left(\frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} g(x, t)dx \right) = 0.$$

Note that $Q = 1$. Now using part (c) and a change of variable $y = x/\sqrt{4kt}$,

$$p(t) = \frac{1}{Q} \int_{\mathbb{R}} x^2 S(x, t) - m^2 = \frac{4kt}{\sqrt{\pi}} \int_{\mathbb{R}} y^2 e^{-y^2} dy.$$

Hence $p(0) = 0$, and so $p(t) = 2kt$.

(e) Using the same change of variable as in part (d),

$$\begin{aligned}
\int_{-\sqrt{p(t)}}^{\sqrt{p(t)}} S(x, t)dx &= \int_{-\sqrt{2kt}}^{\sqrt{2kt}} S(x, t)dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} e^{-y^2} dy \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-y^2} dy \\
&= \operatorname{erf} \left(\frac{1}{\sqrt{2}} \right).
\end{aligned}$$

□

Exercise 3.3.9. Solve the initial-value problem

$$u_t - ku_{xx} + \gamma u = 0, \quad u(x, 0) = f(x).$$

Solution. Let $v(x, t) := e^{\gamma t}u(x, t)$. Then

$$v_t - kv_{xx} = \gamma e^{\gamma t}u + e^{\gamma t}u_t - ke^{\gamma t}u_{xx} = e^{\gamma t}(u_t - ku_{xx} + \gamma u) = 0,$$

so v satisfies the heat equation. So

$$u(x, t) = \frac{e^{-\gamma t}}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} f(y) dy.$$

□

3.4. Boundary value problems on the half-line.

Exercise 3.4.2. Let u be a solution of either

$$u_t = ku_{xx} \text{ for } x, t > 0, \quad u(x, 0) = f(x) \text{ for } x > 0, \quad u(0, t) = 0 \text{ for } t > 0$$

$$\text{or } u_t = ku_{xx} \text{ for } x, t > 0, \quad u(x, 0) = f(x) \text{ for } x > 0, \quad u_x(0, t) = 0 \text{ for } t > 0.$$

Assume that $u(x, t) \geq 0$ and that $u, u_x, u_{xx} \rightarrow 0$ rapidly as $x \rightarrow \infty$.

- Compute the rate of change of the total heat energy $\int_0^\infty u(x, t) dx$.
- For which boundary conditions (Dirichlet or Neumann) is this quantity conserved? Why?
- Assume the Dirichlet boundary condition. If the temperature in the bar $u(x, t) > 0$ for x near zero, does the total amount of heat increase or decrease? If the temperature outside the bar is greater than the temperature inside, does the total amount of heat increase or decrease?

Solution.

$$\begin{aligned} \text{(a)} \quad \frac{d}{dt} \int_0^\infty u(x, t) dx &= k \int_0^\infty u_{xx}(x, t) dx = k \left(\lim_{a \rightarrow \infty} u_x(a, t) - u_x(0, t) \right) \\ &= -ku_x(0, t) \\ &= \begin{cases} -ku_x(0, t) & \text{for Dirichlet bound. cond.,} \\ 0 & \text{for Neumann bound. cond.} \end{cases} \end{aligned}$$

- The rate of change of the total heat energy is conserved for the the Neumann boundary conditions. This is because, as page 95 explains, the Neumann conditions mean that the end of the bar is insulated so that no heat can flow, i.e. Flux = 0.
- For fixed t , the Dirichlet boundary condition $u(0, t) = 0$ and the fact that $u(x, t) > 0$ near 0 together imply that u is increasing near 0. In other words $u_x(0, t) > 0$, giving

$$\frac{d}{dt} \int_0^\infty u(x, t) dx = -ku_x(0, t) < 0.$$

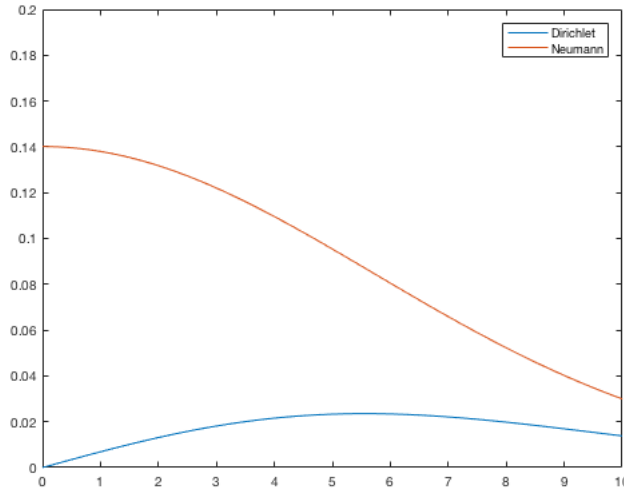
So the total amount of heat energy is decreasing in this situation.

If the temperature outside the bar is greater than the temperature inside the bar, then u is decreasing near zero, i.e. $u_x(x, t) < 0$, so the integral above is positive, i.e. the total heat energy is increasing.

□

Exercise 3.4.3. The program `heat2` solves the initial-boundary-value problems of Dirichlet and Neumann. Which solution decays faster? Find constants C_D, γ_D, C_N and γ_N such that the solution of the Dirichlet boundary-value problem decays like $C_D t^{-\gamma_D}$ at $x = 1$ and the solution of the Neumann boundary-value problem decays like $C_N t^{-\gamma_N}$ at $x = 1$.

Solution. Plotting the solutions at time $t = 15$, it appears that the solution of the Dirichlet boundary-value problem decays faster than that of the Neumann boundary-value problem.



In the code for `heat2`, the vector x is given as $x = -10 : .05 : 10$, so the 221st entry of the vector x corresponds to when $x = 1$, i.e., $x(221) = 1$.

Using time values $t = 10, 11$ in the `heat2` program, we find the values $u(1, 10)$ and $u(1, 11)$ for the solution to the Dirichlet IBVP by the code `dirch(221)` since the 221st entry of the vector `dirch` is the value of $u(1, t)$ at a given time t . We find that $u(1, 10) = .0123$ and $u(1, 11) = .0107$. So

$$\frac{C_D 10^{\gamma_D}}{C_D 11^{\gamma_D}} = \frac{u(1, 10)}{u(1, 11)} = \frac{0.0123}{0.0107} \implies \gamma_D = \log_{11/10} \left(\frac{0.0123}{0.0107} \right) \approx 1.46,$$

which gives $C_D \approx 0.35$ or $C_D \approx .31$. In a similar manner we find that $\gamma_N \approx .43$ and $C_N \approx .44$ or $C_N \approx .43$. \square

Exercise 3.4.5.

(a) Starting with the solution formula for the IBVP

$$u_t = ku_{xx} \text{ for } x, t > 0, \quad u(x, 0) = f(x) \text{ for } x > 0, \quad u(0, t) = 0 \text{ for } t > 0,$$

show that the solution for this IBVP with initial data $f(x) \equiv U$ is given by

$$u(x, t) = U - 2U \int_x^\infty S(y, t) dy.$$

(b) Verify directly that this function u solves the above IBVP.(c) Make the change of variable $z = y/\sqrt{4kt}$, and show that $u(x, t) \rightarrow 0$ for each x as $t \rightarrow \infty$.*Solution.*

(a) Recall that in Exercise 3.3.4 we proved

$$\begin{aligned} \int_{\mathbb{R}} S(x-y, t) f(y) dy &= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} f(y) dy \\ &= \frac{\alpha}{2} \left(1 - \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) + \frac{\beta}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) \end{aligned}$$

where $f(x) = \alpha$ when $x > 0$ and $f(x) = \beta$ when $x < 0$. Applying this formula to the case $f(x) = U$ for all $x \in \mathbb{R}$, we get

$$\begin{aligned} U \int_{\mathbb{R}} S(x-y, t) dy &= \frac{U}{2} \left(1 - \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) + \frac{U}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) \\ &= \frac{U}{2} (1 + 1) \\ &= U, \end{aligned}$$

which means $\int_{\mathbb{R}} S(x-y, t) dy = 1$. Notice that by the change of variable $z = x-y$,

$$\int_{-\infty}^0 S(x-y, t) dy = - \int_{\infty}^x S(z, t) dz = \int_x^{\infty} S(z, t) dz.$$

By the change of variable $z = x+y$,

$$\int_0^{\infty} S(x+y, t) dy = \int_x^{\infty} S(z, t) dz.$$

Putting this all together, we obtain

$$\begin{aligned} u(x, t) &= U \int_0^{\infty} [S(x-y, t) - S(x+y, t)] dy \\ &= U \left(\int_{\mathbb{R}} S(x-y, t) - \int_{-\infty}^0 S(x-y, t) - \int_0^{\infty} S(x+y, t) dy \right) \\ &= U \left(1 - \int_x^{\infty} S(z, t) dz - \int_x^{\infty} S(z, t) dz \right) \\ &= U \left(1 - 2 \int_x^{\infty} S(z, t) dz \right) \end{aligned}$$

(b) Since S satisfies the heat equation,

$$u_t(x, t) = -2U \int_x^\infty S_t(y, t) dy = -2Uk \int_x^\infty S_{xx}(y, t) dy.$$

Also, $u_{xx}(x, t) = 2US_x(x, t)$. So

$$\begin{aligned} u_t(x, t) &= -2Uk \int_x^\infty S_{xx}(y, t) dy = -2Uk \left(\lim_{z \rightarrow \infty} S_x(z, t) - S_x(x, t) \right) \\ &= -2Uk(-S_x(x, t)) \\ &= ku_{xx}(x, t). \end{aligned}$$

Moreover, $u(x, 0) = U - 0 = U$ and $u(0, t) = U - 2U(1/2) = 0$.

(c) Making the change of variable, we have $u(x, t) = U - \frac{2U}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-z^2} dz$. So

$$\lim_{t \rightarrow \infty} u(x, t) = U - \frac{2U}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz = U - \frac{2U}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 0.$$

□

Exercise 3.4.6.

(a) Using the solution of exercise 5, show that the solution of the inhomogeneous IBVP

$$v_t - kv_{xx} = 0 \quad x, t > 0, \quad v(x, 0) = 0, x > 0, \quad v(0, t) = U, t > 0,$$

is given by $v(x, t) = 2U \int_x^\infty S(y, t) dy$.

(b) Show that $v(x, t) \rightarrow U$ for each x as $t \rightarrow \infty$.

(c) Write the solution v in terms of the error function $\operatorname{erf}(x)$.

Solution.

(a) The desired solution has the form $v(x, t) = U - u(x, t)$, so we need to check that this formula for $v(x, t)$ solves the given IBVP. We have

$$v_t = -u_t = -ku_{xx} = kv_{vv}, \quad v(0, t) = U - u(0, t) = U - 0 = U,$$

and $v(x, 0) = U - u(x, 0) = U - f(x) = U - U = 0$. So $v(x, t)$ is a solution to the given IBVP.

(b) $\lim_{t \rightarrow \infty} v(x, t) = U - \lim_{t \rightarrow \infty} u(x, t) = U - 0 = U$.

(c) Using change of variable $z = y/\sqrt{4kt}$, we get

$$\begin{aligned} v(x, t) &= 2U \int_x^\infty e^{-\frac{y^2}{4kt}} dy = \frac{2U}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-z^2} dz \\ &= \frac{2U}{\sqrt{\pi}} \left(\int_0^\infty e^{-z^2} dz - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz \right) \\ &= \frac{2U}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right) \\ &= U \left(1 - \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right). \end{aligned}$$

□

HOMEWORK 6

3. Diffusion.**3.5. Diffusion and nonlinear wave motion.****Exercise 3.5.5.** The nonlinear equation

$$u_t - ku_{xx} = f(u)$$

is called a reaction-diffusion equation. The nonlinear term $f(u)$ represents the reaction of chemicals while the term ku_{xx} as usual represents diffusion.

- Let $f(u) = u(1 - u)$. What constant solutions are there?
- Look for a traveling wave solution $u(x, t) = \varphi(x - ct)$. What second-order nonlinear ODE must φ satisfy?
- Write this second-order ODE as a first order system by introducing the new dependent variable $\psi = \varphi'$. Find the critical points of this system. Linearize the system about each of these critical points, and discuss the stability of each (this will depend on the relationship between c and k).
- It can be proved that for each $c < 0$, there is a unique trajectory $s \rightarrow (\varphi_c(s), \psi_c(s))$ which approaches $(1, 0)$ as $s \rightarrow \infty$, and approaches $(0, 0)$ as $s \rightarrow -\infty$. Sketch the graph of $\varphi_c(s)$ for different values of c .

Solution.

- A constant solution would have to satisfy $u(1 - u) = 0$, so $u = 0$ or $u = 1$ are the only constant solutions.
- Looking for a solution of the form $u(x, t) = \varphi(x - ct)$, φ would need to satisfy the ODE

$$-c\varphi' - k\varphi'' = \varphi(1 - \varphi).$$

- Setting $\psi = \varphi'$, we obtain the system

$$\begin{bmatrix} \varphi' \\ \psi' \end{bmatrix} = \begin{bmatrix} \psi \\ \frac{1}{k}(-\varphi(1 - \varphi) - c\psi) \end{bmatrix}$$

The critical points of this system are $(0, 0)$ and $(1, 0)$. Letting $f = (\varphi', \psi')$,

$$Df = \begin{bmatrix} 0 & 1 \\ \frac{1}{k}(2\varphi - 1) & \frac{-c}{k} \end{bmatrix}.$$

The linearized system for $(0, 0)$ is

$$\begin{bmatrix} \varphi' \\ \psi' \end{bmatrix} = Df|_{(0,0)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ \frac{1}{k}(-\varphi - c\psi) \end{bmatrix}$$

The eigenvalues for $Df|_{(0,0)}$ are

$$\lambda_{\pm} = \frac{-c \pm \sqrt{c^2 - 4k}}{2k},$$

Note that $|c| > \sqrt{c^2 - 4k}$.

- If $c > 0$, then

$$(3.5.5.1) \quad c = |c| > \sqrt{c^2 - 4k} \implies -c + \sqrt{c^2 - 4k} < 0$$

- If $c^2 - 4k \geq 0$, we have real eigenvalues, $\lambda_- < 0$, and 3.5.5.1 implies $\lambda_+ < 0$. Hence $(0, 0)$ is an asymptotically stable critical point.

- If $c^2 - 4k < 0$ then we have imaginary eigenvalues and $\operatorname{Re}(\lambda_{\pm}) = -c/2k < 0$, so $(0, 0)$ is an asymptotically stable critical point.
- If $c < 0$ then
 - (3.5.5.2) $-c = |c| > \sqrt{c^2 - 4k} \implies -c - \sqrt{c^2 - 4k} > 0$
 - If $c^2 - 4k \geq 0$, we have real eigenvalues, $\lambda_+ > 0$, and 3.5.5.2 implies $\lambda_- > 0$. Hence $(0, 0)$ is an unstable critical point.
 - If $c^2 - 4k < 0$ then we have imaginary eigenvalues and $\operatorname{Re}(\lambda_{\pm}) > 0$, so $(0, 0)$ is an unstable critical point.
- If $c = 0$, then $\operatorname{Re}\lambda_{\pm} = \operatorname{Re}(\pm i/\sqrt{k}) = 0$ so $(0, 0)$ is a stable critical point.

The linearized system for $(1, 0)$ is

$$\begin{bmatrix} \varphi' \\ \psi' \end{bmatrix} = Df|_{(1,0)} \begin{bmatrix} \varphi - 1 \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ \frac{1}{k}(\varphi - 1 - c\psi) \end{bmatrix}.$$

The eigenvalues for $Df|_{(0,0)}$ are

$$\lambda_{\pm} = \frac{-c \pm \sqrt{c^2 + 4k}}{2k},$$

Since $(\lambda_+)(\lambda_-) = -1/k$, we have an unstable saddle at $(1, 0)$.

(d)

□

PROJECT 3.7

Write a MATLAB program to implement the Explicit Finite Difference Scheme (3.33):

$$u_{j,n+1} = (1 - 2s)u_{j,n} + s(u_{j+1,n} + u_{j-1,n}).$$

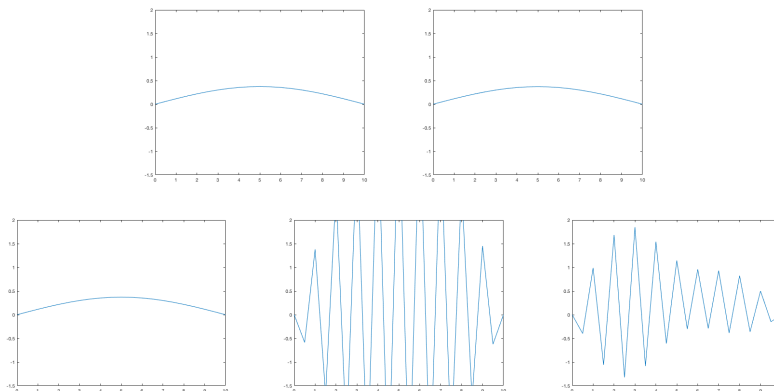
Set $k = 1$ and use the boundary conditions $u = 0$ at $x = 0$ and $x = 10$. Write an mfile `f.m` for the initial data. Try out your program with initial data $\sin(\frac{\pi x}{10})$.

Exercise 3.7.6. (15pts) Fix $\Delta x = 0.5$ and then experiment with various values of Δt to see when the scheme becomes stable. Your observations should agree with the results of section 3.6.

- (5pts) What is the condition on step size for the numerical scheme to be stable in this case?
- (5pts) Print the graphs of the numerical solution for time $t = 10$ with $\Delta t = 0.5, 0.2, 0.125, 0.1, 0.0005$.
- (5pts) Do you observe that the scheme becomes stable for a certain step size?

Solution.

- We must have $\Delta t \leq \frac{\Delta x^2}{2}$, so in our case, this means $\Delta t \leq .125$.
- The following plots correspond to the desired time steps in numerical order, i.e. $\Delta t = 0.0005, 0.1, 0.125, 0.2, 0.5$



- Yup, at $\Delta t = .125$, which corresponds to our result in (a).

MATLAB Code for Exercise 3.7.6, Part (b):

```
delt=.5;
delx=.5;
x=0:delx:10;
xsteps=10/delx;
T=10;
t=0:delt:T;
tsteps=T/delt;
s=delt/delx^2;

u=zeros(xsteps+1,tsteps+1);
```

```

for j=1:xsteps+1
    u(j,1)=f(x(j));
end
for n=1:tsteps
    u(1,n+1) = 0;
    u(xsteps+1,n+1) = 0;
    for j=2:xsteps
        u(j,n+1)=(1-2*s)*u(j,n)+s*(u(j+1,n)+u(j-1,n));
    end
end
plot(x,u(:,tsteps+1))
axis([0 10 -1.5 2])

```

□

Exercise 3.7.7. (15pts) Compare your computed results with the exact solution

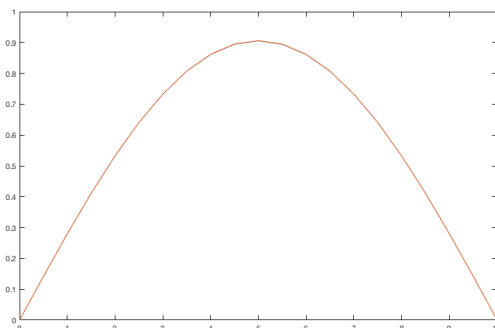
$$\sin\left(\frac{\pi x}{10}\right)e^{-\left(\frac{10}{\pi}\right)^2 t}$$

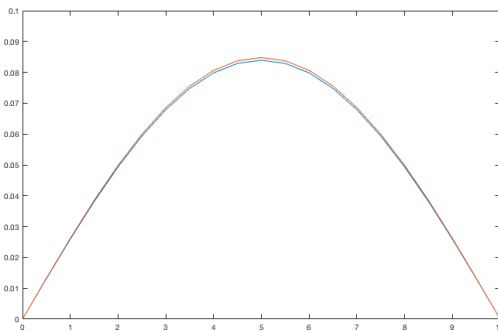
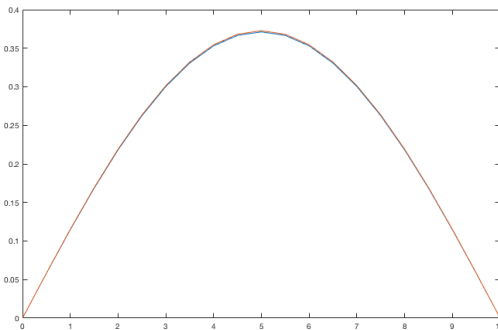
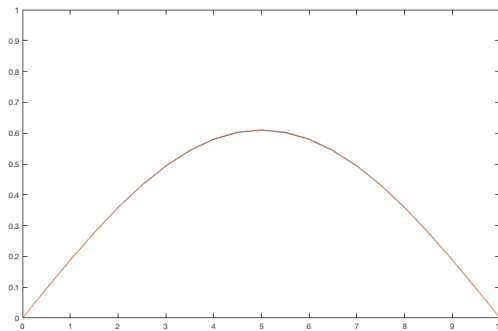
at various times with $\Delta t = 0.125$. Find the maximum error at each time.

- (5pts) Print the graphs of the numerical solution and the exact solution on the same figures for time $t = 1, 5, 10, 25$.
- (5pts) Calculate the maximum error for each of these times. That is $\max(Exact(x, t) - Approximation(x, t))$.
- (5pts) What happens to the maximum error as time increases?

Solution.

- The following plots of the numerical and exact solutions correspond to times $t = 1, 5, 10, 25$, in order. Note the difference in the vertical axis scales for the last two plots from that of the first two plots.





(b) Using the following MATLAB code at $T = 1, 5, 10, 25$,

```
T=25;
delt=.125; delx=.5; x=0:delx:10; xsteps=10/delx; t=0:delt:T; tsteps=T/delt; s=delt/delx^2;
u=zeros(xsteps+1,tsteps+1);
for j=1:xsteps+1
    u(j,1)=f(x(j));
end
for n=1:tsteps
    u(1,n+1) = 0;
    u(xsteps+1,n+1) = 0;
    for j=2:xsteps
```

```

    u(j,n+1)=(1-2*s)*u(j,n)+s*(u(j+1,n)+u(j-1,n));
end end
y=sin(pi*x/10)*(exp(-(pi/10)^2*T));
maxerr=max(transpose(y)-u(:,(tsteps)+1)) ,

```

we obtain the following maximum errors:

t	Maximum Error
1	3.7009×10^{-4}
5	0.0012
10	0.0015
25	8.6179×10^{-4}

- (c) As time increases from $t = 1$ to $t = 15$, the error doesn't have a uniform behavior. From $t = 1$ to $t = 5$, the error increases, but then decreases again between $t = 10$ to $t = 25$.

MATLAB Code for Exercise 3.7.7, Part (a):

```

delt=.125; delx=.5; x=0:delx:10; xsteps=10/delx; T=10; t=0:delt:T; tsteps=T/delt; s=delt/delx^2;

u=zeros(xsteps+1,tsteps+1);

for j=1:xsteps+1
    u(j,1)=f(x(j));
end
for n=1:tsteps
    u(1,n+1) = 0;
    u(xsteps+1,n+1) = 0;
for j=2:xsteps
    u(j,n+1)=(1-2*s)*u(j,n)+s*(u(j+1,n)+u(j-1,n));
end
end
plot(x,u(:,tsteps+1),x,sin(pi*x/10)*(exp(-(pi/10)^2*T)))
axis([0 10 0 .1])

```

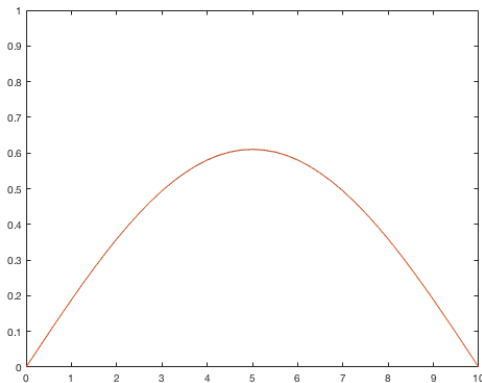
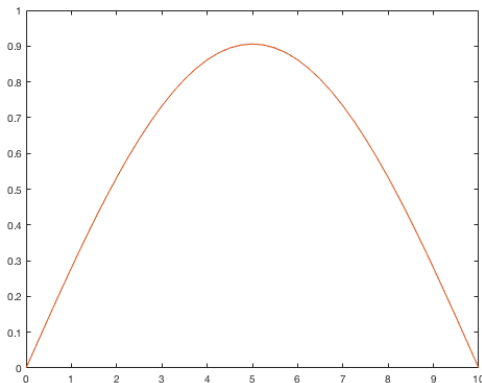
□

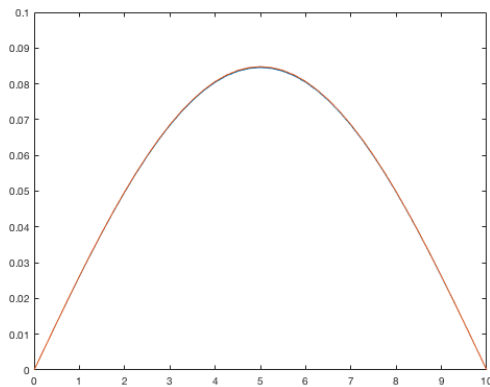
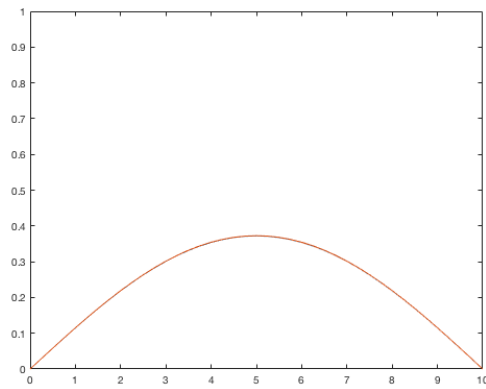
Exercise 3.7.8. (15pts) Now reduce the spatial size to $\Delta x = 0.25$ and make $\Delta t \leq \frac{\Delta x^2}{2}$. Again compare the computed solution and then exact solution at the same times you did before when $\Delta x = 0.5$. Is the error smaller? By how much?

- (5pts) Print the graphs of the numerical solution and the exact solution on the same figures for times $t = 1, 5, 10, 25$.
- (5pts) Calculate the maximum error for each of these times. That is $\max(\text{Exact}(x, t) - \text{Approximation}(x, t))$.
- (5pts) Compare the maximum errors in Q2 with Q3. Which max error is smaller and by how much?

Solution. For $\Delta x = 0.25$, we must have $\Delta t \leq .03125$. So we use $\Delta t = .03125$.

- The following plots of the numerical and exact solutions correspond to times $t = 1, 5, 10, 25$, in order. Only in the last plot can we (slightly) see the difference in the solution graphs. Note the difference in the vertical axis scales for the last plot from that of the first three plots.





- (b) Using the code in Exercise 3.7.7 Part (b), except with `delt=.03125` and `delx=.25`, and at $T = 1, 5, 10, 25$, we obtain the following maximum errors:

t	Maximum Error
1	9.2078×10^{-5}
5	3.1016×10^{-4}
10	3.7861×10^{-4}
25	2.1521×10^{-4}

- (c) The maximum error at times $t = 1, 5, 10, 25$ for $\Delta x = .25$ is smaller than those times for $\Delta x = .5$. The difference in maximum errors are:

t	Difference in Max Error between $\Delta x = .25$ and $\Delta x = .5$
1	2.7801×10^{-4}
5	9.537×10^{-4}
10	0.0011
25	6.4659×10^{-4}

□

HOMEWORK 7

4. Boundary Value Problems for the Heat Equation.**4.2. Convergence of the eigenfunction expansions.**

Exercise 4.2.2. Let u be a solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 & 0 < x < L, t > 0 \\ u(x, 0) &= f(x) & 0 \leq x \leq L, \\ u(0, t) &= u(L, t) = 0 & t \geq 0 \end{aligned}$$

with initial data $f(x) \geq 0$ and $f(0) = f(L) = 0$. Use the maximum (or minimum) principle to show that $u(x, t) \geq 0$ for $0 \leq x \leq L, t \geq 0$.

Proof. Let $T > 0$ and define

$$Q_T := \{(x, t) \in \mathbb{R}^2 : 0 < x < L, 0 < t < T\},$$

and

$$\Gamma_T := \{(x, t) \in \mathbb{R}^2 : x = 0, L, 0 \leq t \leq T\} \cup \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq L, t = 0\}.$$

The initial data implies

$$\min_{\Gamma_T} u \in \{\min_x \{u(x, 0) = f(x)\}, u(0, t) = u(L, t) = 0\} = \{0\},$$

for all $T > 0$, i.e. $\min_{\Gamma_T} u = 0$ for all $T > 0$.

By the minimum principle, $\min_{\overline{Q_T}} u = \min_{\Gamma_T} u = 0$ for all $T > 0$. So $u(x, t) \geq 0$ on the region $\{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq L, t \geq 0\}$, as desired. \square

Exercise 4.2.3. Show that the eigenvalue problem $-\varphi''(x) = \lambda\varphi(x), \varphi(0) = \varphi(L) = 0$ has no complex solution λ with $\text{Im}(\lambda) \neq 0$.

Proof. The characteristic equation for the eigenvalue problem has roots $\pm i\sqrt{\lambda}$, so the solution is

$$\varphi(x) = Ae^{xi\sqrt{\lambda}} + Be^{-xi\sqrt{\lambda}}.$$

The initial condition $\varphi(0) = 0$ implies

$$\varphi(x) = 2A(e^{xi\sqrt{\lambda}} - e^{-xi\sqrt{\lambda}}) = 2Ai(\sin(x\sqrt{\lambda})).$$

Assuming we don't have the trivial solution, $0 = \varphi(L)$ implies $\sin(L\sqrt{\lambda}) = 0$, which is true if and only if $L\sqrt{\lambda} = \pi n$ for $n \in \mathbb{Z}^+$. So $\lambda = (\pi n/L)^2 \in \mathbb{R}_{\geq 0}$. \square

Exercise 4.2.4. Consider a solution of the IBVP

$$\begin{aligned} u_t - ku_{xx} &= 0 & 0 < x < L, t > 0 \\ u(x, 0) &= f(x) & 0 \leq x \leq L, \\ u(0, t) = u(L, t) &= 0 & t \geq 0 \end{aligned}$$

with initial data $f(x) = 5 \sin(2\pi x/L) - 2 \sin(3\pi x/L) + 3 \sin(5\pi x/L)$.

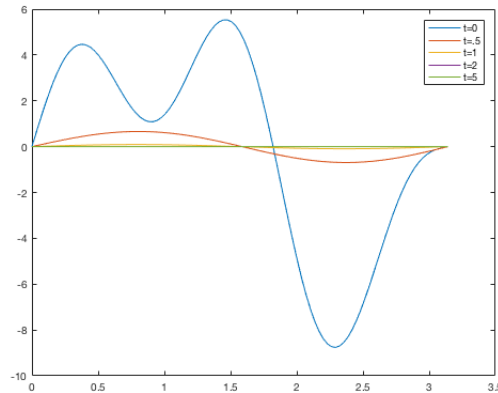
- (a) Find the solution. It consists of three terms.
 (b) Set $L = \pi$ and $k = 1$. Make an mfile `u.m` for the solution $u(x, t)$. Plot snapshots of the solution for $t = 0, .5, 1, 2, 5$. Which term in the sum is a good approximation to the solution when $t = 5$? Why?

Solution.

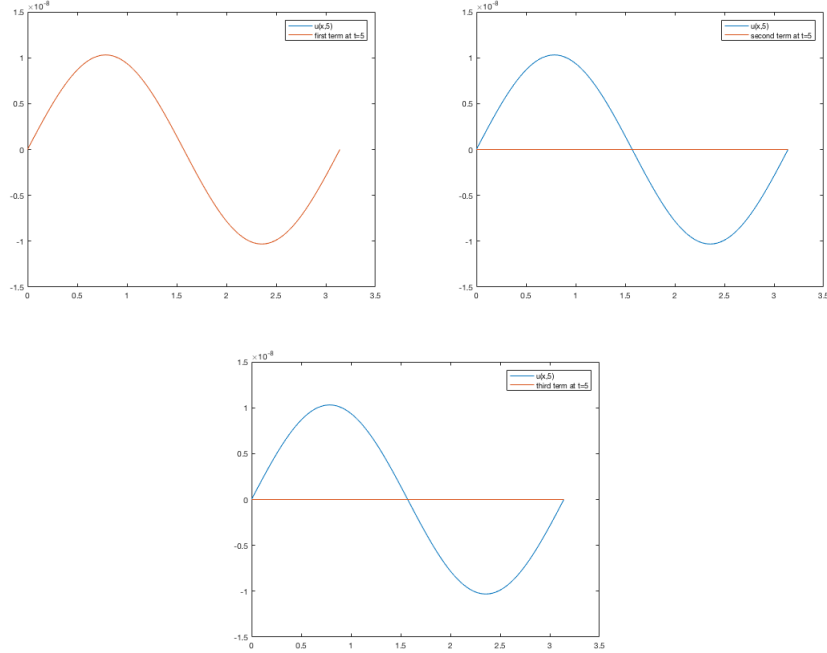
- (a) The solution is $u(x, t) = \sum_1^\infty A_n \varphi(x) \psi_n(t)$. The initial data $u(x, 0) = f(x)$ implies $A_2 = 5, A_3 = -2, A_5 = 3$ and $A_n = 0$ for all $n \in \mathbb{N} - \{2, 3, 5\}$. So

$$\begin{aligned} u(x, t) &= 5 \sin\left(\frac{2\pi x}{L}\right) e^{-\left(\frac{2\pi}{L}\right)^2 kt} \\ &\quad - 2 \sin\left(\frac{3\pi x}{L}\right) e^{-\left(\frac{3\pi}{L}\right)^2 kt} \\ &\quad + 3 \sin\left(\frac{5\pi x}{L}\right) e^{-\left(\frac{5\pi}{L}\right)^2 kt}. \end{aligned}$$

- (b) Snapshots of the solution for $t = 0, .5, 1, 2, 5$.



Below, we have plotted $u(x, 5)$ with each of the terms in the solution at $t = 5$:



The first graph clearly shows that the first term best approximates the solution when $t = 5$. This is because the exponential decreases as n increases. \square

Exercise 4.2.6. Compute the coefficients of A_n of the eigenfunction expansion for the function $f(x) = 1$ for $0 \leq x \leq L/2$, $f(x) = 0$ for $L/2 < x \leq L$. This function has a jump at $x = L/2$. Will the series converge uniformly?

Proof. We have

$$A_n = \frac{2}{L} \int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-2 \cos\left(\frac{n\pi}{2}\right) + 2}{n\pi} = \begin{cases} \frac{2}{(2k+1)\pi} & \text{if } n = 2(2k+1) \text{ for some } k \in \mathbb{Z}_{\geq 0} \\ \frac{2}{(2k+1)\pi} & \text{if } n = 2k+1 \text{ for some } k \in \mathbb{Z}_{\geq 0} \\ 0 & \text{else} \end{cases}$$

The series diverges since

$$\sum_{n=1}^{\infty} |A_n| = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} > \sum_{n=1}^{\infty} \frac{1}{4n-1} = \infty.$$

\square

Exercise 4.2.8. Suppose that the initial data $f(x)$ is C^1 , with $f(0) = f(L) = 0$, so that the coefficients A_n satisfy the summability condition

$$\sum_{n=1}^{\infty} |A_n| < \infty.$$

Show that the solution $u(x, t)$ of

$$\begin{aligned} u_t - ku_{xx} &= 0 & 0 < x < L, t > 0 \\ u(x, 0) &= f(x) & 0 \leq x \leq L, \\ u(0, t) = u(L, t) &= 0 & t \geq 0 \end{aligned}$$

tends to zero exponentially as $t \rightarrow \infty$:

$$|u(x, t)| \leq Ce^{-\lambda_1 kt},$$

where C is a constant. Can you show this type of result even when f is only piecewise continuous? Hint: Let $t_0 > 0$, and show that the estimate can be made for $t \geq t_0$.

Solution. Let $C = \sum_{n=1}^{\infty} |A_n|$. Then

$$|u(x, t)| \leq \sum_{n=1}^{\infty} |A_n| |\varphi_n(x)| |\psi_n(t)| \leq \sum_{n=1}^{\infty} |A_n| e^{-\lambda_1 kt} = Ce^{-\lambda_1 kt}.$$

Setting $g(x) = u(x, t_0)$ for $t_0 > 0$, the solution of the IBVP

$$\begin{aligned} u_t - ku_{xx} &= 0 & 0 < x < L, t > t_0 \\ u(x, t_0) &= g(x) & 0 \leq x \leq L, \\ u(0, t) = u(L, t) &= 0 & t \geq t_0 \end{aligned}$$

will have the same long term behavior at the solution of our original IBVP, and so the same □

Exercise 4.2.15. Another way to prove the uniqueness of the solutions of

$$\begin{aligned} u_t - ku_{xx} &= 0 & 0 < x < L, t > 0 \\ u(x, 0) &= f(x) & 0 \leq x \leq L, \\ u(0, t) &= u(L, t) = 0 & t \geq 0. \end{aligned}$$

(a) Show

$$\int_0^L u^2(x, T) dx + 2k \int_0^T \int_0^L u_x^2(x, t) dx dt = \int_0^L u^2(x, 0) dx.$$

(b) Use this equation to prove the uniqueness of the solution.

Proof.

(a) Multiplying the equation by u , and integrating in x from 0 to L , then integrating by parts, we have

$$\begin{aligned} 0 &= \int_0^L uu_t dx - k \int_0^L u_{xx} u dx \\ &= \frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx - k \left(uu_x \Big|_0^L - \int_0^L u_x^2 dx \right) \\ (u(0, t) = u(L, t) = 0) \quad &= \frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx + k \int_0^L u_x^2 dx \end{aligned}$$

Integrating in t , we get

$$\begin{aligned} 0 &= \frac{1}{2} \int_0^L u^2 dx + k \int_0^L u_x^2 dx \\ &= \frac{1}{2} \int_0^T \frac{d}{dt} \int_0^L u^2 dx dt + k \int_0^T \int_0^L u_x^2 dx dt \\ &= \frac{1}{2} \left(\int_0^L u^2 dx \right) \Big|_0^T + k \int_0^T \int_0^L u_x^2 dx dt \\ &= \frac{1}{2} \int_0^L u^2(x, T) dx - \frac{1}{2} \int_0^L u^2(x, 0) dx + k \int_0^T \int_0^L u_x^2 dx dt. \end{aligned}$$

(b) If u and v are two solutions of the IBVP, then so is $u - v$, with initial data $(u - v)(x, 0) = 0$ and $(u - v)(0, t) = (u - v)(L, t) = 0$. This, together with the formula from (a), we have

$$\int_0^L (u - v)^2(x, T) dx + 2k \int_0^T \int_0^L (u - v)_x^2 dx dt = 0.$$

Since $(u - v)^2 \geq 0$ and $(u - v)_x^2 \geq 0$, then each term above is 0. In particular,

$$\int_0^L (u - v)^2(x, T) dx = 0,$$

which implies $(u - v)^2(x, T) = 0$ for all T , i.e., $u = v$.

□

HOMEWORK 8

4. Boundary Value Problems for the Heat Equation.

4.3. Symmetric boundary conditions.

Exercise 4.3.1. Let u be a solution of $u_t - ku_{xx} = 0$ with the Neumann boundary conditions, $u_x(0, t) = u_x(L, t) = 0$.

(a) Show that the average

$$\bar{u}(t) = \frac{1}{L} \int_0^L u(x, t) dx = \frac{1}{L} \int_0^L f dx = \bar{f}$$

for all $t \geq 0$.

(b) Show that $u(x, t) - \bar{f}$ tends to zero exponentially as $t \rightarrow \infty$.

Solution.

(a) We have

$$\frac{d}{dt} \bar{u}(t) = \frac{k}{L} \int_0^L u_{xx} dx = \frac{k}{L} u_x \Big|_0^L = \frac{k}{L} (u_x(L, t) - u_x(0, t)) = 0.$$

So

$$\bar{u}(t) = \frac{1}{L} \int_0^L u(x, 0) dx = \frac{1}{L} \int_0^L f dx = \bar{f}.$$

(b) We have $\bar{f} = A_0/2$,

$$A_n = \frac{1}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad \text{and} \quad u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \varphi_n(x).$$

Moreover, there exists an M so that for all n , $|A_n| \leq M$. So

$$\begin{aligned} |u(x, t) - \bar{f}| &\leq \sum_{n=1}^{\infty} |A_n| e^{-\lambda_n kt} |\varphi_n(x)| \\ &\leq \sum_{n=1}^{\infty} |A_n| e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left| \cos\left(\frac{n\pi x}{L}\right) \right| \\ &\leq M \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \\ &= M \sum_{n=1}^{\infty} e^{-n^2 \left(\frac{\pi}{L}\right)^2 kt} \\ &= M \sum_{n=1}^{\infty} \left(e^{-n \left(\frac{\pi}{L}\right)^2 kt} \right)^n \\ &\leq M \sum_{n=1}^{\infty} \left(e^{-\left(\frac{\pi}{L}\right)^2 kt} \right)^n \\ &= -M + \frac{M}{1 - e^{-\left(\frac{\pi}{L}\right)^2 kt}}, \end{aligned}$$

which tends to 0 exponentially as $t \rightarrow \infty$.

□

Exercise 4.3.2. Let $u(x, t)$ be the solution of $u_t - ku_{xx} = 0$ with Neumann boundary conditions $u_x(0, t) = u_x(L, t) = 0$, and the initial data

$$f(x) = \begin{cases} 1, & 0 \leq x \leq L/2 \\ 0, & L/2 < x \leq L \end{cases}.$$

- (a) Find the coefficients A_n in the appropriate eigenfunction expansion.
 (b) Verify the conclusions of Exercise 4.3.1.

Solution.

- (a) We have

$$A_0 = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{0\pi x}{L}\right) dx = \frac{2}{L} \int_0^{L/2} f(x) dx = 1,$$

and for $n \geq 1$

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L} \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \Big|_0^{L/2} \\ &= \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \\ &= \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{2}{n\pi} & \text{if } n = 2m + 1, m \text{ odd} \\ \frac{2}{n\pi} & \text{if } n = 2m + 1, m \text{ even} \end{cases}. \end{aligned}$$

- (b) We have

$$\bar{f} = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \int_0^{L/2} dx = \frac{1}{2}$$

and

$$\begin{aligned} \bar{u}(t) &= \left(\frac{A_0}{2} \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \cos\left(\frac{n\pi x}{L}\right) \right) dx \\ &= \frac{1}{2} + \frac{1}{L} \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{2} + \frac{1}{L} \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \Big|_0^L dx \\ &= \frac{1}{2}. \end{aligned}$$

Moreover, $|A_n| \leq 2/n\pi$, so

$$\begin{aligned} |u(x, t) - \bar{f}| &\leq \sum_{n=1}^{\infty} |A_n| e^{-(\frac{n\pi}{L})^2 kt} \left| \cos\left(\frac{n\pi x}{L}\right) \right| \\ &\leq \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-\lambda_n kt} \\ &= e^{-\lambda_1 kt} \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{\lambda_1 kt} e^{-\lambda_n kt} \\ &= e^{-\lambda_1 kt} \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{(\lambda_1 - \lambda_n) kt}. \end{aligned}$$

Since $0 < \lambda_1 < \lambda_2 < \dots$, then $\lambda_n - \lambda_{n+1} < 0$ for all n , so

$$\frac{e^{(\lambda_1 - \lambda_{n+1}) kt}}{e^{(\lambda_1 - \lambda_n) kt}} = e^{(\lambda_n - \lambda_{n+1}) kt} < 1,$$

hence by the ratio test our series converges, say $\sum_{n=1}^{\infty} \frac{2}{n\pi} e^{(\lambda_1 - \lambda_n) kt} = M$.
Then

$$|u(x, t) - \bar{f}| \leq M e^{-\lambda_1 kt},$$

so $|u(x, t) - \bar{f}|$ tends to 0 exponentially.

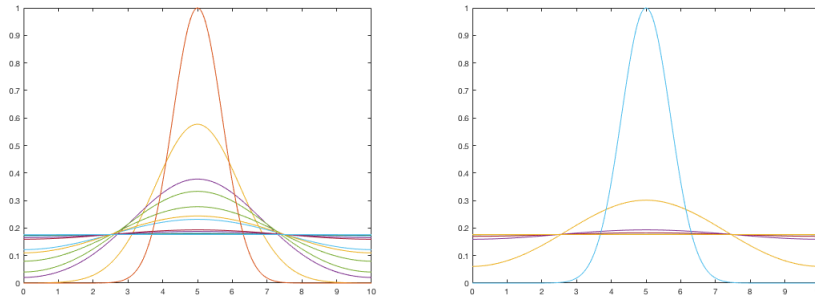
□

Exercise 4.3.7. Run `heat4` with initial data $f(x) = \exp[-(x - 5)^2]$ and $q(x) = 0$, $k = 1$ and $k = 5$.

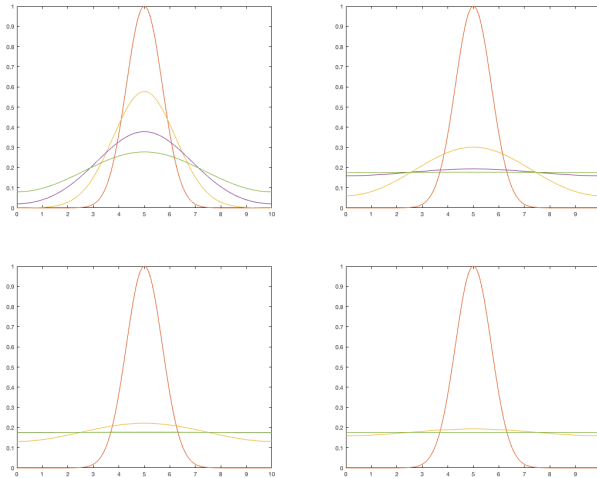
- (a) Compute the integral $(1/10) \int_0^{10} f(x) dx$ (using the error function). Does this value agree with the asymptotic value, as $t \rightarrow \infty$, of the solution as seen in the graphs of the solution?
- (b) For which value of k is the convergence more rapid? Does this show up in the estimate of Exercise 4.3.1?

Solution.

- (a) After a change of variables $y = x - 5$, we get $\frac{1}{10} \int_0^{10} f(x) dx = \frac{\sqrt{\pi}}{10} \operatorname{erf}(5) = .1772$. We see from the following plots that as $t \rightarrow \infty$, $u \rightarrow .1772$, and this agrees with the value of the integral we found above. The first plot is when $k = 1$ and the second is when $k = 5$.



- (b) We see from the following plots that as k increases, the convergence is more rapid. In Exercise 4.3.1, our estimate shows that $|u(x, t) - \bar{f}|$ decays exponentially as $k \rightarrow \infty$. The plots are given in the order $k = 1, 5, 10, 15$.



□

HOMEWORK 9

4. Boundary Value Problems for the Heat Equation.

4.4. Inhomogeneous problems and asymptotic behavior.

Exercise 4.4.1. Consider the IBVP

$$u_t - ku_{xx} = q(x, t), \quad u(x, 0) = f(x)$$

with $f(x) = 0$, boundary conditions $u(0, t) = u(L, t) = 0$, and

$$q(x, t) = e^{-t} \sin(\pi x/L) - \sin(3\pi x/L).$$

- (a) Solve this problem. Note that the solution will contain only a few terms.
 (b) Change the equation to

$$u_t - ku_{xx} + 2u = q,$$

and solve with the same boundary conditions, initial conditions, and the same q as in part (a).

Solution.

- (a) We know that

$$u(x, t) = \sum_{n=1}^{\infty} [A_n e^{-\lambda_n k t} + \int_0^t q_n(s) e^{-\lambda_n k(t-s)} ds] \varphi_n(x),$$

where $\varphi_n = \sin\left(\frac{n\pi x}{L}\right)$, $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$, and $q_n(s) = \frac{\langle q(x, s), \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$. Since $f(x) = 0$, $A_n = 0$ for all n . Notice that $q(x, t) = e^{-t} \varphi_1(x) - \varphi_3(x)$. We have

$$q_n(s) = \frac{\langle q(x, s), \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = e^{-s} \frac{\langle \varphi_1, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} - \frac{\langle \varphi_3, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}.$$

Recall that φ_n is orthogonal to φ_m for $n \neq m$, so if $n \neq 1, 3$, then $q_n = 0$. So

$$\begin{aligned} q_1(s) &= e^{-s} \frac{\langle \varphi_1, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle} - \frac{\langle \varphi_3, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle} = e^{-s}, \\ q_3(s) &= e^{-s} \frac{\langle \varphi_1, \varphi_3 \rangle}{\langle \varphi_3, \varphi_3 \rangle} - \frac{\langle \varphi_3, \varphi_3 \rangle}{\langle \varphi_3, \varphi_3 \rangle} = -1. \end{aligned}$$

Hence

$$u(x, t) = \sin\left(\frac{\pi x}{L}\right) \int_0^t e^{-\lambda_1 k(t-s)-s} ds + \sin\left(\frac{3n\pi x}{L}\right) \int_0^t e^{-\lambda_3 k(t-s)} ds.$$

- (b) Suppose, just as at the beginning of this section, that we can expand u and q in terms of the eigenfunctions $\varphi_n(x)$:

$$u(x, t) = \sum_1^{\infty} u_n(t) \varphi_n(x) \quad \text{and} \quad q(x, t) = \sum_1^{\infty} q_n(t) \varphi_n(x)$$

where

$$q_n(t) = \frac{\langle q(x, t), \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}.$$

Substituting this into our PDE,

$$\begin{aligned} 0 &= u_t - ku_{xx} + 2u - q \\ &= \sum_1^{\infty} u'_n(t)\varphi_n(x) - ku_n(t)\varphi''(x) + 2u_n(t)\varphi_n(x) - q_n(t)\varphi_n(x) \\ &= \sum_1^{\infty} [u'_n(t) - ku_n(t)\lambda_n + 2u_n(t) - q_n(t)]\varphi_n(x). \end{aligned}$$

Because the eigenfunctions φ_n are orthogonal, the above implies

$$u'_n(t) - (k\lambda_n + 2)u_n(t) = q_n(t)$$

for all n . As in the computation at the beginning of this section, we get that $u_n(0) = A_n = 0$ for all n . Solving the above ODE, we have

$$u_n(t) = \int_0^t e^{-(2+k\lambda_n)(t-s)} q_n(s) ds.$$

From our computation in part (a), we have $q_1(s) = e^{-s}$, $q_3(s) = -1$ and $q_n = 0$ if $n \neq 1, 3$. So

$$\begin{aligned} u(x, t) &= \sum_1^{\infty} \left[\int_0^t e^{-(2+k\lambda_n)(t-s)} q_n(s) ds \right] \varphi_n(x) \\ &= \sin\left(\frac{\pi x}{L}\right) \int_0^t e^{-(2+k\lambda_1)(t-s)-s} q_1(s) ds \\ &\quad - \sin\left(\frac{3\pi x}{L}\right) \int_0^t e^{-(2+k\lambda_3)(t-s)} ds. \end{aligned}$$

□

Exercise 4.4.2. Solve the problem

$$u_t - ku_{xx} = q, \quad u(x, 0) = 0$$

with $q(x, t) = \sin(t)r(x)$ and with boundary conditions $u(0, t) = u(L, t) = 0$. How does the solution behave as $t \rightarrow \infty$? Does it oscillate, or does it tend to zero?

Solution. We have

$$q_n(s) = \frac{\langle \sin(s)r(x), \varphi_n(x) \rangle}{\langle \varphi_n(x), \varphi_n(x) \rangle} = \sin(s) \frac{\langle r(x), \varphi_n(x) \rangle}{\langle \varphi_n(x), \varphi_n(x) \rangle} = \sin(s) \frac{2}{L} \langle r(x), \varphi_n(x) \rangle.$$

So

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} \left[\int_0^t q_n(s) e^{-\lambda_n k(t-s)} ds \right] \varphi_n(x) \\ &= \sum_{n=1}^{\infty} \left[\int_0^t \sin(s) e^{-\lambda_n k(t-s)} ds \right] \langle r(x), \varphi_n(x) \rangle \frac{2}{L} \varphi_n(x) \\ &= \sum_{n=1}^{\infty} \left[\frac{e^{-t\lambda_n k} + \lambda_n k \sin(t) - \cos(t)}{(\lambda_n k)^2 + 1} \right] \langle r(x), \varphi_n(x) \rangle \frac{2}{L} \varphi_n(x). \end{aligned}$$

If $r = 0$, then $u(x, t) = 0$, so $u \rightarrow 0$ as $t \rightarrow \infty$. If $r \neq 0$, u oscillates as $t \rightarrow \infty$, since $e^{-t\lambda_n k} \rightarrow 0$ as $t \rightarrow \infty$, and its other terms with t oscillate as $t \rightarrow \infty$. Since $r(x)$ doesn't depend on t , nothing changes if we assume that

$$r(x) = \begin{cases} x, & 0 \leq x \leq \frac{L}{2} \\ L - x & \frac{L}{2} \leq x \leq L \end{cases}.$$

In which case, we get

$$q_n(s) = \sin(s) \frac{\langle r(x), \varphi_n(x) \rangle}{\langle \varphi_n(x), \varphi_n(x) \rangle} = \sin(s) \frac{4L}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right).$$

Whence

$$u(x, t) = \sum_{n=1}^{\infty} \left[\frac{e^{-t\lambda_n k} + \lambda_n k \sin(t) - \cos(t)}{(\lambda_n k)^2 + 1} \right] \frac{4L}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

□

Exercise 4.4.3. Consider the heat problem

$$u_t - ku_{xx} = x, \quad 0 < x < L, t > 0$$

$$u(0, t) = 1, \quad u(L, t) = 2, \quad t \geq 0$$

$$u(x, 0) = 1 + (x/L)^2, \quad 0 \leq x \leq L.$$

- Find the steady-state solution $U(x)$ to this problem. Look for U in the form $U(x) = Ax^3 + Bx + C$; find the coefficients A, B, C .
- Write the desired solution $u(x, t) = v(x, t) + U(x)$, and determine the IBVP satisfied by v . What is the initial condition of v ?
- Find v in terms of an eigenfunction expansion. You do not need to calculate the coefficients A_n . Show that $v(x, t) \rightarrow 0$ as $t \rightarrow \infty$. Finally, write the complete expression for u .

Solution.

- We have $C = U(0) = 1$ and

$$x = U_t - kU_{xx} = -kU_{xx} = -6kAx \implies A = -\frac{1}{6K}.$$

Then

$$AL^3 + BL + 1 = U(L) = 2 \implies -\frac{L^3}{6K} + BL = 1. \implies B = \frac{1}{L} + \frac{L^2}{6K}.$$

- We have $v(x, t) = u(x, t) - U(x)$ and v satisfies the following IBVP:

$$v_t - kv_{xx} = 0 \quad 0 < x < L, t > 0$$

$$v(0, t) = 0, \quad v(L, t) = 0, \quad t \geq 0$$

$$v(x, 0) = 1 + (x/L)^2 - U(x).$$

- Now,

$$v(x, t) = \sum_{n=1}^{\infty} A_n \varphi_n(x) \psi_n(t)$$

and in Exercise 4.3.1(a), we showed that $v(x, t)$ decays exponentially as $t \rightarrow \infty$, since there exists M so that $|A_n| \leq M$ for all n , since $u(x, 0) - U(x)$ is continuous and $[0, L]$ is compact. We have

$$u(x, t) = \sum_{n=1}^{\infty} A_n \varphi_n(x) \psi_n(t) + U(x)$$

and $\lim_{t \rightarrow \infty} u(x, t) = U(x)$.

□

6. Fourier Series and Fourier Transform.

6.1. Fourier Series.

Exercise 6.1.2. Verify the equation

$$c_{\pm n} = \frac{1}{2}[A_n \mp iB_n], \quad n = 1, 2, \dots$$

Solution. We use that sine is odd and cosine is even to obtain

$$\begin{aligned} \frac{1}{2}[A_n \mp iB_n] &= \frac{1}{2L} \int_{-L}^L f(x) \left(\cos\left(\frac{n\pi x}{L}\right) \mp i \sin\left(\frac{n\pi x}{L}\right) \right) \\ &= \frac{1}{2L} \int_{-L}^L f(x) \left(\cos\left(\frac{(\mp n)\pi x}{L}\right) + i \sin\left(\frac{(\mp n)\pi x}{L}\right) \right) \\ &= \frac{1}{2L} \int_{-L}^L f(x) e^{i\frac{(\mp n)\pi x}{L}} \\ &= c_{\pm n}. \end{aligned}$$

□

Exercise 6.1.3. Suppose that $f(x)$ is a C^1 function with period $2L$. Let d_n be the complex Fourier coefficient of f' and c_n the complex Fourier coefficient of f . Show that

$$d_n = \left(\frac{in\pi}{L}\right) c_n.$$

Solution.

$$\begin{aligned} d_n &= \frac{1}{2L} \int_{-L}^L f'(x) e^{-in\pi x/L} dx \\ &= \frac{1}{2L} \left(\left(e^{-in\pi x/L} f(x) \right) \Big|_{-L}^L - \left(\frac{-in\pi}{L} \right) \int_{-L}^L f(x) e^{-in\pi x/L} dx \right) \\ &= \frac{1}{2L} (e^{-in\pi} f(L) - e^{in\pi} f(-L)) + \left(\frac{in\pi}{L} \right) c_n \\ &= \frac{f(L)}{2L} (e^{-in\pi} - e^{in\pi}) + \left(\frac{in\pi}{L} \right) c_n \quad (\text{since } f(L) = f(-L)) \\ &= \left(\frac{in\pi}{L} \right) c_n, \end{aligned}$$

where the last equality follows since

$$\begin{aligned} e^{-in\pi} - e^{in\pi} &= (\cos(-n\pi) + i \sin(-n\pi)) - (\cos(n\pi) + i \sin(n\pi)) \\ &= \cos(-n\pi) - \cos(n\pi) \\ &= 0. \end{aligned}$$

□

HOMEWORK 10

6. Fourier Series and Fourier Transform.**6.2. Convergence of Fourier Series.**

Exercise 6.2.3. Let $\tilde{\varphi}_n(x) = \frac{1}{\sqrt{2L}} \exp[(in\pi x/L)]$ be the normalized eigenfunctions (note that $\|\tilde{\varphi}_n(x)\| = 1$ for all n). Let

$$\tilde{c}_n = \langle f, \tilde{\varphi}_n \rangle = \int_{-L}^L f(x) \overline{\tilde{\varphi}_n(x)} dx.$$

Use Parseval's equality to show that

$$\|f\|^2 = \sum_{-\infty}^{\infty} |\tilde{c}_n|^2.$$

Solution. We have

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx,$$

and since $\overline{\tilde{\varphi}_n(x)} = \frac{1}{\sqrt{2L}} e^{-in\pi x/L}$, we have

$$\tilde{c}_n = \frac{1}{\sqrt{2L}} \int_{-L}^L f(x) e^{-in\pi x/L} dx = \sqrt{2L} c_n.$$

Hence $|\tilde{c}_n|^2 = 2L|c_n|^2$, and so by Parseval's equality,

$$\|f\|^2 = 2L \sum_{-\infty}^{\infty} |c_n|^2 = \sum_{-\infty}^{\infty} |\tilde{c}_n|^2.$$

□

Exercise 6.2.5. Use data choice (1) in the program `fseries`. In example 3 of Section 4.2 we expanded f in terms of a sine series:

$$f(x) = \sum_1^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad B_n = \frac{4L}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right),$$

and that the series converged uniformly.

- (a) Instead expand f in a cosine series

$$f(x) = \frac{A_0}{2} + \sum_1^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

Calculate the coefficients A_n .

- (b) Does the cosine series converge uniformly?
 (c) Run the program `fseries` with data choice (1). This will sum the Fourier series (with $L = 2$) of the odd or even extension of f to $[-2, 2]$. Sum the series with the number of terms $N = 4, 8, 16, 32$ and the odd extension to get the sine series. How rapidly does σ_N^2 decrease? Make an analytic estimate of p , such that $\sigma_N^2 \approx N^{-p}$. Then check your power p with the numbers computed by the program `fseries`.
 (d) Run the program `fseries` with data choice (1) and sum the cosine series by choosing an even extension. Let the number of terms $N = 4, 8, 16, 32$. Does this series fit the data differently? How rapidly does σ_N^2 decrease?

Solution.

- (a) We have

$$\begin{aligned} A_n &= \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} \\ &= \frac{2}{L} \int_0^{L/2} x \cos\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int_{L/2}^L (L-x) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2L}{(n\pi)^2} \left(2 \cos\left(\frac{n\pi}{L}\right) - \cos(n\pi) - 1\right). \end{aligned}$$

- (b) Since $|A_n| \leq C(n^{-2})$, the series converges uniformly to f .
 (c) Since $|B_n| \leq C(n^{-2})$

$$\sigma_N^2 = \sum_{|n| \geq N+1} |B_n|^2 \leq C \sum_{N+1}^{\infty} \frac{1}{n^4} \approx C \int_N^{\infty} \frac{1}{x^4} dx = O(N^{-3}).$$

So σ_N^2 decreases at a rate on the order of N^{-3} , and so we suspect that $p = 3$. Using MATLAB's values for σ_N^2 , we get

$$\begin{aligned} 4^{-p} = \sigma_4^2 &= .00076620 \implies p = 5.175 \\ 8^{-p} = \sigma_8^2 &= .00010376 \implies p = 4.4118 \\ 16^{-p} = \sigma_{16}^2 &= .00001326 \implies p = 4.0505 \\ 32^{-p} = \sigma_{32}^2 &= .00000167 \implies p = 3.839. \end{aligned}$$

So it seems $p \rightarrow 3$ as $N \rightarrow \infty$.

(d) Since $|A_n| \leq C(n^{-2})$

$$\sigma_N^2 = \sum_{|n| \geq N+1} |A_n| \leq C \sum_{N+1}^{\infty} \frac{1}{n^4} \approx C \int_N^{\infty} \frac{1}{x^4} dx = O(N^{-3}).$$

So σ_N^2 decreases at a rate on the order of N^{-3} , and so we suspect that $p = 3$. Using MATLAB's values for σ_N^2 , we get

$$\begin{aligned} 4^{-p} = \sigma_4^2 = .00120548 &\implies p = 4.848 \\ 8^{-p} = \sigma_8^2 = .0019155 &\implies p = 4.1166 \\ 16^{-p} = \sigma_{16}^2 = .00002594 &\implies p = 3.808 \\ 32^{-p} = \sigma_{32}^2 = .0000332 &\implies p = 3.6404. \end{aligned}$$

So it seems $p \rightarrow 3$ as $N \rightarrow \infty$. □

6.3. The Fourier transform.

Exercise 6.3.1. Show that f is even and real valued if and only if \bar{f} is even and real valued.

Proof. If f is even and real valued then

$$\hat{f}(-\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi} dx = - \int_{\infty}^{-\infty} f(-y)e^{-iy\xi} dy = \int_{-\infty}^{\infty} f(y)e^{-iy\xi} dy = \hat{f}(\xi),$$

where we set $x = -y$ at the second equality. Also,

$$\begin{aligned} \overline{\hat{f}(\xi)} &= \int_{-\infty}^{\infty} \overline{f(x)e^{-ix\xi}} dx = \int_{-\infty}^{\infty} f(x)\overline{e^{-ix\xi}} dx \\ &= \int_{-\infty}^{\infty} f(x)e^{ix\xi} dx \\ &= \int_{-\infty}^{\infty} f(y)e^{-iy\xi} dx \\ &= \hat{f}(\xi). \end{aligned}$$

So \hat{f} is even and real-valued. Conversely, assuming \hat{f} is even and real valued,

$$f(-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-ix\xi} d\xi = -\frac{1}{2\pi} \int_{\infty}^{-\infty} \hat{f}(-v)e^{ixv} dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(v)e^{ixv} dv = f(x),$$

where we set $\xi = -v$ at the second equality. Also,

$$\begin{aligned} \overline{f(x)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{f}(\xi)e^{ix\xi}} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{e^{ix\xi}} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-ix\xi} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(v)e^{ixv} dv \\ &= f(x), \end{aligned}$$

so f is even and real-valued. □

Exercise 6.3.2. Verify rule (5):

$$\widehat{\tau_a f}(\xi) = \exp(-ia\xi)\hat{f}(\xi).$$

Proof. Making the substitution $y = x - a$,

$$\begin{aligned}\widehat{\tau_a f}(\xi) &= \int_{-\infty}^{\infty} f(x-a)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f(y)e^{-i(y+a)\xi}dy \\ &= e^{-ia\xi} \int_{-\infty}^{\infty} f(y)e^{-iy\xi}dy \\ &= e^{-ia\xi}\hat{f}(\xi).\end{aligned}$$

□

Exercise 6.3.3. Let

$$f(x) = \begin{cases} 1/2, & |x-a| \leq b \\ 0, & |x-a| > b \end{cases}.$$

Find \hat{f} . What is the rate of decay of $\hat{f}(\xi)$ as $\xi \rightarrow \infty$?

Solution.

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx = \frac{1}{2} \int_{a-b}^{a+b} e^{-ix\xi}dx \\ &= -\frac{1}{2i\xi} \left(e^{-i(a+b)\xi} - e^{-i(a-b)\xi} \right) \\ &= \frac{e^{-ia\xi}}{2i\xi} (e^{ib\xi} - e^{-ib\xi}) \\ &= \frac{\sin(b\xi)}{\xi e^{ia\xi}}.\end{aligned}$$

Hence $|\hat{f}(\xi)| \leq \xi^{-1}$ and so $\hat{f}(\xi) = O(\xi^{-1})$.

□

Exercise 6.3.5. Let

$$f_n(x) = \begin{cases} 0, & |x| > 1/n \\ n/2, & |x| < 1/n \end{cases}.$$

(a) Find $\hat{f}_n(\xi)$.

(b) Show that $\lim_{n \rightarrow \infty} \hat{f}_n(\xi) = 1$ for all ξ .

Since formally $f_n(x) \rightarrow \delta(x)$ as $n \rightarrow \infty$, this calculation indicates that we may define the Fourier transform of the generalized function $\delta(x)$ by

(i) $\hat{\delta}(\xi) = \lim_{n \rightarrow \infty} \hat{f}_n(\xi) \equiv 1$.

This is consistent with another formal definition

(ii) $\hat{\delta}(\xi) = \int \delta(x) \exp(-ix\xi) dx \equiv 1$.

(c) Find the Fourier transform of the generalized function $\delta(x-a)$ using both (i) and (ii) above and show that they are again consistent, and also consistent with the general rule (5) as above.

Solution.

(a)

$$\begin{aligned} \hat{f}_n(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} e^{-ix\xi} dx = -\frac{n}{2i\xi} \left(e^{-i(1/n)\xi} - e^{-i(-1/n)\xi} \right) \\ &= \frac{n}{2i\xi} \left(e^{i(1/n)\xi} - e^{-i(1/n)\xi} \right) \\ &= \frac{n}{\xi} \sin\left(\frac{\xi}{n}\right). \end{aligned}$$

(b)

$$\lim_{n \rightarrow \infty} \hat{f}_n(\xi) = \lim_{n \rightarrow \infty} \frac{\sin(\frac{\xi}{n})}{\frac{\xi}{n}} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

(c) Using (i),

$$\widehat{\tau_a \delta}(\xi) = \lim_{n \rightarrow \infty} \widehat{\tau_a f_n}(\xi) = \lim_{n \rightarrow \infty} e^{-ia\xi} \hat{f}_n(\xi) = e^{-ia\xi}.$$

Using (ii),

$$\begin{aligned} \widehat{\tau_a \delta}(\xi) &= \int_{-\infty}^{\infty} \delta(x-a) e^{-ix\xi} dx \\ (y = x-a) &= \int_{-\infty}^{\infty} \delta(y) e^{-i(y+a)\xi} dy \\ &= e^{-ia\xi} \int_{-\infty}^{\infty} \delta(y) e^{-iy\xi} dy \\ &= e^{-ia\xi} \hat{\delta}(\xi) \\ &= e^{-ia\xi}. \end{aligned}$$

These are consistent with rule (5).

□

HOMEWORK 11

6. Fourier Series and Fourier Transform.**6.5. The discrete Fourier Transform.**

Exercise 6.5.1. Show that if $f(x)$ has period 2π and is real-valued, then the Fourier coefficients satisfy $c_{-k} = \bar{c}_k$, and the DFT coefficients satisfy $d_{N-k} = \bar{d}_k$ for $0 \leq k < N/2$.

Solution. We have

$$c_{-k} = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{ikx} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{e^{-ikx}} dx = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x) e^{-ikx}} dx = \bar{c}_k,$$

and for $W = e^{2\pi i/N}$,

$$\begin{aligned} d_{N-k} &= \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \overline{W^{(N-k)j}} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \overline{W^{Nj} W^{-kj}} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \overline{W^{-kj}} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) W^{kj} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \overline{f(x_j) \overline{W^{kj}}} \\ &= \bar{d}_k \end{aligned}$$

□

Exercise 6.5.2. Let

$$f(x) = 2e^{ix} + 1 + 3e^{-ix}.$$

Here the Fourier coefficients are $c_1 = 3$, $c_0 = 1$, and $c_{-1} = 2$. Use

$$d_k = c_k + \sum_{m=-\infty}^{\infty} c_{k+mN}, \quad 0 \leq k < N/2$$

and

$$d_{N-k} = c_{-k} + \sum_{m=-\infty}^{\infty} c_{-k+mN}, \quad 1 \leq k \leq N/2$$

to find the DFT coefficients for $N = 2$, then for $N = 4$.

Solution.

- $N = 2$:

$$d_0 = c_0 + \sum_{m=-\infty}^{\infty} c_{m2} = c_0 = 1$$

$$d_1 = d_{2-1} = c_{-1} + \sum_{m=-\infty}^{\infty} c_{-1+m2} = c_{-1} + c_1 = 5$$

- $N = 4$:

$$d_0 = c_0 + \sum_{m=-\infty}^{\infty} c_{m4} = c_0 = 1$$

$$d_1 = c_1 + \sum_{m=-\infty}^{\infty} c_{1+m4} = c_1 = 3$$

$$d_2 = d_{4-2} = c_{-2} + \sum_{m=-\infty}^{\infty} c_{-2+m4} = 0$$

$$d_3 = d_{4-1} = c_{-1} + \sum_{m=-\infty}^{\infty} c_{-1+m4} = c_{-1} = 2.$$

□

Exercise 6.5.3. Write an mfile, `h.m`, for

$$\hat{f}(\xi) \approx h_N(\xi) = \sum_{j=0}^{N-1} f(x_j) e^{-ix_j \xi} \Delta x.$$

Write it as a function `h(s,N)`. Let $f(x) = 1$ for $0 < x < 2$ and $f(x) = 0$ elsewhere.

- Compute the Fourier transform of f , and plot it on the interval $-4\pi \leq \xi \leq 4\pi$. Let the transform variable be `s`.
- The function mfile for this kind of discontinuous function is

```
function y = f(x)
y = (x<2) - (x<= 0)
```

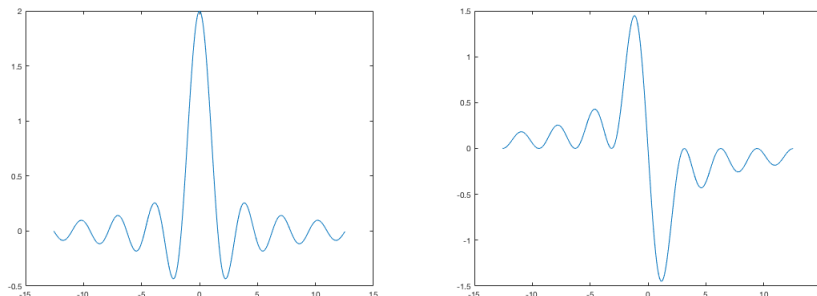
Using the mfiles `h.m` and `f.m`, plot the graphs of h_N for various values of N against the graph of \hat{f} . Describe how the period of h_N lengthens, and how the fit gets better as N increases. Note that, unless told otherwise, MATLAB plots the real part of a complex valued function.

Solution.

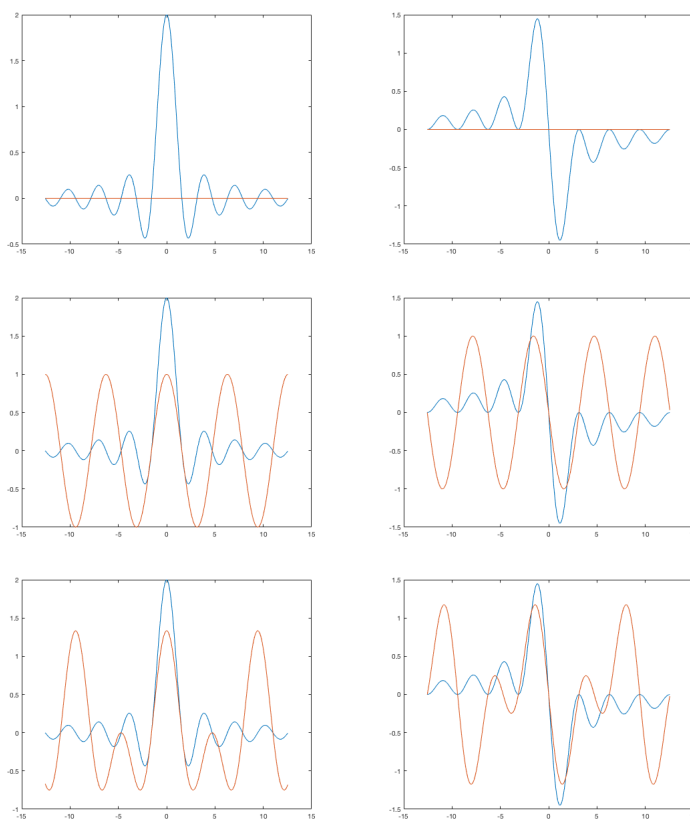
(a) We have

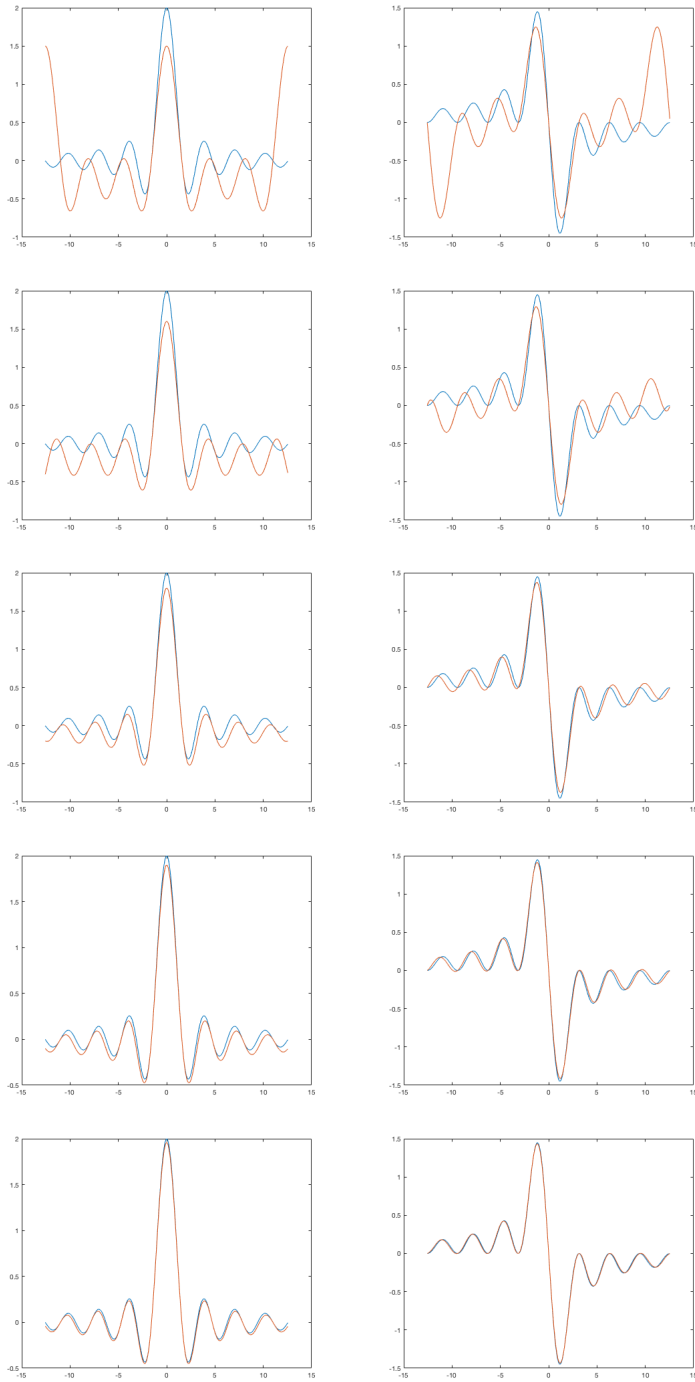
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi} dx = \int_0^2 e^{-ix\xi} dx = \frac{1 - e^{-i2\xi}}{i\xi}$$

Real and imaginary parts of \hat{f} , respectively:



(b) The left column displays the real parts of the plots for \hat{f} and h_N , and the right column displays the imaginary parts of said plots. The rows correspond to $N = 1, 2, 3, 4, 5, 10, 20, 50$:





As N increases, the period of h_N lengthens, and hence h_N better fits the plot for \hat{f} .

□

6.6. *The fast Fourier Transform (FFT).*

Exercise 6.6.1. Let $f(x) = x$ on $[0, 2\pi]$, and then assume f is extended periodically to \mathbb{R} .

- Calculate the complex Fourier coefficients c_k of f according to (6.36).
- Make the mfile `f.m` for this function. Run program `fast` with $N = 4, 8, 16, 32, 64, 256$. For each N compare the DFT coefficients with the Fourier coefficients. The program automatically prints out d_0 . To see the four DFT coefficients d_1, d_2, d_3, d_4 , enter the command `d(1:4)`. Note the rate of convergence of d_k to c_k for a fixed k as $N \rightarrow \infty$. In particular, note how slowly $\text{Re}(d_k) \rightarrow 0$.
- Because the periodic extension of f is discontinuous, the Fourier series for f will converge (at $x = 0$) to the average value $(f(0+) + f(2\pi-))/2 = \pi$. After the command `u=f(xsample)` in the script mfile above, insert the command `u(1)=pi`. Does this make the DFT coefficients converge more rapidly to the Fourier coefficients?

Solution.

- (a) For $k \neq 0$,

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} x e^{-ikx} dx = \frac{1}{2\pi} \left[\frac{x e^{-ikx}}{-ik} \Big|_0^{2\pi} + \frac{1}{ik} \int_0^{2\pi} e^{-ikx} dx \right] = \frac{i}{k}.$$

For $k = 0$,

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_0^{2\pi} x dx = \pi.$$

- (b) Let d_k^N denote the k th DFT coefficient corresponding to a fixed N .

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
d_k^4	2.356	-0.7854+ 0.75854i	-0.7854+ 0.0000i	-0.7854- 0.7854i	
d_k^8	2.7489	-0.3927+ 0.9481i	-0.3927+ 0.3927i	-0.3927+ 0.1627i	-0.3927+ 0.0000i
d_k^{16}	2.9452	-0.1963+ 0.9871i	-0.1963+ 0.4740i	-0.1963+ 0.2939i	-0.1963+ 0.1963i
d_k^{32}	3.0434	-0.0982+ 0.9968i	-0.0982+ 0.4936i	-0.0982+ 0.3236i	-0.0982+ 0.2370i
d_k^{64}	3.0925	-0.0491+ 0.9992i	-0.0491+ 0.4984i	-0.0491+ 0.3309i	-0.0491+ 0.2468i
d_k^{256}	3.1293	-0.0123+ 0.9999i	-0.0123+ 0.4999i	-0.0123+ 0.3332i	-0.0123+ 0.2498i
c_k	π	i	$i/2$	$i/3$	$i/4$

For fixed k , we see that $d_k^N \rightarrow c_k$ as N increases.

(c)

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
d_k^4	3.1416	0.0000 + 0.7854 <i>i</i>	0.0000 + 0.0000 <i>i</i>	0.0000 - 0.7854 <i>i</i>	
d_k^8	3.1416	0.0000 + 0.9481 <i>i</i>	0.0000 + 0.3927 <i>i</i>	0.0000 + 0.1627 <i>i</i>	0.0000 + 0.0000 <i>i</i>
d_k^{16}	3.1416	0.0000 + 0.9871 <i>i</i>	0.0000 + 0.4740 <i>i</i>	0.0000 + 0.2939 <i>i</i>	0.0000 + 0.1963 <i>i</i>
d_k^{32}	3.1416	0.0000 + 0.9968 <i>i</i>	0.0000 + 0.4936 <i>i</i>	0.0000 + 0.3236 <i>i</i>	-0.0000+ 0.2370 <i>i</i>
d_k^{64}	3.1416	-0.0000+ 0.9992 <i>i</i>	0.0000 + 0.4984 <i>i</i>	-0.0000+ 0.3309 <i>i</i>	0.0000 + 0.2468 <i>i</i>
d_k^{256}	3.1416	-0.0000+ 0.9999 <i>i</i>	0.0000 + 0.4999 <i>i</i>	0.0000 + 0.3332 <i>i</i>	0.0000 + 0.2498 <i>i</i>
c_k	π	i	$i/2$	$i/3$	$i/4$

We have $\text{Re}(d_k^N)$ converging faster than before to $\text{Re}(c_k)$ as N increases, but the same rate of convergence for the imaginary parts as before.

□

HOMEWORK 12

5. Waves Again.**5.3.** *The wave equation without boundaries.***Exercise 5.3.3.** Consider the IVP

$$u_{tt} = c^2 u_{xx}, \quad x, t \in \mathbb{R}$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}$$

with $f(x) = 0$ and

$$g(x) = \begin{cases} e^{-x}, & x > 0 \\ -e^x, & x < 0 \end{cases}$$

- Sketch the graph of g , how is it similar to the data of exercise 2? Do you expect the solutions to be similar?
- The data g has a singularity at $x = 0$. In the x, t plane, sketch two characteristics emanating from $x = 0$. They divide the x, t plane for ($t \geq 0$) into three regions. The singularity in the initial data will propagate along these lines. You can see that the solution has a jump in u_x there.
- Find the solution using d'Alembert's formula. You will need to evaluate the integral in three different ways, yielding three different formulas, one for each of the three regions.
- Show that the solution of this IVP can also be represented

$$u(x, t) = F(x - ct) + G(x + ct),$$

where

$$F(x) = \begin{cases} \frac{1}{2c}(e^{-x} - 1), & x \geq 0 \\ \frac{1}{2c}(e^x - 1), & x < 0 \end{cases}$$

and $G(x) = -F(x)$.

- Assume that $c = 1$. Write an mfile `bigf.m` for F . Then use the command `plot(x, bigf(x-t)-bigf(x+t))` to plot snapshots of the solution on $[-5, 5]$ for several values of $t > 0$.

Solution.

(a)

The graphs are similar as $|x| \rightarrow \infty$, but different around 0, so we'd expect the solutions to be similar.

(b)

(c) We have

$$\int_{x-ct}^{x+ct} g(y)dy = \begin{cases} \int_{x-ct}^{x+ct} -e^y dy, & x < -ct, t \geq 0 \\ \int_0^{x+ct} e^{-y} dy + \int_{x-ct}^0 -e^y dy, & -ct < x < ct, t \geq 0 \\ \int_{x-ct}^{x+ct} e^{-y} dy, & x > ct, t \geq 0 \end{cases}$$

$$= \begin{cases} e^{x-ct} - e^{x+ct}, & x < -ct, t \geq 0 \\ -e^{-(x+ct)} + e^{x-ct}, & -ct < x < ct, t \geq 0 \\ e^{-(x-ct)} - e^{-(x+ct)}, & x > ct, t \geq 0 \end{cases}.$$

Since $f(x) = 0$,

$$u(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy$$

$$= \begin{cases} \frac{1}{2c}(e^{x-ct} - e^{x+ct}), & x < -ct, t \geq 0 \\ \frac{1}{2c}(-e^{-(x+ct)} + e^{x-ct}), & -ct < x < ct, t \geq 0 \\ \frac{1}{2c}(e^{-(x-ct)} - e^{-(x+ct)}), & x > ct, t \geq 0 \end{cases}$$

(d) With the change of variables $z = x - ct$ and $w = x + ct$, we have

$$F(z) = F(x - ct) = \begin{cases} \frac{1}{2c}(e^{-(x-ct)} - 1), & x - ct \geq 0 \\ \frac{1}{2c}(e^{x-ct} - 1), & x - ct < 0 \end{cases}$$

and

$$G(w) = -F(w) = -F(x + ct) = \begin{cases} \frac{1}{2c}(1 - e^{-(x+ct)}), & x + ct \geq 0 \\ \frac{1}{2c}(1 - e^{x+ct}), & x + ct < 0 \end{cases}.$$

When $x < -ct, t \geq 0$, then $z = x - ct < 0$ and $w = x + ct < 0$, so

$$F(z) + G(w) = \frac{1}{2c}(e^{x-ct} - e^{x+ct}).$$

When $-ct < x < ct, t \geq 0$, then $z = x - ct < 0$ and $w = x + ct > 0$, so

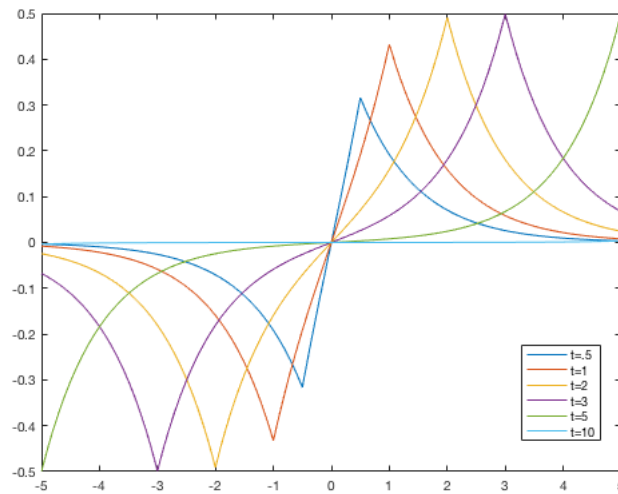
$$F(z) + G(w) = \frac{1}{2c}(-e^{-(x+ct)} + e^{x-ct}).$$

When $x > ct, t \geq 0$, then $z = x - ct > 0$ and $w = x + ct > 0$, so

$$F(z) + G(w) = \frac{1}{2c}(e^{-(x-ct)} - e^{-(x+ct)}).$$

These match $u(x, t)$ on all three regions.

(e)



□

Exercise 5.3.5. Let $F(x) = \exp[-(x+5)^2]$ and $G(x) = \exp[-(x-5)^2]$. Assuming that $c = 1$, let $u(x, t) = F(x-t) + G(x+t)$.

- What is the data pair $f(x) = u(x, 0)$, and $g(x) = u_t(x, 0)$?
- Write the mfile `bigf.m` for this $F(x)$. Plot the solution on $[-10, 10]$ for values of t ranging from 1 to 7. In particular observe snapshots at $t = 4.5, 5$, and 5.5 . What happens as the two waves interact? Are the waves intact after this interaction?

Solution.

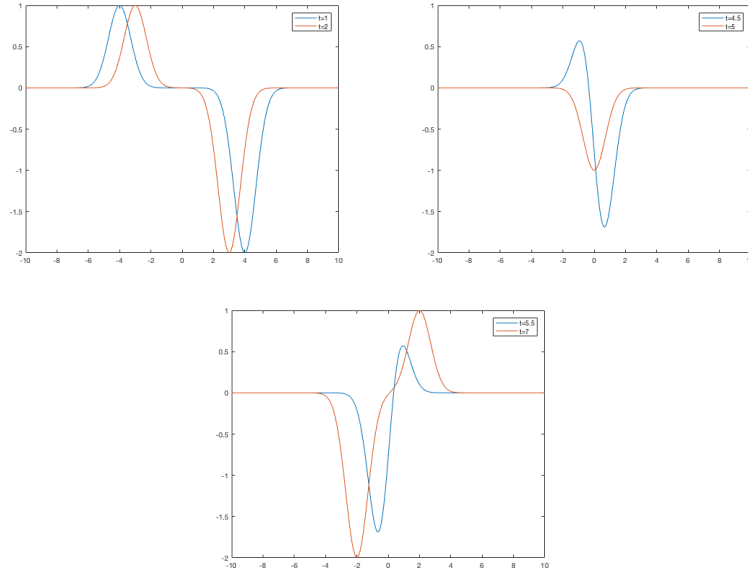
- We have

$$f(x) = u(x, 0) = F(x) + G(x) = e^{-(x+5)^2} - 2e^{-(x-5)^2}$$

and

$$g(x) = u_t(x, 0) = -F'(x) + G'(x) = 2(x+5)e^{-(x+5)^2} + 4(x-5)e^{-(x-5)^2}.$$

-



As the two waves interact at $t = 5$, they become one wave, and after the interaction, they continue to move intact.

□

Exercise 5.3.7.

- (a) If a solution of the wave equation is in the form $u(x, t) = F(x - ct) + G(x + ct)$, show that the energy is

$$e(t) = e(0) = c^2 \int_{\mathbb{R}} [(F'(x))^2 + (G'(x))^2] dx,$$

provided this integral converges.

- (b) Which of the solutions in exercises 1-6 is a finite-energy solution?

Solution.

- (a) We have

$$\begin{aligned} u_t^2(x, t) &= -c^2 F'^2(x - ct) - 2c^2 F'(x - ct)G'(x + ct) + c^2 G'^2(x + ct) \\ &= c^2 [F'^2(x - ct) - 2F'(x - ct)G'(x + ct) + G'^2(x + ct)] \end{aligned}$$

and

$$c^2 u_x^2(x, t) = c^2 (F'(x) + G'(x))^2 = c^2 [F'^2(x) + 2F'(x)G'(x) + G'^2(x)].$$

Hence

$$\begin{aligned} e(t) &= \frac{1}{2} \int_{\mathbb{R}} c^2 [F'^2(x - ct) - 2F'(x - ct)G'(x + ct) + G'^2(x + ct)] \\ &\quad + c^2 [F'^2(x) + 2F'(x)G'(x) + G'^2(x)] dx \\ &= \frac{1}{2} \int_{\mathbb{R}} 2c^2 [F'^2(z) + G'^2(z)] dz \\ &= c^2 \int_{\mathbb{R}} [(F'(x))^2 + (G'(x))^2] dx. \end{aligned}$$

(b) For Exercise 5.3.3, since $f(x) = 0$ and

$$\int_{\mathbb{R}} g^2(x) dx = - \int_{-\infty}^0 e^{2y} dy + \int_0^{\infty} e^{-2y} dy = -\frac{1}{2} + \frac{1}{2} = 0,$$

and so by Theorem 5.2, $e(t)$ is finite.

For Exercise 5.3.5, we find that

$$e(t) = \int_{\mathbb{R}} [(F'(x))^2 + (G'(x))^2] dx = 5\sqrt{\frac{\pi}{2}} < \infty.$$

□

Exercise 5.3.9. Consider the equation

$$u_{tt} + 2du_t - u_{xx} + d^2u = 0 \quad x, t \in \mathbb{R}$$

with initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}.$$

- (a) Let $v(x, t) = \exp(dt)u(x, t)$. Show that v satisfies $v_{tt} - v_{xx} = 0$. What is the initial data of v ?
- (b) Solve for v using d'Alembert's formula. Then find $u(x, t) = \exp(-dt)v(x, t)$.
- (c) If $\int_{\mathbb{R}} |f(x)| dx$, $\max |f(x)|$, and $\int_{\mathbb{R}} |g(x)| dx$ are all finite, show that $|u(x, t)| \leq C \exp(-dt)$, where C is a constant determined by the initial data f, g .

Solution.

(a) We have

$$\begin{aligned} v_{tt} - v_{xx} &= d^2 e^{dt} u + d e^{dt} u_t + d e^{dt} u_t + e^{dt} u_{tt} - e^{dt} u_{xx}(x, t) \\ &= e^{dt} (u_{tt} + 2du_t - u_{xx} + d^2 u) \\ &= 0 \end{aligned}$$

(b) We have

$$v(x, t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{\mathbb{R}} df(y) + g(y) dy$$

and so

$$u(x, t) = e^{-dt} \left(\frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{\mathbb{R}} df(y) + g(y) dy \right)$$

- (c) Based on the hypothesis in (c), the expression in parenthesis immediately above this sentence is finite in absolute value. So $|u(x, t)| \leq C e^{-dt}$.

□

HOMEWORK 13

5. Waves Again.**5.4. Boundary value problems on the half-line.****Exercise 5.4.1.** Consider the IBVP

$$(5.30) \quad u_{tt} - c^2 u_{xx} = 0, \quad x > 0, t \in \mathbb{R}$$

$$(5.30) \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \geq 0$$

$$(5.31) \quad u(0, t) = 0, \quad t \in \mathbb{R}$$

with $g(x) \equiv 0$. In this case we have

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct) - f(ct - x)].$$

Let the function f be

$$f(x) = \begin{cases} 0 & 0 \leq x < 2 \\ 1 & 2 \leq x < 4 \\ 0 & x > 4 \end{cases}$$

The initial data has singularities at $x = 2$ and $x = 4$.

- (a) In the quarter plane $\{(x, t) : x > 0, t > 0\}$, sketch the characteristics emanating from $x = 2$ and $x = 4$. At what times do the leftward leaning characteristics reach the boundary $x = 0$? Draw rightward leaning characteristics from these points on the t axis. These are the reflected characteristics.
- (b) The collection of characteristics emanating from $x = 2, x = 4$ and the reflected characteristics emanating from the t axis divide the quarter-plane into nine regions. Use the formula above to find the values of the (weak) solution u in each of these nine regions. When using this formula, keep in mind that you are assuming that $f(x) = 0$ when $x < 0$.
- (c) If you are an observer standing at $x = 6$, what motion of the string do you see as t increases?

Solution.

(a)

The leftward characteristics have equations $x = -ct + x_0$ or $t = (x_0 - x)/c$, and hence intercept the t axis at $t = x_0/c$ (for $x_0 = 2$ and $x_0 = 4$).

(b) In Region 1, we have

$$-2 < x - ct < 2, \quad x + ct < 2, \quad -2 < ct - x < 2$$

$$\text{so } u(x, t) = 1/2(0 + 0 - 0) = 0.$$

In Region 2, we have

$$2 < x - ct < 4, \quad 2 < x + ct < 4, \quad -4 < ct - x < -2$$

$$\text{so } u(x, t) = 1/2(1 + 1 - 0) = 1.$$

In Region 3, we have

$$x - ct > 4, \quad x + ct > 4, \quad -4 < ct - x$$

$$\text{so } u(x, t) = 1/2(0 + 0 - 0) = 0.$$

In Region 4, we have

$$-2 < x - ct < 2, \quad 2 < x + ct < 4, \quad -2 < ct - x < 2$$

$$\text{so } u(x, t) = 1/2(1 + 0 - 0) = 1/2.$$

In Region 5, we have

$$2 < x - ct < 4, \quad 4 < x + ct, \quad -4 < ct - x < -2$$

$$\text{so } u(x, t) = 1/2(0 + 1 - 0) = 1/2.$$

In Region 6, we have

$$-4 < x - ct < -2, \quad 2 < x + ct < 4, \quad 2 < ct - x < 4$$

$$\text{so } u(x, t) = 1/2(1 + 0 - 1) = 0.$$

In Region 7, we have

$$-2 < x - ct < 2, \quad 4 < x + ct, \quad -2 < ct - x < 2$$

$$\text{so } u(x, t) = 1/2(0 + 0 - 0) = 0.$$

In Region 8, we have

$$x - ct < -4, \quad 4 < x + ct, \quad 4 < ct - x$$

$$\text{so } u(x, t) = 1/2(0 + 0 - 0) = 0.$$

In Region 9, we have

$$-4 < x - ct < -2, \quad 4 < x + ct, \quad 2 < ct - x < 4$$

$$\text{so } u(x, t) = 1/2(0 + 0 - 1) = -1/2.$$

(c) Using (b), we see that the observer would see the motion as follows:

$$u(6, t) = \begin{cases} 0 & 0 < t < 2/c \\ 1/2 & 2/c < t < 4/c \\ 0 & 4/c < t < 8/c \\ -1/2 & 8/c < t < 10/c \\ 0 & t > 10/c \end{cases}$$

□

Exercise 5.4.4.

(a) Make a calculation similar to that made in Section 5.3 to show that the energy

$$e(t) = \frac{1}{2} \int_0^\infty [u_t^2 + c^2 u_x^2] dx$$

is conserved for solutions of the (5.30), (5.31), and of (5.30) with the Neumann boundary condition.

(b) What energy quantity is conserved by solutions of (5.30) with the Robin condition ($u_x(0, t) - hu(0, t) = 0$)?

Solution. First, we have

$$\begin{aligned} \frac{d}{dt} e(t) &= \int_0^\infty [u_{tt}u_t + c^2 u_x u_{xt}] dx \\ &= \int_0^\infty u_{tt}u_t dx + c^2 \left(u_x u_t \Big|_0^\infty - \int_0^\infty u_{xx} u_t dx \right) \\ &= \int_0^\infty u_t (u_{tt} - c^2 u_{xx}) dx + c^2 u_x u_t \Big|_0^\infty \\ &= c^2 u_x u_t \Big|_0^\infty. \end{aligned}$$

(a) Assuming (5.30), (5.31), and that $u_x \rightarrow 0$ as $x \rightarrow \infty$, we get that

$$u_x u_t \Big|_0^\infty = 0,$$

so our calculation above shows that energy is conserved.

Now assuming (5.30) and the Neumann boundary condition and that $u_x \rightarrow 0$ as $x \rightarrow \infty$, we again get that

$$u_x u_t \Big|_0^\infty = 0,$$

so our calculation above shows that energy is conserved.

(b) Assuming that $u_x \rightarrow 0$ as $x \rightarrow \infty$, and the Robin conditions, we get

$$\begin{aligned} \frac{d}{dt} e(t) &= c^2 u_x u_t \Big|_0^\infty = c^2 (u_x(0, t) u_t(0, t)) \\ &= c^2 (hu(0, t) u_t(0, t)) \\ &= -\frac{c^2 h}{2} \frac{d}{dt} u^2(0, t), \end{aligned}$$

so that $\tilde{e}(t) = e(t) + (c^2 h/2) u^2(0, t)$ is the conserved energy.

□

Exercise 5.4.11. Consider the IBVP

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0, & x > 0 \\ u(x, 0) &= u_t(x, 0) = 0, \\ u_x(0, t) &= h(t),\end{aligned}$$

- (a) Again look for the solution in the form of a wave moving to the right down the string $u(x, t) = F(x - ct)$. Impose the boundary condition, deriving a simple ODE that F must satisfy. Solve the ODE for F as an integral of h .
- (b) Suppose that $h(t) = t/(1+t^2)$ for $t > 0$ and $h(t) = 0$ for $t < 0$. Find $u(x, t)$. What is $\lim_{t \rightarrow \infty} u(0, t)$?

Solution.

- (a) We have $h(t) = u_x(0, t) = F'(-ct)$. So with $y = -ct$,

$$F(s) = F(s) - F(0) = \int_0^s F'(y) dy = \int_0^s h(-y/c) dy.$$

- (b) When $s > 0$, then $h = 0$ so $F = 0$. If $h(-y/c) \neq 0$, then $-y/c > 0$ so when $s < 0$,

$$\begin{aligned}(w = 1 + y^2/c) \quad F(s) &= \int_0^s h(-y/c) dy \\ &= \int_0^s \frac{-y/c}{1 + y^2/c^2} dy \\ &= -\frac{c}{2} \int_c^{1+s^2/c^2} \frac{1}{w} dw \\ &= -\frac{c}{2} \ln |1 + s^2/c|\end{aligned}$$

So $u = 0$ when $x \geq ct$, and when $x < ct$, we have

$$u(x, t) = F(x - ct) = -\frac{c}{2} \ln |1 + (x - ct)^2/c^2|$$

and

$$\lim_{t \rightarrow \infty} u(0, t) = \lim_{t \rightarrow \infty} -\frac{c}{2} \ln |1 + (-ct)^2/c^2| = \lim_{t \rightarrow \infty} -\frac{c}{2} \ln |1 + t^2| = -\infty.$$

□

HOMEWORK 14

5. Waves Again.

5.5. Boundary value problems on a finite interval.

Exercise 5.5.1. Solve the IBVP

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, & 0 < x < L, t \in \mathbb{R} \\ u(0, t) &= u(L, t) = 0, & t \in \mathbb{R} \\ u(x, 0) &= f(x), & u_x(x, 0) = g(x), & 0 < x < L \end{aligned}$$

with initial data

$$f(x) = 7 \sin(2\pi x/L) - 2 \sin(5\pi x/L), \quad g(x) = -4 \sin(3\pi x/L).$$

Think!! You will not need the full expansion of the solution. What frequencies are present in the solution? What is the period of the motion?

Solution. Since

$$\sum_1^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) = 7 \sin(2\pi x/L) - 2 \sin(5\pi x/L),$$

we see that $A_n = 0$ if $n \neq 2$ or $n \neq 5$, and $A_2 = 7, A_5 = -2$. Since

$$\sum_1^{\infty} c(n\pi/L) B_n \sin\left(\frac{n\pi x}{L}\right) = g(x) = -4 \sin(3\pi x/L),$$

we see that $c(n\pi/L) B_n = 0$ if $n \neq 3$ and $c(3\pi/L) B_3 = -4$ gives $B_3 = -4L/3c\pi$. So

$$\begin{aligned} u(x, t) &= \sum_1^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) \\ &= 7 \cos\left(\frac{c2\pi t}{L}\right) \sin\left(\frac{2\pi x}{L}\right) - 2 \cos\left(\frac{c5\pi t}{L}\right) \sin\left(\frac{5\pi x}{L}\right) \\ &\quad + \frac{-4L}{3c\pi} \sin\left(\frac{c3\pi t}{L}\right) \sin\left(\frac{3\pi x}{L}\right) \end{aligned}$$

The frequencies (in cycles per second) which are present in the solution are $v_2 = c/L, v_3 = c3/2L, v_5 = c5/2L$. The period of the motion is $2L/c$ in t . \square

Exercise 5.5.2. Solve the IBVP as in Exercise 5.5.1 with homogeneous Neumann boundary conditions $u_x(0, t) = u_x(L, t) = 0$. Take $f(x) = 0$ and $g(x) = x$. Use the formulas

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx & n = 0, 1, \dots \\ \omega_n B_n &= \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx, & n = 1, 2, \dots \\ B_0 &= \frac{2}{L} \int_0^L g(x) dx. \end{aligned}$$

What frequencies $\omega_n = c(n\pi/L)$ are present in the solution?

Solution. We see immediately from the formula that $A_n = 0$ for all n since $f(x) = 0$. Then

$$\begin{aligned} B_0 &= \frac{2}{L} \int_0^L x dx = L. \\ \frac{cn\pi}{L} B_n &= \frac{2}{L} \int_0^L x \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L} \frac{L}{n\pi} \left(x \sin\left(\frac{n\pi x}{L}\right) \Big|_0^L - \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx \right) \\ &= \frac{2L}{(n\pi)^2} (\cos(n\pi) - 1). \end{aligned}$$

So

$$B_n = \frac{2L^2}{(n\pi)^3} (\cos(n\pi) - 1) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{2} \\ -\frac{4L^2}{(n\pi)^3} & \text{if } n \equiv 1 \pmod{2} \end{cases}.$$

Hence the frequencies (in cycles per second) which are present in the solution are $v_n = c(n/2L)$ for $n \equiv 1 \pmod{2}$. The solution is

$$u(x, t) = \frac{Lt}{2} + \sum_1^{\infty} -\frac{4L^2}{(n\pi)^3} \sin\left(\frac{cn\pi t}{L}\right) \cos\left(\frac{n\pi x}{L}\right)$$

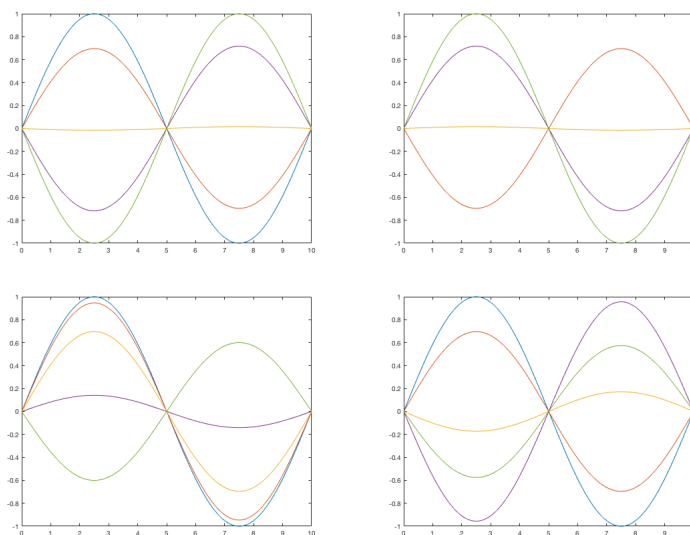
□

Exercise 5.5.3. Consider the IBVP as in Exercise 5.5.1 with initial data $f(x) = \sin(2\pi x/10)$ and $g(x) = 0$.

- What is the period of the motion?
- Write mfiles `f.m` and `g.m` with array-smart. Now run program `wave1` with data choice 3, $d = k = 0$. The time step for `wave1` is $\Delta t = .025$. Choose the number of time steps `n1`, `n2`, `n3`, `n4` so that `snap1` is 1/8 period, `snap2` is 1/4 period, etc. Make a number of plots so that you get a complete picture of the motion. What are the node(s)?
- Do the same for the initial data of exercise 1.

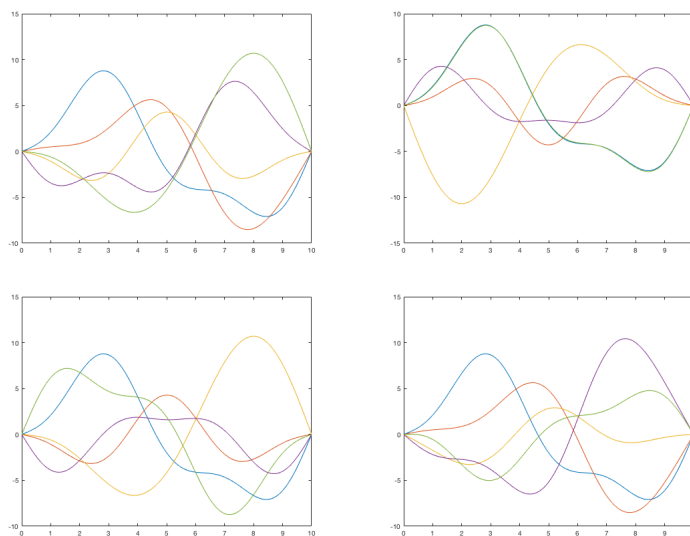
Solution.

- Based on our initial data, we get that $A_2 = 1$ and $A_n = 0$ for all $n \neq 2$. So the period of motion is $1/\nu_2 = 1/(2c/2L) = L/c = 10/1 = 10$.
-



The nodes are at $x = 0, 5, 10$.

(c)



The nodes are at $x = 0, 10$.

□

Exercise 5.5.12. Solve the inhomogeneous equation

$$u_{tt} - c^2 u_{xx} = q(x, t)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

with boundary conditions $u(0, t) = u(0, L) = 0$.

- (a) Set $f(x) = g(x) = 0$ and $q(x, t) = e^t \sin(2\pi x/L)$. You will not need a full eigenfunction expansion.

(b) Again set $f(x) = g(x) = 0$ and now set

$$q(x, t) = \sin(2\pi x/L) \text{ for } 0 \leq t \leq T, \quad q(x, t) = 0 \text{ for } t > T.$$

Solve the equation. What happens to the solution as $t \rightarrow \infty$? Does it damp out, or does it continue to oscillate?

Solution. (a) We have

$$e^t \sin\left(\frac{2\pi x}{L}\right) = q(x, t) = \sum_1^\infty q_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

so $q_2(t) = e^t$. Since $f(x) = g(x) = 0$, we get

$$\begin{aligned} u(x, t) &= \sin\left(\frac{2\pi x}{L}\right) \frac{L}{2c\pi} \int_0^t \sin\left(\frac{c2(t-s)\pi}{L}\right) e^s ds \\ &= \sin\left(\frac{2\pi x}{L}\right) \frac{L}{2c\pi} \left(\frac{1}{1 + \left(\frac{2c\pi}{L}\right)^2} \right) \left(\frac{2c\pi}{L} e^t - \sin\left(\frac{2c\pi t}{L}\right) - \frac{2c\pi}{L} \cos\left(\frac{2c\pi t}{L}\right) \right) \end{aligned}$$

(b) We have

$$\sin\left(\frac{2\pi x}{L}\right) = q(x, t) = \sum_1^\infty q_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

so $q_2(t) = 1$. Since $f(x) = g(x) = 0$, we get

$$\begin{aligned} u(x, t) &= \sin\left(\frac{2\pi x}{L}\right) \frac{L}{2c\pi} \int_0^t \sin\left(\frac{c2(t-s)\pi}{L}\right) ds = \frac{L^2}{2(c\pi)^2} \sin\left(\frac{2\pi x}{L}\right) \sin^2\left(\frac{c\pi t}{L}\right), \\ &\text{when } 0 \leq t \leq T, \text{ and} \\ u(x, t) &= \sin\left(\frac{2\pi x}{L}\right) \frac{L}{2c\pi} \int_0^T \sin\left(\frac{c2(t-s)\pi}{L}\right) ds = \frac{L^2}{2(c\pi)^2} \sin\left(\frac{2\pi x}{L}\right) \sin^2\left(\frac{c\pi T}{L}\right) \\ &\text{when } t > T. \text{ So as } t \rightarrow \infty, \text{ the solution continues to oscillate.} \end{aligned}$$

□

Exercise 5.5.14. Calculate the conserved energy quantity $e(t)$ for the solution of

$$u_{tt} - c^2 u_{xx} = 0$$

with boundary conditions

$$u_x(0, t) = hu(0, t), \quad u_x(L, t) = 0, \quad t \in \mathbb{R}$$

Solution. Assuming that $u_x \rightarrow 0$ as $x \rightarrow \infty$, we get

$$\frac{d}{dt} e(t) = c^2 u_x u_t \Big|_0^\infty = c^2 (u_x(0, t) u_t(0, t)) = c^2 (hu(0, t) u_t(0, t)) = -\frac{c^2 h}{2} \frac{d}{dt} u^2(0, t),$$

so that $\tilde{e}(t) = e(t) + (c^2 h/2) u^2(0, t)$ is the conserved energy. □