ALGEBRA II – MIDTERM

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Exercise 1. Prove that the following are equivalent for an algebraic set V, where K is algebraically closed.

(a) V is connected in the Zariski topology.
(b) $K[V]$ cannot be written as $K[V] \cong R_1 \times R_2$ for two proper ideals $R_1, R_2 \subseteq K[V]$.

Proof. Suppose $K[V] \cong R_1 \times R_2$ for two proper ideals $R_1, R_2 \subseteq K[V]$. So $R_1, R_2 \subseteq K[V]$ are such that $R_1 \cap R_2 = \mathcal{I}(V)$ and $R_1 + R_2 = K[V]$. Since K is algebraically closed, the equality $R_1 + R_2 = K[V]$ occurs if and only if

$$\mathcal{Z}(R_1) \cap \mathcal{Z}(R_2) = \mathcal{Z}(R_1 + R_2) = \varnothing,$$

so $\mathcal{Z}(R_1)$ and $\mathcal{Z}(R_2)$ are disjoint, and since they are also contained in V, we have

$$\mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2) \subseteq V.$$

From $R_1 \cap R_2 = \mathcal{I}(V)$, we obtain

$$\mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2) = \mathcal{Z}(R_1 \cap R_2) = \mathcal{Z}(\mathcal{I}(V)) \supseteq V$$

Hence $V = \mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2)$ and hence (a) $\Longrightarrow$ (b).

Conversely, if $V = \mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2)$ for two ideals $R_1, R_2 \subseteq K[V]$, then R_1 and R_2 are proper ideals since $\mathcal{Z}(R_1)$ and $\mathcal{Z}(R_2)$ are disjoint. From $V = \mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2)$, we have

$$\mathcal{I}(V) = \mathcal{I}(\mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2)) = \mathcal{I}(\mathcal{Z}(R_1)) \cap \mathcal{I}(\mathcal{Z}(R_2)) \supseteq R_1 \cap R_2,$$

since $R_i \subseteq \mathcal{I}(\mathcal{Z}(R_i))$ for $i = 1, 2$. And since we already have $\mathcal{I}(V) \subseteq R_1 \cap R_2$, we have $\mathcal{I}(V) = R_1 \cap R_2$, and therefore $K[V] \cong R_1 \times R_2$. Hence (b) $\Longrightarrow$ (a) $\square$

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Exercise 2. Let R be a commutative ring and $I, J \subseteq R$ ideals. The *ideal quotient* is defined as

$$(I : J) = \{r \in R \mid rJ \subset I\}.$$

Prove the following statements which kind of explain the name.

(a) Show that $\mathcal{Z}(I) - \mathcal{Z}(J) \subseteq \mathcal{Z}((I : J))$.
(b) Show that if K is algebraically closed and I is radical then $\mathcal{Z}((I : J))$ is exactly the Zariski closure of $\mathcal{Z}(I) - \mathcal{Z}(J)$.
(c) Show that if V and W are affine algebraic sets, then $(\mathcal{I}(V) : \mathcal{I}(W)) = \mathcal{I}(V - W)$.

Proof.

(a) Let $a \in \mathcal{Z}(I) - \mathcal{Z}(J)$ and $f \in (I : J)$. Since a is not in $\mathcal{Z}(J)$, there exists $h \in J$ with $h(a) \neq 0$. Since $fh \in I$, then $f(a)h(a) = 0$, which implies $f(a) = 0$, i.e., $a \in \mathcal{Z}((I : J))$.

(b) We need to show $\mathcal{Z}((I : J))$ is the smallest algebraic set containing $\mathcal{Z}(I) - \mathcal{Z}(J)$, i.e., for any set S of polynomials over K for which $\mathcal{Z}(I) - \mathcal{Z}(J) \subseteq \mathcal{Z}(S)$, we have $\mathcal{Z}((I : J)) \subseteq \mathcal{Z}(S)$. Since $\mathcal{Z}$ is inclusion reversing, it suffices to show

$$S \subseteq (I : J). \tag{2.1}$$

Let $h \in S$ and $g \in J$. We need to show that $hg \in I$, proving (2.1).

Suppose there exists $b \in \mathcal{Z}(I)$ for which $h(b)g(b) \neq 0$. In particular, $b \in \mathcal{Z}(I) - \mathcal{Z}(J)$, so $hg \notin \mathcal{I}(\mathcal{Z}(I) - \mathcal{Z}(J))$. On the other hand, for all $c \in \mathcal{Z}(S)$, $h(c)g(c) = 0$ since $h \in S$. So

$$hg \notin \mathcal{I}(\mathcal{Z}(I) - \mathcal{Z}(J)) \quad \text{and} \quad hg \in \mathcal{I}(\mathcal{Z}(S)).$$

But and $\mathcal{I}$ is inclusion reversing, meaning

$$\mathcal{I}(\mathcal{Z}(S)) \subseteq \mathcal{I}(\mathcal{Z}(I) - \mathcal{Z}(J)).$$

So we have a contradiction, meaning $h(b)g(b) = 0$ for all $b \in \mathcal{Z}(I)$. Hence

$$hg \in \mathcal{I}(\mathcal{Z}(I)) = \sqrt{I} = I,$$

where the first equality follows from Hilbert's Nullstellensatz (since K is algebraically closed) and the second equality follows since I is radical.

(c) If $V = W$ then $(\mathcal{I}(V) : \mathcal{I}(W)) = K[\mathbb{A}^n]$, since $\mathcal{I}(W)$ is an ideal, and $\mathcal{I}(V - W) = \mathcal{I}(\varnothing) = K[\mathbb{A}^n]$, so we are done. Assume $V \neq W$.

" $\subseteq$ " :

Let $f \in (\mathcal{I}(V) : \mathcal{I}(W))$ and $a \in V - W$. If $g(a) = 0$ for all $g \in \mathcal{I}(W)$, then $\mathcal{I}(W) \subseteq \mathcal{I}(\{a\})$ which implies

$$a \in \{a\} \subseteq \mathcal{Z}(\mathcal{I}(\{a\})) \subseteq \mathcal{Z}(\mathcal{I}(W)) = W,$$

which gives a contradiction. So there exists $h \in \mathcal{I}(W)$ with $h(a) \neq 0$. Then $f(a)h(a) = 0$ since $fh \in \mathcal{I}(V)$, and since $h(a) \neq 0$, we must have $f(a) = 0$. Hence $f \in \mathcal{I}(V - W)$.

" $\supseteq$ " :

Let $f \in \mathcal{I}(V - W)$, let $g \in \mathcal{I}(W)$, and let $a \in V$. If $a \in W$, then $f(a)g(a) = 0$ since $g(a) = 0$. If $a \in V - W$, then $f(a)g(a) = 0$ since $f(a) = 0$. In either case, $fg \in \mathcal{I}(V)$, so $f \in (\mathcal{I}(V) : \mathcal{I}(W))$.

□

Exercise 3. For any point in an algebraic set $x \in X$, let $\mathfrak{m}_x \subset K[X]$ be the maximal ideal of all functions on X vanishing at x. Let $\varphi\colon V \to W$ be a morphism of algebraic sets, $v \in V$, and $w = \varphi(v)$. Check on your own that $\widetilde{\varphi}(\mathfrak{m}_w) \subseteq \mathfrak{m}_v$, and therefore that φ induces a map $f\colon \mathfrak{m}_w/\mathfrak{m}_w^2 \to \mathfrak{m}_v/\mathfrak{m}_v^2$ (don't include this in what you turn in). Assume now that K is algebraically closed, so $K[X]/\mathfrak{m}_x \simeq K$ for any algebraic set X over K and point $x \in X$. Check that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a K-vector space and the map f above is K-linear (again don't include this in what you turn in).

In one of the presentations, it is shown that if φ is an isomorphism of algebraic sets, then each f as described above is an isomorphism of K-vector spaces. We use the contrapositive of this below to show that certain algebraic sets can't be isomorphic, since it's relatively much easier to show that two vector spaces are not isomorphic.

(a) Let $X = \mathbb{A}^1_K$. Find the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ for any point $x \in X$.
(b) Let $V = \mathcal{Z}(y^2 - x^3) \subset \mathbb{A}^2$. Determine a point $v \in V$ which you suspect is not like any point of $\mathbb{A}^1$. Compute $\mathfrak{m}_v/\mathfrak{m}_v^2$ and use what you found in (a) to determine that $V \not\simeq \mathbb{A}^1$.
(c) Do the same for $W = \mathcal{Z}(y^2 - x^3 - x^2) \subset \mathbb{A}^2$.
(d) Use what you learned above to show that $\mathcal{Z}(xy - z^2) \not\simeq \mathbb{A}^2$.

Solution. Let's be really fancy by stating and proving more general facts:

Lemma 3.1. *Let $n \in \mathbb{Z}^+$ and K be an algebraically closed field. Let $f \in K[x_1, \ldots, x_n]$ be a polynomial with no linear term, and for which (f) is a radical ideal. Suppose $V := \mathcal{Z}(f) \subseteq \mathbb{A}^n_K$ contains $0_{\mathbb{A}^n_K}$. Then $\mathfrak{m}_0/\mathfrak{m}_0^2$ is a K-vector space of dimension n.*

Proof. Write $\overline{x}_i$ for the image of x_i in $K[V]$. Since (f) is radical and K is algebraically closed, then $\mathcal{I}(\mathcal{Z}(f)) = (f)$, so $K[V] = K[\mathbb{A}^n_K]/(f)$.

Consider a nonconstant monomial term $m := cx_1^{e_1} \cdots x_n^{e_n}$ in $K[\underline{x}]$, where $c \in K$ and $e_i \in \mathbb{Z}_{\geq 0}$. For any i, $\overline{mx}_i + \mathfrak{m}_0^2 = \overline{0} + \mathfrak{m}_0^2$. So if $f_1, \ldots, f_n \in K[\mathbb{A}^n_K]$ and $\overline{f}_1\overline{x}_1 + \cdots + \overline{f}_n\overline{x}_n$ is in $\mathfrak{m}_0$, then

$$\sum_{i=1}^{n} \overline{f}_i\overline{x}_i + \mathfrak{m}_0^2 = \sum_{i=1}^{n} a_i\overline{x}_i + \mathfrak{m}_0^2$$

where a_i is the constant term of f_i for all $1 \leq i \leq n$. This shows

$$\mathrm{Span}_K\{\overline{x}_1 + \mathfrak{m}_0^2, \ldots, \overline{x}_n + \mathfrak{m}_0^2\} = \mathfrak{m}_0/\mathfrak{m}_0^2.$$

Our assumptions on f ensure that, for all $1 \leq i \leq n$,

$$\overline{x}_i + \mathfrak{m}_0^2 \notin \mathrm{Span}_K\{\overline{x}_j + \mathfrak{m}_0^2 : j \neq i\}$$

Indeed, if the this were not true for some $\overline{x}_i + \mathfrak{m}_0^2$, then we have an expression

$$\overline{x}_i + \sum_{j \neq i} a_j\overline{x}_j + \mathfrak{m}_0^2 = \overline{0} + \mathfrak{m}_0^2,$$

for some $\{a_j\} \subset K$, which means

$$x_i + \sum_{j \neq i} a_j x_j \in (x_i x_j)_{1 \leq i,j \leq n} + (f)$$

However, the ideals $(x_i x_j), (f) \subset K[\mathbb{A}^n_k]$ do not contain polynomials with linear terms, implying that the above expression is equal to 0_K. But since the x_i are algebraically independent over K, this means $a_i = 0$ for all $1 \leq i \leq n$. Hence $\{\overline{x}_i + \mathfrak{m}_0^2\}_{i=1}^m$ is a linearly independent set, and hence a K-basis for $\mathfrak{m}_0/\mathfrak{m}_0^2$. □

Lemma 3.2. *Let $m \in \mathbb{Z}^+$, let K be an algebraically closed field, let $W = \mathbb{A}_K^m$. Then for all $w \in W$, $\mathfrak{m}_w/\mathfrak{m}_w^2$ is a K-vector space of dimension m.*

Proof. Write $w = (w_1, \ldots, w_m)$. Define a map

$$\begin{aligned} \varphi_w : W &\longrightarrow W \\ (u_1, \ldots, u_m) &\longmapsto (u_1 - w_1, \ldots, u_n - w_m). \end{aligned}$$

Now φ_w is a morphism since it is defined in terms of the polynomials $\{y_i - w_i\}_{i=1}^m$ in $K[y_1, \ldots, y_m]$. Moreover, φ_w has an obvious inverse morphism, and so φ_w is an isomorphism. Since $\varphi(w) = 0_{\mathbb{A}_K^m}$, the isomorphism φ_w induces an isomorphism of K-vector spaces, $\mathfrak{m}_0/\mathfrak{m}_0^2 \cong \mathfrak{m}_w/\mathfrak{m}_w^2$. Using $f = 0$ and $n = m$ in Lemma 3.1, we know $\mathfrak{m}_0/\mathfrak{m}_0^2 \cong \mathfrak{m}_w/\mathfrak{m}_w^2$, has dimension m. □

Corollary 3.3. *Let $m, n \in \mathbb{Z}^+$ be distinct, let K be an algebraically closed field, let $V \subseteq \mathbb{A}_K^n$ be an algebraic set as in Lemma 3.1, let $W = \mathbb{A}_K^m$. Then $V \ncong W$.*

Proof. By Lemma 3.1 and Lemma 3.2, for any $w \in W$,

$$\dim_K(\mathfrak{m}_0/\mathfrak{m}_0^2) = n \quad \text{and} \quad \dim_K(\mathfrak{m}_w/\mathfrak{m}_w^2) = m.$$

Since $n \neq m$, then $V \ncong W$ by the discussion in the exercise statement. □

The exercise now follows:

(a) Lemma 3.2 implies m_x/m_x^2 has dimension 1 over K.
(b) Since $f(x, y) = y^2 - x^3$ satisfies the criteria of Lemma 3.1, we can apply Corollary 3.3.
(c) Since $f(x, y) = y^2 - x^3 - x^2$, satisfies the criteria of Lemma 3.1, we can apply Corollary 3.3.
(d) Since $f(x, y, z) = xy - z^2$, satisfies the criteria of Lemma 3.1, we can apply Corollary 3.3.

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