Homework for Homological Algebra

Nicholas Camacho
Department of Mathematics
University of Iowa
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Beware: Some solutions may be incorrect!

HOMOLOGICAL ALGEBRA

NICHOLAS CAMACHO

1. Homework 1

Exercise 1.1.

(a) Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor (covariant or contravariant) and $f : A \to B$ is an isomorphism in $\mathcal{C}$. Show that $F(f)$ is an isomorphism in $\mathcal{D}$.

(b) Show that if f is an isomorphism, then f is both a monomorphism and an epimorphism in $\mathcal{C}$. How about the converse? (Prove it or give a counter-example).

(c) Let $\mathcal{C}$ be a concrete category. Prove that every injective morphism in $\mathcal{C}$ is a monomorphism, and every surjective morphism in $\mathcal{C}$ is an epimorphism. Prove that in R-Mod, or Mod-R, every monomorphism is injective and every epimorphism is surjective. Give an example of a concrete category with non-surjective epimorphisms.

Proof.

(a) If $f' \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ is such that $f \circ f' = \mathrm{id}_B$ and $f' \circ f = \mathrm{id}_A$, then in $\mathcal{D}$

$$\mathrm{id}_{F(B)} = F(f) \circ F(f') \quad \text{and} \quad \mathrm{id}_{F(A)} = F(f') \circ F(f)$$

if F is covariant. Similar proof if F is contravariant.

(b) If $g, g' \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ and $f \circ g = f \circ g'$, then

$$g = \mathrm{id}_A \circ g = f' \circ f \circ g = f' \circ f \circ g' = \mathrm{id}_A \circ g' = g',$$

so f is a monomorphism. Similarly, if $h, h' \in \mathrm{Hom}_{\mathcal{C}}(B, Y)$ and $h \circ f = h' \circ f$, then

$$h = h \circ \mathrm{id}_B = h \circ f \circ f' = h' \circ f \circ f' = h' \circ \mathrm{id}_B = h',$$

so f is an epimorphism.

The converse is not true. Consider the category of Rings with 1 (morphisms sending 1 to 1). The inclusion map $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is non-surjective, but is an epimorphism and a monomorphism: If $h, h' : \mathbb{Q} \to R$ for any ring R and $hf = h'f$, then $h = h'$ since

$$h\left(\frac{a}{b}\right) = \frac{hf(a)}{hf(b)} = \frac{h'f(a)}{h'f(b)} = h'\left(\frac{a}{b}\right).$$

If $g, g' : R \to \mathbb{Z}$ and $fg = fg'$, then $f(g(r)) = f(g'(r))$ implies $g(r) = g'(r)$ for all $r \in R$ since f is injective. So $g = g'$.

(c) **In a concrete category $\mathcal{C}$:**

If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is injective and $f \circ g = f \circ g'$ for $g, g' \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, then for all $x \in X$ we have $f(g(x)) = f(g'(x))$, which implies $g(x) = g'(x)$ for all $x \in X$, and hence $g = g'$. So f is a monomorphism.

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If $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is surjective and $h \circ f = h' \circ g$ for $h, h' \in \mathrm{Hom}_{\mathcal{C}}(B,Y)$, then for all $b \in B$ there exists $a \in A$ so that $f(a) = b$. So we have $h(b) = h(f(a)) = h'(f(a)) = h'(b)$, and hence $h = h'$. So f is an epimorphism.

In R-Mod :

If $f \in \mathrm{Hom}_R(A,B)$ is a monomorphism, let $i : \mathrm{Ker}(f) \hookrightarrow A$ be the inclusion. Then $f \circ i = f \circ \mathbf{0}$ so $i = \mathbf{0}$, i.e. f is injective. If $f \in \mathrm{Hom}_R(A,B)$ is an epimorphism and $p : B \to B/\operatorname{im} f$ is the natural projection, then $pf = \mathbf{0} = \mathbf{0}f$, which implies $p = \mathbf{0}$, so $B = \operatorname{im} f$.

Counter-example:

The example given in part (b) is an example of a concrete category with a non-surjective epimorphism.

□

Exercise 1.2. Let F : R-Mod $\to$ Ab be an additive functor (covariant or contravariant). Suppose $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is a split short exact sequence of (unital) left R-modules. Prove that $F(B) \cong F(A \oplus C)$ and that $F(A \oplus C) \cong F(A) \oplus F(C)$.

Proof. Since the sequence is short exact, $B \cong A \oplus C$, and so the isomorphism $F(B) \cong F(A \oplus C)$ follows from Exercise 1.1, part (a).

Let $p_A : A \oplus C \to A$ and $p_C : A \oplus C \to A$ be the projection maps, and let $i_A : A \to A \oplus C$ and $i_C : C \to A \oplus C$ be the inclusions. First assume that F is covariant. Define maps

$$\begin{aligned} f : F(A \oplus C) &\to F(A) \oplus F(C) \\ x &\mapsto (F(p_A)(x), F(p_C)(x)), \quad \text{and} \\ g : F(A) \oplus F(C) &\to F(A \oplus C) \\ (u,v) &\mapsto F(i_A)(u) + F(i_C)(v). \end{aligned}$$

Since they are defined in terms of morphisms in Ab, both f and g are themselves group homomorphisms. Since $p_A i_A = \mathrm{id}_A$ and $p_C i_C = \mathrm{id}_C$, we get

$$\begin{aligned} fg(u,v) &= f(Fi_A(u) + Fi_C(v)) \\ &= (Fp_A(Fi_A(u) + Fi_C(v)), Fp_C(Fi_A(u) + Fi_C(v))) \\ &= (Fp_A Fi_A(u) + Fp_A Fi_C(v), Fp_C Fi_A(u) + Fp_C Fi_C(v)) \\ &= (u + F(p_A i_C)(v), F(p_C i_A)(u) + v). \end{aligned}$$

Let $\mathbf{0}$ denote the zero map in R-Mod or Ab. Since F is additive, $F(\mathbf{0}) = \mathbf{0}$. Hence it follows that $fg = \mathrm{id}_{F(A) \oplus F(C)}$. Using the additivity of F and the fact that $i_A p_A + i_C p_C = \mathrm{id}_{A \oplus C}$, we get

$$gf(x) = Fi_A Fp_A(x) + Fi_C Fp_C(x) = F(i_A p_A + i_C p_C)(x) = x.$$

So $gf = \mathrm{id}_{F(A \oplus C)}$, which shows that $F(A \oplus C) \cong F(A) \oplus F(C)$.

If F is contravariant, the following maps give the desired isomorphism:

$$f : x \mapsto (F(i_A)(x), F(i_C)(x)) \quad \text{and} \quad g : (u,v) \mapsto F(p_A)(u) + F(p_C)(v).$$

□

2. Homework 2

Exercise 2.1. Let $\mathcal{C}$ = R-Mod or Mod-R, let $(I, \leq)$ be a directed partially ordered set, and let K be a cofinal subset (for all $i \in I$, there exists $k \in K$ with $i \leq k$).

(a) If $(\{A_i\}_{i\in I}, \{\varphi_j^i\}_{i\leq j})$ is a direct system in $\mathcal{C}$ with index set I, prove that $(\{A_k\}_{k\in K}, \{\varphi_\ell^k\}_{k\leq \ell})$ is a direct system in $\mathcal{C}$ with index set K. Moreover prove that the direct limits of both these direct systems are isomorphic. Show that this may be false if I is not directed.

(b) Same question, but for an inverse system.

Proof of (a). Since $K \subset I$, $\{A_k\}_{k\in K}$ is a collection of modules in $\mathcal{C}$, and moreover, $\varphi_{k_3}^{k_1} = \varphi_{k_3}^{k_2}\varphi_{k_2}^{k_1}$ for all $k_1 \leq k_2 \leq k_3$ in K. So $(\{A_k\}_{k\in K}, \{\varphi_\ell^k\}_{k\leq\ell})$ is a direct system in $\mathcal{C}$ with index set K.

Let $(\iota_j : A_j \to \coprod_{i\in I} A_i)$ be the inclusions for the coproduct (direct sum). Define

$$S := \left\langle \{\iota_j\varphi_j^i a_i - \iota_i a_i \mid i \leq j \text{ in I, and } a_i \in A_i\} \right\rangle \subset \coprod A_i.$$

Now let $\alpha := (\alpha_j : A_j \to \coprod A_i/S)_{j\in I}$ be the collection of morphisms defined by precomposing the natural projection with the inclusions. Let $\varinjlim A_k$ be the direct limit of the direct system which is indexed over K. We know that

$$\varinjlim A_i = \left(\coprod A_i \big/ S,\ \alpha\right),$$

and hence to show $\varinjlim A_i \cong \varinjlim A_k$, we will show that $\varinjlim A_i$ satisfies the universal mapping property of $\varinjlim A_k$. To that end, suppose $X \in \mathrm{Ob}(\mathcal{C})$ and $(f_k : A_k \to X)_{k\in K}$ is a collection of morphisms in $\mathcal{C}$ satisfying $f_k = f_\ell\varphi_\ell^k$ for all $k \leq \ell$ in K.

Let $i \in I$. Since K is cofinal, there exists $k_i \in K$ such that $i \leq k_i$. Define a map

$$\psi : \coprod_{i\in I} A_i \longrightarrow X, \quad (a_i)_i \longmapsto \sum_{i\in I} f_{k_i}\varphi_{k_i}^i a_i.$$

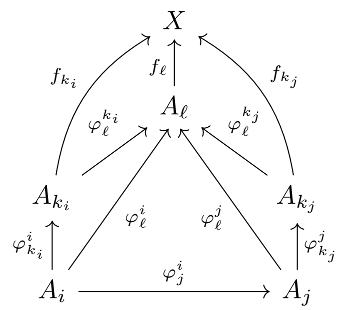
Since all but finitely many coordinates of $(a_i)_i$ are zero, ψ is well-defined. Also, ψ is a module homomorphism since it is defined in terms of module homomorphisms. Notice that for $i \leq j$ in I,

$$\iota_j\varphi_j^i a_i - \iota_i a_i =: (\tilde{a}_m)_{m\in I} \quad \text{where} \quad \tilde{a}_m = \begin{cases} \varphi_j^i a_i & \text{if } m = j, \\ -a_i & \text{if } m = i, \\ 0 & \text{if } m \neq i, j. \end{cases}$$

So

$$\psi(\tilde{a}_m)_m = f_{k_j}\varphi_{k_j}^j\varphi_j^i a_i - f_{k_i}\varphi_{k_i}^i a_i. \tag{2.1.1}$$

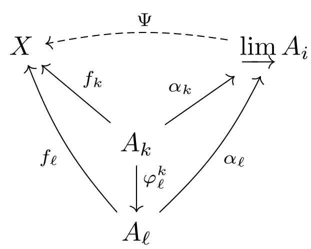
Now, the hypostheses that I is directed and K is cofinal together imply that there exists $\ell \in K$ so that $k_i \leq \ell$ and $k_j \leq \ell$. Hence we get the following diagram in $\mathcal{C}$:



Notice that since k_i, k_j and ℓ are all in K, the top two triangles commute. Hence the entire diagram commutes. In particular,

$$f_{k_j}\varphi^j_{k_j}\varphi^i_j = f_{k_i}\varphi^i_{k_i},$$

and so Equation 2.1.1 becomes $\psi(\tilde{a}_m)_m = 0$. Since elements of the form $(\tilde{a}_m)_m$ generate S, we have $\psi(S) = 0$, and hence ψ induces a well defined morphism $\Psi : \coprod A_i/S \longrightarrow X$. Moreover, for ℓ in K and $a_\ell \in A_\ell$, $\Psi\alpha_\ell a_\ell = f_{k_\ell}\varphi^\ell_{k_\ell} a_\ell = f_\ell a_\ell$, so Ψ makes the diagram commute:

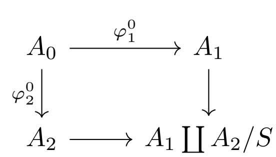


Now if $\tilde{\Psi} : \varinjlim A_i \to X$ is another morphism in $\mathcal{C}$ making the diagram commute, then $\tilde{\Psi}\alpha_{k_i} = f_{k_i}$ for all $k_i \in K$, and so

$$\Psi((a_i)i + S) = \sum_i f_{k_i}\varphi^i_{k_i} a_i = \sum_i \tilde{\Psi}\alpha_{k_i}\varphi^i_{k_i} a_i = \tilde{\Psi}\sum_i \alpha_i a_i = \tilde{\Psi}((a_i)_i + S).$$

Therefore, Ψ is unique, and hence $\varinjlim A_i \cong \varinjlim A_k$. □

Consider $I = \{0, 1, 2\}$ with partial order $0 < 1$ and $0 < 2$. We get the pushout as our direct limit, so $\varinjlim A_i = A_1 \coprod A_2/S$ where $S := \{(\varphi^0_1(a_0), -\varphi^0_2(a_0)) : a_0 \in A_0\}$.



The subset $K = \{1, 2\}$ is cofinal and its associated direct system has direct limit $\varinjlim A_k = A_1 \coprod A_2$, which is not isomorphic to $\varinjlim A_i = A_1 \coprod A_2/S$, (unless of course $S = 0$, in which case φ^0_1 and φ^0_1 are both the zero map. So just assume they're not).

Exercise 2.2. Let R, S be rings, let $(I, \leq)$ be a partially ordered set.

(a) If $(\{A_i\}_{i\in K}, \{\varphi^i_j\}_{i\leq j})$ is a direct system in R-Mod with index set I, prove that there is an exact sequence in R-Mod

$$\coprod_{i\in I}\coprod_{\substack{j\in I\\ i\leq j}} B_{ij} \xrightarrow{f} \coprod_{i\in I} A_i \xrightarrow{p} \varinjlim A_i \to 0$$

where $B_{ij} = A_i$ for all $i \leq j$.

If $(\{C_i\}_{i\in K}, \{\psi^j_i\}_{i\geq j})$ is an inverse system in $\mathcal{C}$ with index set I, prove that there is an exact sequence

$$0 \to \varprojlim C_i \xrightarrow{\iota} \prod_{i\in I} C_i \xrightarrow{g} \prod_{i\in I}\prod_{\substack{j\in I\\ i\leq j}} D_{ij}$$

where $D_{ij} = C_i$ for all $i \leq j$.

(b) Let F : R-Mod $\longrightarrow$ S-Mod be an additive left exact functor.

If F is covariant and preserves direct products, prove that F preserves inverse limits.

If F is contravariant and converts direct sums into direct products, prove that F converts direct limits into inverse limits.

Proof.

(a) First, p is the natural projection, and ι is inclusion. Let $\iota_j : A_j \to \coprod_{i\in I} A_i$ be the jth inclusion for the coproduct. Define f by the rule

$$((a_{ij})_{j\in I, i\leq j})_{i\in I} \longmapsto \sum_{i\in I}\sum_{\substack{j\in I\\ i\leq j}} \iota_j\varphi^i_j a_{ij} - \iota_i a_{ij}.$$

The sum is well-defined since we are working over coproducts, and so only finitely many components of tuples are nonzero. By definition of $\varinjlim A_i$, an element $(a_i)_i \in \coprod A_i$ is in $\ker p$ if and only if it is a finite sum of elements of the form $\iota_j\varphi^i_j a_i - \iota_i a_i$, and so $\operatorname{im} f = \ker p$. Next, define g by the rule

$$(c_i)_{i\in I} \longmapsto \left((\psi^j_i c_j - c_i)_{j\in I, i\leq j}\right)_{i\in I}.$$

An element $(c_i)_i \in \prod C_i$ is in $\ker g$ is and only if $\psi^j_i c_j = c_i$ for all $i \leq j$ in I. This is precisely the definition of the elements of $\varprojlim C_i$.

(b)

□

Exercise 2.3. Let G be a group and let $\mathcal{N}$ be the family of all normal subgroups of finite index in G.

(a) If $N' \subseteq N$ in $\mathcal{N}$ then there is a homomorphism $\psi_N^{N'} : G/N' \to G/N$. Show that the family of all such quotients together with the maps $\psi_N^{N'}$ forms an inverse system over $\mathcal{N}$ where $N \leq N'$ iff $N' \subseteq N$.

(b) The inverse limit of the system in (a), $\varprojlim G/N$, is denoted by $\hat{G}$ and is called the *profinite completion* of G. There is a natural homomorphism $f : G \to \hat{G}$ sending g to $(gN)_{N\in\mathcal{N}}$. Show that f is injective if and only if G is *residually finite*, i.e. $\bigcap_{N\in\mathcal{N}} N = \{1_G\}$.

(c) Write down the profinite completion of $\mathbb{Z}$, viewed as a subgroup under addition.

Proof.

(a) The map $\psi_N^{N'} : G/N' \to G/N$ given by $gN' \mapsto gN$ is a well-defined group homomorphism since $N' \subseteq N$. Moreover, when $N \leq N' \leq N''$

$$\psi_N^{N'} \psi_{N'}^{N''}(gN'') = \psi_N^{N'}(gN') = gN = \psi_N^{N''}(gN''),$$

so $(\{G/N\}_{N\in\mathcal{N}}, \{\psi_N^{N'}\}_{N\leq N'})$ forms an inverse system over $\mathcal{N}$.

(b) Follows from the fact that

$$\ker f = \{g : (gN)_{N\in\mathcal{N}} = (N)_{N\in\mathcal{N}}\} = \{g : g \in N \ \forall N \in \mathcal{N}\} = \bigcap_{N\in\mathcal{N}} N$$

(c) The normal subgroups of $\mathbb{Z}$ with finite index are all nonzero subgroups since $\mathbb{Z}$ is abelian, i.e. $\mathcal{N} = \{n\mathbb{Z} \mid n \in \mathbb{Z}^+\}$. Moreover

$$n|m \iff m\mathbb{Z} \subseteq n\mathbb{Z} \iff n\mathbb{Z} \leq m\mathbb{Z}$$

So

$$\begin{aligned}
\varprojlim \mathbb{Z}/n\mathbb{Z} &= \left\{ (a_n + n\mathbb{Z})_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} \mathbb{Z}/n \,\middle|\, \psi_{n\mathbb{Z}}^{m\mathbb{Z}}(a_m + n\mathbb{Z}) = a_n + n\mathbb{Z}, \forall n|m \right\} \\
&= \left\{ (a_n + n\mathbb{Z})_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} \mathbb{Z}/n \,\middle|\, a_m + n\mathbb{Z} = a_n + n\mathbb{Z}, \ \forall n|m \right\} \\
&= \left\{ (a_n + n\mathbb{Z})_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+} \mathbb{Z}/n \,\middle|\, a_m - a_n \in n\mathbb{Z}, \ \forall n|m \right\}.
\end{aligned}$$

□

3. Homework 3

Exercise 3.1.

(a) Show that the $\mathbb{Z}$-module $\mathbb{Z}/2$ does not have a projective cover. (You can use without proof that over a PID every projective module, whether or not it is finitely generated, is free.)

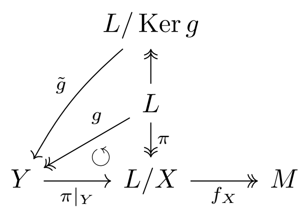
(b) Let R be a left Artinian ring, and let M be a finitely generated left R-module. Prove that M has a projective cover.

Proof.

(a) Suppose there exists an essential epimorphism $\epsilon : P \twoheadrightarrow \mathbb{Z}/2$ for a projective, hence free, $\mathbb{Z}$-module $P \cong \bigoplus_{i\in I} \mathbb{Z}$, where I is some index set. Since ϵ is not the zero map, at least of the group generators for $\bigoplus_i \mathbb{Z}$ maps to $1+2\mathbb{Z}$, say x. But $\epsilon(\langle 3x\rangle) = \mathbb{Z}/2$ and $\langle 3x\rangle \lneq \bigoplus_i \mathbb{Z}$, contradicting that ϵ is essential.

(b) Let $L \in \mathrm{Ob}({}_{\mathrm{R}}\mathbf{proj})$ with an epimorphism $f : L \twoheadrightarrow M$. Let $\mathcal{S}$ be the set of all submodules $N \subseteq \mathrm{Ker}(f)$ so that $f_N : L/N \to M$ is an essential epimorphism. Now $\mathcal{S} \neq \varnothing$ since $L/\mathrm{Ker}(f) \cong M$. Since R is Artinian and L is finitely generated, then L is Artinian and hence $\mathcal{S}$ has a minimal element, say X. It remains to show that L/X is projective.

If $\pi : L \twoheadrightarrow L/X$, then $f = \pi \circ f_X$. So if π is essential, then f is essential, and we are done. If not, let $Y \subsetneq L$ be a minimal submodule such that $\pi(Y) = L/X$. Then $\pi|_Y : Y \twoheadrightarrow L/X$ is essential, and since L is projective, we find a surjective map $g : L \twoheadrightarrow Y$ so that $\pi = \pi|_Y \circ g$.



Since g is surjective, then $\tilde{g}$ is an isomorphism. So the composition $f_X \circ \pi|_Y \circ \tilde{g}$ is an essential epimorphism. By the minimality of X in $\mathcal{S}$, $X \subseteq \mathrm{Ker}\, g$. Now if $\ell \in \mathrm{Ker}\, g$, then $\ell + X = \pi(\ell) = 0 + X$, so $X = \mathrm{Ker}\, g$. Hence the map $h : L/X \to L$ given by $\ell + X \mapsto g(\ell)$ is a well-defined R-module homomorphism. Moreover

$$(\pi \circ h)(\ell + X) = \pi(g(\ell)) = \pi|_Y(g(\ell)) = \pi(\ell) = \ell + X,$$

hence $\pi \circ h = \mathrm{id}_{L/X}$, and so $X \oplus L/X \cong L$, implying L/X is projective.

□

Exercise 3.2. This exercise gives an example of a direct system $(\{B_i\}_{i\in I}, \{\varphi^i_j\}_{i\leq j})$ of right R-modules over a directed partially ordered set $(I, \leq)$ such that $\varinjlim B_i$ is flat, but not all B_i are flat.

Let k be a field and let $R = k[x, y]$ be the polynomial ring over k in two commuting variables x, y.

(a) Let $\mathfrak{m} = (x, y)$ be the maximal ideal of R. Prove that $\mathfrak{m}$ is not a flat R-module by showing that the inclusion map $\iota : \mathfrak{m} \to R$ does not stay injective when tensoring with $\mathfrak{m}$ over R.

(b) Let $I = \{1, 2\}$ with $1 < 2$, so I is a directed partially ordered set. Consider the direct system $m \hookrightarrow R$ of R-modules, indexed by I. Show that the direct limit of this direct system is isomorphic to R. Since R is flat over R but $\mathfrak{m}$ is not flat over R, this gives an example of the desired kind.

Proof.

(a) The map $\mathfrak{m} \otimes_R \mathfrak{m} \xrightarrow{\mathrm{id}_\mathfrak{m} \otimes\, \iota} \mathfrak{m} \otimes_R R$ sends $x \otimes y - y \otimes x$ to

$$x \otimes y - y \otimes x = xy \otimes 1 - yx \otimes 1 = (xy - yx) \otimes 1 = 0.$$

We claim $x \otimes y - y \otimes x$ is not zero in $\mathfrak{m} \otimes_R \mathfrak{m}$, so $\mathrm{id}_\mathfrak{m} \otimes \iota$ is not injective, i.e., $\mathfrak{m}$ is not a flat R-module.

Let $\overline{a} := a + \mathfrak{m}^2$, for all $a \in \mathfrak{m}$. From the natural map $\pi : \mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$, we obtain

$$\mathfrak{m} \otimes_R \mathfrak{m} \xrightarrow{\mathrm{id}_\mathfrak{m} \otimes\, \pi} \mathfrak{m} \otimes_R \mathfrak{m}/\mathfrak{m}^2,$$

which sends $x \otimes y - y \otimes x$ to the element $x \otimes \overline{y} - y \otimes \overline{x}$. Hence to prove our claim, it suffices to show that $x \otimes \overline{y} - y \otimes \overline{x}$ is not zero in $\mathfrak{m} \otimes_R \mathfrak{m}/\mathfrak{m}^2$.

Let $ax + by \in \mathfrak{m}$, where $a, b \in R$ have constant terms a_0, b_0, respectively. Since $\mathfrak{m}^2$ contains all monomials of degree at least 2, then

$$\overline{ax + by} = a_0\overline{x} + b_0\overline{y},$$

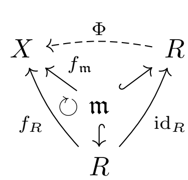
and hence $\overline{x}$ and $\overline{y}$ span $\mathfrak{m}/\mathfrak{m}^2$ over k. Moreover if $a_0x + b_0y \in \mathfrak{m}^2$, then $a_0x + b_0y = 0_K$, implying $a_0 = b_0 = 0$ and hence $\mathfrak{m}/\mathfrak{m}^2 = k\overline{x} \oplus k\overline{y}$.

Notice that in $\mathfrak{m} \otimes_R \mathfrak{m}/\mathfrak{m}^2$, a simple tensor $(ax + by) \otimes (c_0\overline{x} + d_0\overline{y})$, where $a, b \in R, c_0, d_0 \in k$, equals $(a_0x + b_0y) \otimes (c_0\overline{x} + d_0\overline{y})$ where a_0, b_0 are the constant terms of a, b, respectively. It follows that $\pi \otimes \mathrm{id}_{\mathfrak{m}/\mathfrak{m}^2}$ is an isomorphism. Since $- \otimes_R \mathfrak{m}/\mathfrak{m}^2$ preserves direct limits, we get

$$\begin{aligned}\mathfrak{m} \otimes_R \mathfrak{m}/\mathfrak{m}^2 &\cong \mathfrak{m}/\mathfrak{m}^2 \otimes_R \mathfrak{m}/\mathfrak{m}^2 \cong (k\overline{x} \oplus k\overline{y}) \otimes_R (k\overline{x} \oplus k\overline{y}) \\ &\cong k(\overline{x} \otimes_R \overline{x}) \oplus k(\overline{x} \otimes_R \overline{y}) \oplus k(\overline{y} \otimes_R \overline{x}) \oplus k(\overline{y} \otimes_R \overline{y}).\end{aligned}$$

Via this isomorphism, $x \otimes \overline{y} - y \otimes \overline{x} \longmapsto \overline{x} \otimes \overline{y} - \overline{y} \otimes \overline{x}$, the latter of which cannot be zero since $\overline{x} \otimes \overline{y}$ and $\overline{y} \otimes \overline{x}$ are members of a k-basis.

(b) Given a diagram



define $\Phi := f_R$. Then Φ makes the diagram commute and is unique, so $R \cong \varinjlim(\mathfrak{m} \hookrightarrow R)$.

$\square$

Exercise 3.3.

(a) Given two exact sequences of R-modules (all left or all right R-modules)

$$0 \to B \to E^0 \to E^1 \to \cdots \to E^n \to X \to 0 \quad \text{and}$$
$$0 \to B \to D^0 \to D^1 \to \cdots \to D^n \to Y \to 0,$$

where all E^i and D^i are injective, prove that

$$X \oplus D^n \oplus E^{n-1} \oplus D^{n-2} \oplus \cdots \cong Y \oplus E^n \oplus D^{n-1} \oplus E^{n-2} \oplus \cdots$$

(b) Given two exact sequences of R-modules (all left or all right R-modules)

$$0 \to K \to P_n \to P_{n-1} \to \cdots \to P_0 \to B \to 0 \quad \text{and}$$
$$0 \to L \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to B \to 0,$$

where all P_i and Q_i are projective, prove that

$$K \oplus Q_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \cdots \cong L \oplus P_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \cdots$$

Proof.

(a) By induction on n. The case $n = 0$ is the dual statement of Schanuel's Lemma. Now suppose the statement is true for $n > 0$. For all i, let $f^i : E^{i-1} \to E^i$ and $g^i : D^{i-1} \to D^i$. From the dual statement of Schanuel's Lemma, using the short exact sequences

$$0 \to B \xrightarrow{f^0} E^0 \twoheadrightarrow E^0/\operatorname{Ker} f^1 \to 0 \quad \text{and}$$
$$0 \to B' \xrightarrow{g^0} D^0 \twoheadrightarrow D^0/\operatorname{Ker} g^1 \to 0,$$

we have $E^0 \oplus (D^0/\operatorname{Ker} g^1) \cong D^0 \oplus (E^0/\operatorname{Ker} f^1)$. This gives sequences

$$0 \to D^0 \oplus \frac{E^0}{\operatorname{Ker} f^1} \xrightarrow{\begin{pmatrix} \mathrm{id}_{D^0} & 0 \\ 0 & \tilde{f}^1 \end{pmatrix}} D^0 \oplus E^1 \xrightarrow{(\,0\ f^2\,)} E^2 \to \cdots \to E^{n+1} \to X \to 0$$

$$0 \to E^0 \oplus \frac{D^0}{\operatorname{Ker} g^1} \xrightarrow{\begin{pmatrix} \mathrm{id}_{E^0} & 0 \\ 0 & \tilde{g}^1 \end{pmatrix}} E^0 \oplus D^1 \xrightarrow{(\,0\ g^2\,)} D^2 \to \cdots \to D^{n+1} \to Y \to 0$$

Note that $D^0 \oplus E^1$ and $E^0 \oplus D^1$ are injective, since the product of injective modules is injective. Moreover we have

$$\begin{aligned}
\operatorname{Ker} \begin{pmatrix} \mathrm{id}_{D^0} & 0 \\ 0 & \tilde{f}^1 \end{pmatrix} &\cong \operatorname{Ker} \tilde{f}^1 \cong \operatorname{Ker} f^1/\operatorname{Ker} f^1 = 0,\\
\operatorname{Ker} (\,0\ f^2\,) &= D^0 \oplus \operatorname{Ker} f^2\\
&= D^0 \oplus \operatorname{Im} f^1\\
&\cong D^0 \oplus \frac{\operatorname{Im} f^1 + \operatorname{Ker} f^1}{\operatorname{Ker} f^1}\\
&= \operatorname{Im} \begin{pmatrix} \mathrm{id}_{D^0} & 0 \\ 0 & \tilde{f}^1 \end{pmatrix},\\
\operatorname{Im} (\,0\ f^2\,) &\cong \operatorname{Im} f^2 = \operatorname{Ker} f^3,
\end{aligned}$$

with similar statements for the second sequence. So the sequences are exact and we are in the situation of our induction hypothesis, hence

$$X \oplus D^{n+1} \oplus E^n \oplus D^{n-1} \oplus \cdots \cong Y \oplus E^{n+1} \oplus D^n \oplus E^{n-1} \oplus \cdots .$$

(b) By induction on n. The case $n = 0$ is Schanuel's Lemma. Now suppose the statement is true for $n > 0$. For all i, let $f_i : P_i \to P_{i-1}$ and $g_i : Q_i \to Q_{i-1}$. From Schanuel's Lemma, using the short exact sequences

$$0 \to \operatorname{Im} f_1 \hookrightarrow P_0 \xrightarrow{f_0} B \to 0 \quad \text{and}$$
$$0 \to \operatorname{Im} g_1 \hookrightarrow Q_0 \xrightarrow{g_0} B' \to 0,$$

we have $\operatorname{Im} f_1 \oplus Q_0 \cong \operatorname{Im} g_1 \oplus P_0$. This gives sequences

$$0 \to K \to P_{n+1} \to \cdots \to P_2 \xrightarrow{\left(\begin{smallmatrix} f_2 \\ 0 \end{smallmatrix}\right)} P_1 \oplus Q_0 \xrightarrow{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & \mathrm{id}_{Q_0} \end{smallmatrix}\right)} \operatorname{Im} f_1 \oplus Q_0 \to 0$$

$$0 \to L \to Q_{n+1} \to \cdots \to Q_2 \xrightarrow{\left(\begin{smallmatrix} g_2 \\ 0 \end{smallmatrix}\right)} Q_1 \oplus P_0 \xrightarrow{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & \mathrm{id}_{P_0} \end{smallmatrix}\right)} \operatorname{Im} g_1 \oplus P_0 \to 0$$

Note that $Q_1 \oplus P_0$ and $P_1 \oplus Q_0$ are projective since the coproduct of projective modules is projective. Moreover we have

$$\begin{aligned} \operatorname{Im} \begin{pmatrix} f_1 & 0 \\ 0 & \mathrm{id}_{Q_0} \end{pmatrix} &= \operatorname{Im} f_1 \oplus Q_0, \\ \operatorname{Im} \left(\begin{smallmatrix} f_2 \\ 0 \end{smallmatrix}\right) &\cong \operatorname{Im} f_2 = \operatorname{Ker} f_1 \cong \operatorname{Ker} \begin{pmatrix} f_1 & 0 \\ 0 & \mathrm{id}_{Q_0} \end{pmatrix}, \\ \operatorname{Ker} \left(\begin{smallmatrix} f_2 \\ 0 \end{smallmatrix}\right) &\cong \operatorname{Ker} f_2 = \operatorname{Im} f_3, \end{aligned}$$

with similar statements for the second sequence. So the sequences are exact and we are in the situation of our induction hypothesis, hence

$$K \oplus Q_{n+1} \oplus P_n \oplus Q_{n-1} \oplus \cdots \cong L \oplus P_{n+1} \oplus Q_n \oplus P_{n-1} \oplus \cdots .$$

$\square$

4. Homework 4

Exercise 4.1. A full subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is said to be a *skeleton* of $\mathcal{C}$ if every object in $\mathcal{C}$ is isomorphic to exactly one object in $\mathcal{C}_0$. Assuming the Axiom of Choice (e.g., using the Gödel-Bernays system), prove that $\mathcal{C}$ and $\mathcal{C}_0$ are equivalent categories. Show that $\mathcal{C}$ and $\mathcal{C}_0$ need not be isomorphic categories (give an example).

Proof. For all $X \in \mathrm{Ob}(\mathcal{C})$, let $\tilde{X}$ denote the object in $\mathcal{C}_0$ for which $X \cong \tilde{X}$, and choose an isomorphism $f_X : X \to \tilde{X}$. Define

$$\begin{aligned} F : \mathcal{C} &\longrightarrow \mathcal{C}_0 \\ X &\longmapsto \tilde{X} \\ (f : X \to X') &\longmapsto Ff := f_{X'} \circ f \circ f_X^{-1} \end{aligned}$$

For all $X, X' \in \mathrm{Ob}(\mathcal{C})$, define

$$\begin{aligned} \hat{F}_{X,X'} : \mathrm{Hom}_{\mathcal{C}_0}(\tilde{X}, \tilde{X}') &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, X') \\ g &\longmapsto f_{X'}^{-1} \circ g \circ f_X. \end{aligned}$$

Then

$$F\hat{F}(g) = F(f_{X'}^{-1} \circ g \circ f_X) = f_{X'} \circ f_{X'}^{-1} \circ g \circ f_X \circ f_X^{-1} = g,$$

and

$$\hat{F}F(f) = \hat{F}(f_{X'} \circ f \circ f_X^{-1}) = f_{X'}^{-1} \circ f_{X'} \circ f \circ f_X^{-1} \circ f_X = f.$$

So F is fully faithful. Moreover, since $\mathcal{C}_0$ is a skeleton of $\mathcal{C}$, if $Y \in \mathcal{C}_0$, there exists $X \in Ob(\mathcal{C})$ so that $\tilde{X} = Y$, hence $Y = F(X)$, implying F is dense. Hence F is an equivalence.

The category $\mathcal{C}$ of finite ordered sets is equivalent to the full subcategory $\mathcal{C}_0$ of finite ordered sets of the form $[n] := \{1 < 2 < \cdots < n\}$ for $n \in \mathbb{Z}^+$. Let $N \in \mathrm{Ob}(\mathcal{C})$ and suppose $|N| = n$. Let N_j denote the jth element in N. Then $N \cong [n]$ via the map $N_j \mapsto j$, and moreover, $[n]$ is the only object in $\mathcal{C}_0$ isomorphic to N, since $[n]$ is the only set in $\mathcal{C}_0$ of cardinality n. So $\mathcal{C}_0$ is a skeleton of $\mathcal{C}$. However, $\mathcal{C} \ncong \mathcal{C}_0$, since any isomorphism would need to uniquely identify the object in $\mathcal{C}_0$ with, say, cardinality n, with an object in $\mathcal{C}$ with, say, cardinality m. But there are many such objects in $\mathcal{C}$, so any choice would leave other objects in $\mathcal{C}$ with cardinality m unaccounted for. ∎

Exercise 4.3. Let $F, F' : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{C} \to \mathcal{C}$ be covariant functors.

(a) Prove: If (F, G) and (F', G) are adjoint pairs, then F and F' are naturally isomorphic.
(b) Prove: If (F, G) and (F, G') are adjoint pairs, then G and G' are naturally isomorphic.
(c) Prove: If F and G are quasi-inverses of each other, then (F, G) and (G, F) are adjoint pairs.

Proof.

(a) For all $X \in \mathrm{Ob}(\mathcal{C})$ and for all $Y \in \mathrm{Ob}(\mathcal{D})$, there exists a natural bijection

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F'(X), Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

For all $X \in \mathrm{Ob}(\mathcal{C})$, define $\eta_X : F(X) \to F'(X)$ by

$$\eta_X := \Phi_{X,F'(X)}(\mathrm{id}_{F'(X)}).$$

Then for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ we have a diagram, each square commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F'(X), F'(X)) & \xrightarrow{\Phi_{X,F'(X)}} & \mathrm{Hom}_{\mathcal{D}}(F(X), F'(X)) \\
\Big\downarrow {\scriptstyle (F'f)_*} & & \Big\downarrow {\scriptstyle (F'f)_*} \\
\mathrm{Hom}_{\mathcal{D}}(F'(X), F'(Z)) & \xrightarrow{\Phi_{X,F'(Z)}} & \mathrm{Hom}_{\mathcal{D}}(F(X), F'(Z)) \\
\Big\uparrow {\scriptstyle (F'f)^*} & & \Big\uparrow {\scriptstyle (Ff)^*} \\
\mathrm{Hom}_{\mathcal{D}}(F'(Z), F'(Z)) & \xrightarrow{\Phi_{Z,F'(Z)}} & \mathrm{Hom}_{\mathcal{D}}(F(Z), F'(Z))
\end{array}$$

So

$$\begin{aligned}
F'f \circ \eta_X = (F'f)_*(\eta_X) &= (\Phi_{X,F'(Z)} \circ (F'f)_*)(\mathrm{id}_{F'(X)}) \\
&= \Phi_{X,F'(Z)}(F'f) \\
&= (\Phi_{X,F'(Z)} \circ (F'f)^*)(\mathrm{id}_{F'(Z)}) \\
&= (Ff)^*(\eta_Z) \\
&= \eta_Z \circ Ff.
\end{aligned}$$

Hence $\eta = \{\eta_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is natural. In a similar manner we may define a natural transformation $\epsilon = \{\epsilon_X\}_{X \in \mathrm{Ob}(\mathcal{C})} : F' \to F$ by

$$\epsilon_X := \Phi^{-1}_{X,F(X)}(\mathrm{id}_{F(X)})$$

Using the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F'(X), F'(X)) & \xrightarrow{\Phi_{X,F'(X)}} & \mathrm{Hom}_{\mathcal{D}}(F(X), F'(X)) \\
\Big\downarrow {\scriptstyle (\epsilon_X)_*} & & \Big\downarrow {\scriptstyle (\epsilon_X)_*} \\
\mathrm{Hom}_{\mathcal{D}}(F'(X), F(X)) & \xrightarrow{\Phi_{X,F(X)}} & \mathrm{Hom}_{\mathcal{D}}(F(X), F(X))
\end{array},$$

we have

$$\begin{aligned}\epsilon_X \circ \eta_X &= ((\epsilon_X)_* \circ \Phi_{X,F'(X)})(\mathrm{id}_{F'(X)})\\ &= (\Phi_{X,F(X)} \circ (\epsilon_X)_*)(\mathrm{id}_{F'(X)})\\ &= \Phi_{X,F(X)}(\epsilon_X)\\ &= \mathrm{id}_{F(X)}\,.\end{aligned}$$

Similarly, $\eta_X \circ \epsilon_X = \mathrm{id}_{F'(X)}$. So η is a natural isomorphism.

(b) For all $X \in \mathrm{Ob}(\mathcal{C})$ and for all $Y \in \mathrm{Ob}(\mathcal{D})$, there exists a natural bijection

$$\Psi_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, G'(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

For all $Y \in \mathrm{Ob}(\mathcal{D})$, define $\eta_Y : G(Y) \to G'(Y)$ by

$$\eta_Y := \Psi_{G'(Y),Y}(\mathrm{id}_{G'(Y)}).$$

Then for all $g \in \mathrm{Hom}_{\mathcal{D}}(Y, W)$ we have a diagram, each square commutative:

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{C}}(G'(Y), G'(Y)) & \xrightarrow{\Psi_{G'(Y),Y}} & \mathrm{Hom}_{\mathcal{C}}(G'(Y), G(Y))\\ {\scriptstyle (G'g)_*}\downarrow & & \downarrow{\scriptstyle (Gg)_*}\\ \mathrm{Hom}_{\mathcal{C}}(G'(Y), G'(W)) & \xrightarrow{\Psi_{G'(Y),W}} & \mathrm{Hom}_{\mathcal{C}}(G'(Y), G(W))\\ {\scriptstyle (G'g)^*}\uparrow & & \uparrow{\scriptstyle (G'g)^*}\\ \mathrm{Hom}_{\mathcal{C}}(G'(W), G'(W)) & \xrightarrow{\Psi_{G'(W),W}} & \mathrm{Hom}_{\mathcal{C}}(G'(W), G(W))\end{array}$$

So

$$\begin{aligned}Gg \circ \eta_Y = (Gg)_*(\eta_Y) &= (\Psi_{G'(Y),W} \circ (G'g)_*)(\mathrm{id}_{G'(Y)})\\ &= \Psi_{G'(Y),W}(G'g)\\ &= (\Psi_{G'(Y),W} \circ (G'g)^*)(\mathrm{id}_{G'(W)})\\ &= (G'g)^*(\eta_W)\\ &= \eta_W \circ G'g.\end{aligned}$$

Hence $\eta = \{\eta_Y\}_{Y\in\mathrm{Ob}(\mathcal{D})}$ is natural. In a similar manner we may define a natural transformation $\epsilon = \{\epsilon_Y\}_{Y\in\mathrm{Ob}(\mathcal{D})} : G'(Y) \to G(Y)$ by

$$\epsilon_Y := \Psi^{-1}_{G'(Y),Y}(\mathrm{id}_{G(Y)}).$$

As in part (a), we have $\eta_Y \circ \epsilon_Y = \mathrm{id}_{G'(Y)}$ and $\epsilon_Y \circ \eta_Y = \mathrm{id}_{G(Y)}$. So η is a natural isomorphism.

(c) There exists a natural isomorphism $\tau : \mathrm{Id}_{\mathcal{C}} \to GF$. Since G is fully faithful, for all $X \in \mathrm{Ob}(\mathcal{C})$ and for all $Y \in \mathrm{Ob}(\mathcal{D})$, there are maps between the classes

$$\mathrm{Hom}_{\mathcal{C}}(GF(X), G(Y)) \longleftrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

which are two-sided inverses to one another. In particular, for all $\beta \in \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$, there exists a unique $\tilde{\beta} \in \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ so that

$$G\tilde{\beta} = \beta \circ \tau_X^{-1}.$$

So for all $X \in \mathrm{Ob}(\mathcal{C})$ and for all $Y \in \mathrm{Ob}(\mathcal{D})$, define maps

$$\begin{aligned} \Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) &\rightleftarrows \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) : \Psi_{X,Y} \\ \alpha &\longmapsto G\alpha \circ \tau_X, \\ \tilde{\beta} &\longleftarrow\!\shortmid \beta. \end{aligned}$$

Then $\Psi_{X,Y}(\Phi_{X,Y}(\alpha)) = \widetilde{G\alpha \circ \tau_X}$. Since $G\alpha = G\alpha \circ \tau_X \circ \tau_X^{-1}$, then $\alpha = \widetilde{G\alpha \circ \tau_X}$. Moreover, $\Phi_{X,Y}(\Psi_{X,Y}(\beta)) = G\tilde{\beta} \circ \tau_X^{-1} = \beta$. Hence (F, G) is an adjoint pair. Exchanging the roles of F and G, we see also that (G, F) is an adjoint pair.

□

5. Homework 5

Exercise 5.1. Let $m \in \mathbb{Z}^+$, and let $R = \mathbb{Z}/m$.

(a) Let $A = \mathbb{Z}/d$ where $d|m$, and let B be an arbitrary R-module. Determine $\operatorname{Tor}_n^R(A, B)$ for all $n \geq 0$.

(b) Let C be an arbitrary R-module, and let $D = \mathbb{Z}/p$ where $p|m$. Determine $\operatorname{Ext}_R^n(C, D)$ for all $n \geq 0$ in terms of $\operatorname{Hom}_R(C, R)$. Moreover, show that if $p^2|m$, then $\operatorname{Ext}_R^n(D, D) \cong D$ for all n.

Proof.

(a) Consider the maps $R \xrightarrow{m/d} R$ and $R \xrightarrow{d} R$, multiplication by m/d and d, respectively. We have $d((m/d)(a + m\mathbb{Z}) = 0 + m\mathbb{Z}$, hence $\operatorname{Im}(m/d) \subseteq \operatorname{Ker}(d)$. If $db + m\mathbb{Z} = 0 + m\mathbb{Z}$, then there exists $c \in \mathbb{Z}$ with $db = cm$. So $(m/d)(c + m\mathbb{Z}) = b + m\mathbb{Z}$, which shows $\operatorname{Ker}(d) = \operatorname{Im}(m/d)$. Similarly, we get $\operatorname{Ker}(m/d) = \operatorname{Im}(d)$.

Now let $R \xrightarrow{\epsilon} A$ be the canonical surjection. Then

$$\epsilon(da + m\mathbb{Z}) = da + d\mathbb{Z} = 0 + m\mathbb{Z},$$

and if $\epsilon(b+m\mathbb{Z}) = 0+d\mathbb{Z}$, then $b = da$ for some $a \in \mathbb{Z}$, so $da+m\mathbb{Z} = b+m\mathbb{Z}$. Hence $\operatorname{Ker}(\epsilon) = \operatorname{Im}(d)$. So we get a periodic free resolution of A:

$$\cdots \xrightarrow{d} R \xrightarrow{m/d} R \xrightarrow{d} R \xrightarrow{\epsilon} A \to 0.$$

For $n \geq 1$, let $\partial_n = m/d$ if n is even, and $\partial_n = d$ if n is odd. Then

$$(P_A)_\bullet : \quad \cdots \xrightarrow{\partial_3} R \xrightarrow{\partial_2} R \xrightarrow{\partial_1} R \to 0.$$

We have commutative diagram, where the vertical arrows are each the map $\sum r_i \otimes b_i \mapsto \sum r_i b_i$:

$$\begin{array}{lccccccccc}
((P_A)_\bullet \otimes_R B): & \cdots \xrightarrow{\partial_3 \otimes \mathrm{id}_B} & R \otimes_R B & \xrightarrow{\partial_2 \otimes \mathrm{id}_B} & R \otimes_R B & \xrightarrow{\partial_1 \otimes \mathrm{id}_B} & R \otimes_R B & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
& \cdots \xrightarrow{\quad d \quad} & B & \xrightarrow{\quad m/d \quad} & B & \xrightarrow{\quad d \quad} & B & \longrightarrow & 0
\end{array}$$

Hence

$$\operatorname{Tor}_0^n(A, B) \cong B/dB \cong A \otimes_R B$$

$$(n \text{ odd}) \qquad \operatorname{Tor}_n^R(A, B) \cong \{b \in B \mid db = 0\} \Big/ (m/d)B$$

$$(n \text{ even}) \qquad \operatorname{Tor}_n^R(A, B) \cong \{b \in B \mid (m/d)b = 0\} \Big/ dB$$

(b) Consider the maps $R \xrightarrow{p} R$ and $R \xrightarrow{m/p} R$, multiplication by p and m/p, respectively. Similarly as in part (a), we get that $\operatorname{Ker}(p) = \operatorname{Im}(m/p)$ and $\operatorname{Ker}(m/p) = \operatorname{Im}(p)$. Since $p|m$, let $m = pa$, and consider the map

$$\iota : D \to R, \quad 1 + p\mathbb{Z} \mapsto a + m\mathbb{Z}.$$

Since $pa+m\mathbb{Z} = 0+m\mathbb{Z}$, then $p\iota = 0$ and hence $\operatorname{Im}(\iota) \subseteq \operatorname{Ker}(p)$. Conversely, if $pb + m\mathbb{Z} = 0 + m\mathbb{Z}$, there exists $c \in \mathbb{Z}$ with $pb = cm$. Then $mb = apb = acm$, implying $b = ac$. So $\iota(c + p\mathbb{Z}) = ac + m\mathbb{Z} = b + m\mathbb{Z}$, and so $\operatorname{Ker}(p) = \operatorname{Im}(\iota)$. So we get a periodic injective resolution of D:

$$0 \to D \xrightarrow{\iota} R \xrightarrow{p} R \xrightarrow{m/p} R \xrightarrow{p} \cdots.$$

For $n \geq 0$, let $\delta^n = p$ if $n \equiv 0 \mod 2$ and let $\delta^n = m/p$ if $n \equiv 1 \mod 2$. Then we get a truncated cochain complex for D

$$E_D^\bullet : \quad 0 \to R \xrightarrow{\delta^0} R \xrightarrow{\delta^1} R \xrightarrow{\delta^2} \cdots .$$

Then

$$\mathrm{Hom}_R(C, E_D^\bullet) : \quad 0 \to \mathrm{Hom}_R(C,R) \xrightarrow{\delta_*^0} \mathrm{Hom}_R(C,R) \xrightarrow{\delta_*^1} \mathrm{Hom}_R(C,R) \xrightarrow{\delta_*^2} \cdots$$

So

$$\mathrm{Ext}_R^0(C,D) = \{\alpha : C \to R \mid p\alpha = 0\} \cong \mathrm{Hom}_R(C,D)$$

$$(n \text{ odd}) \qquad \mathrm{Ext}_R^n(C,D) = \{\beta : C \to R \mid (m/p)\beta = 0\} \big/ p\,\mathrm{Hom}_R(C,R)$$

$$(n \text{ even}) \qquad \mathrm{Ext}_R^n(C,D) = \{\gamma : C \to R \mid p\gamma = 0\} \big/ (m/p)\,\mathrm{Hom}_R(C,R)$$

Now consider the periodic free resolution of D, where ϵ is the canonical surjection

$$\cdots \xrightarrow{p} R \xrightarrow{m/p} R \xrightarrow{p} R \xrightarrow{\epsilon} D \to 0.$$

For $n \geq 1$, let $\partial_n = m/p$ if n is even, and $\partial_n = p$ if n is odd. Then

$$(P_D)_\bullet : \quad \cdots \xrightarrow{\partial_3} R \xrightarrow{\partial_2} R \xrightarrow{\partial_1} R \to 0$$

and we have a commutative diagram, where the vertical arrows are all the map $\alpha \mapsto \alpha(1 + m\mathbb{Z})$:

$$\begin{array}{llllllll}
\mathrm{Hom}_R((P_D)_\bullet, D) : & 0 \longrightarrow & \mathrm{Hom}_R(R,D) & \xrightarrow{\partial_1^*} & \mathrm{Hom}_R(R,D) & \xrightarrow{\partial_2^*} & \mathrm{Hom}_R(R,D) & \longrightarrow \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
 & 0 \longrightarrow & D & \xrightarrow{m/p} & D & \xrightarrow{p} & D & \longrightarrow \cdots
\end{array}$$

Suppose that $p^2 | m$, so $m = p^2c$ for some $c \in \mathbb{Z}$ and $m/p = pc$. Hence

$$\mathrm{Ker}(m/p : D \to D) = \{a + p\mathbb{Z} \mid (pc)a \in p\mathbb{Z}\} = D$$
$$\mathrm{Im}(p : D \to D) = pD = 0$$
$$\text{and} \quad \mathrm{Ker}(p : D \to D) = D$$
$$\mathrm{Im}(m/p : D \to D) = (m/p)D = (pc)D = 0.$$

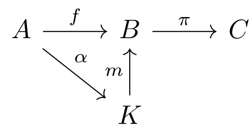
So $\mathrm{Ext}_R^n(D,D) \cong D$ for all $n \geq 0$.

□

Exercise 5.3. Use the definition of abelian category from lecture, i.e. $\mathcal{A}$ is an abelian category if $\mathcal{A}$ is an additive category, every morphism in $\mathcal{A}$ has a kernel and cokernel, every monomorphism in $\mathcal{A}$ is the kernel of its cokernel, and every epimorphism in $\mathcal{A}$ is the cokernel of its kernel.

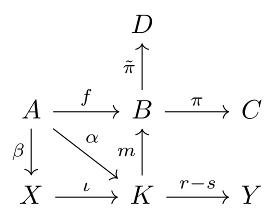
Using only this definition and the definitions of kernel and cokernel in an additive categroy, prove that every morphism $f : A \to B$ in $\mathcal{A}$ factors as $f = m\alpha$ for an empimorphism $\alpha : A \to K$ and a monomorphism $m : K \to B$ (where K is a suitable object in $\mathcal{A}$).

Proof. Let $(C, \pi) = \operatorname{Coker}(f)$ and $(K, m) = \operatorname{Ker}(\operatorname{Coker}(f)) = \operatorname{Ker}(\pi)$. Since $\pi f = 0$ and $(K, m) = \operatorname{Ker} \pi$, there exists a unique α such that $f = m\alpha$.



Since m is a kernel, m is monic. We will show that α is epic.

Suppose $r\alpha = s\alpha$. Let $(X, \iota) = \operatorname{Ker}(r - s)$. Since $(r - s)\alpha = 0$, there exists a unique $\beta : A \to X$ such that $\alpha = \iota\beta$. Let $(D, \tilde{\pi}) = \operatorname{Coker}(m\iota)$.



Since $\tilde{\pi} m\iota = 0$, we have

$$0 = \tilde{\pi} m\iota\beta = \tilde{\pi} m\alpha = \tilde{\pi} f.$$

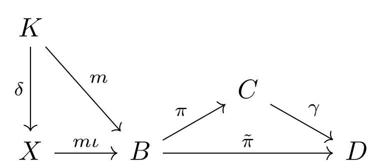
So, since $\operatorname{Coker}(f) = (C, \pi)$, there exists a unique $\gamma : C \to D$ with $\tilde{\pi} = \gamma\pi$. Since $m\iota$ is monic,

$$(X, m\iota) = \operatorname{Ker}(\operatorname{Coker}(m\iota)) = \operatorname{Ker}(\tilde{\pi}).$$

So the equation

$$\tilde{\pi} m = \gamma\pi m = 0,$$

implies that there exists a unique map $\delta : K \to X$ so that $m = m\iota\delta$.



But m is monic, so $\iota\delta = \mathrm{id}_K$. Hence ι is a monic retraction, implying ι is an isomorphism (we proved this fact on a homework last semester in Algebra). Hence $(r - s)\iota = 0$ implies $r = s$, so α is epic. □

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: nicholas-camacho@uiowa.edu