ALGEBRA II – FINAL

NICHOLAS CAMACHO

Exercise 1. Prove that the following are equivalent for a ring R (commutative with identity).

(a) $\operatorname{Spec} R$ is disconnected in the Zariski topology.
(b) R can be written as $R = R_1 \times R_2$ for two proper ideals $R_1, R_2 \subseteq R$.

Lemma 1.1. *Let R be a ring and $N \subset R$ an ideal such that every element of N is nilpotent. If $\bar{e} \in R/N$ is idempotent, then there exists $e \in R$ which is also idempotent and $\bar{e} = e + N$.*

Proof of Exercise 1. (a) $\implies$ (b): Suppose $\operatorname{Spec} R$ is disconnected, so $\operatorname{Spec} R = \mathcal{Z}(I) \sqcup \mathcal{Z}(J)$ for two proper ideals $I, J \subsetneq R$. Since $\mathcal{Z}(I) \sqcup \mathcal{Z}(J) = \mathcal{Z}(I \cap J)$,

$$\sqrt{0} = \bigcap_{P \in \operatorname{Spec} R} P = \mathcal{I}(\operatorname{Spec} R) = \mathcal{I}(\mathcal{Z}(I \cap J)) = \sqrt{I \cap J}.$$

Since $I \cap J \subseteq \sqrt{I \cap J}$, then the images of I and J in $R/\sqrt{0}$ intersect trivially. If $I + J$ is a proper ideal, then $I + J$ is contained in a maximal ideal M, and in particular, M contains both I and J, contradicting that $\mathcal{Z}(I)$ and $\mathcal{Z}(J)$ are disjoint. Hence $I + J = R$, and so $\overline{I} + \overline{J} = R/\sqrt{0}$, where $\overline{I}$ and $\overline{J}$ denote the images of I and J in $R/\sqrt{0}$, respectively. Hence

$$\overline{R} := R/\sqrt{0} = \overline{I} \times \overline{J}.$$

and we find an idempotent[1] $\overline{e} \in R/\sqrt{0}$ so that $\overline{R} = \overline{R}\,\overline{e} \times \overline{R}(\overline{1} - \overline{e})$, with $\overline{e} \in I$ and $\overline{1} - \overline{e} \in J$. By the lemma, $\overline{e}$ corresponds to an idempotent $e \in R$, giving $R = Re \times R(1-e)$.

If $e = 0$, then $\overline{1} = \overline{1} - \overline{e} \in \overline{J}$, so $\overline{J} = \overline{R}$, a contradiction. Similarly, if $e = 1$, then $\overline{1} \in \overline{I}$, again a contradiction. Hence Re and $R(1-e)$ are proper ideals of R.

(b) $\implies$ (a): If $R = R_1 \times R_2$, then

$$\mathcal{Z}(R_1) \cap \mathcal{Z}(R_2) = \mathcal{Z}(R_1 \cup R_1) = \mathcal{Z}(R) = \varnothing$$

and

$$\mathcal{Z}(R_1) \cup \mathcal{Z}(R_2) = \mathcal{Z}(R_1 \cap R_1) = \mathcal{Z}(\varnothing) = \operatorname{Spec} R,$$

and so $\operatorname{Spec} R = \mathcal{Z}(R_1) \sqcup \mathcal{Z}(R_2)$. For $i = 1, 2$, since R_i is a proper ideal, it is contained in a maximal ideal, and hence $\mathcal{Z}(R_i) \neq \varnothing$.

$\square$

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[1] In particular, $\overline{1} = \overline{i} + \overline{j}$ for $\overline{i} \in \overline{I}$ and $\overline{j} \in \overline{J}$. Then $\overline{e} := \overline{i}$ is an idempotent.

Exercise 2. Let $\mathcal{C}$ be a category, and let $\phi_X : X \to Z$ and $\phi_Y : Y \to Z$ be morphisms in $\mathcal{C}$. Let $X \times_Z Y$ be the pullback of ϕ_X and ϕ_Y.

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\widetilde{\phi_Y}} & X \\ \downarrow{\scriptstyle \widetilde{\phi_X}} & & \downarrow{\scriptstyle \phi_X} \\ Y & \xrightarrow{\phi_Y} & Z \end{array}$$

(a) Assume Z is a terminal object of $\mathcal{C}$ (for example, a 1 element set when $\mathcal{C}$ is the category of sets) and that $X \times Y$ exists in $\mathcal{C}$. Prove that $X \times_Z Y \cong X \times Y$.
(b) Suppose that $\mathcal{C}$ is the category of sets, and f and g are inclusions of subsets. Determine $X \times_Z Y$.
(c) Let R be a ring and $f : R \to S$, $g : R \to T$ two R-algebras. Let $Z = \operatorname{Spec} R$, $X = \operatorname{Spec} S$, and $Y = \operatorname{Spec} T$. Consider the morphisms and $\phi_X = f^*$ and $\phi_Y = g^*$. Determine if a pullback of ϕ_X and ϕ_Y always exists in the category of affine schemes. If not, give a counterexample. If so, explicitly describe the pulllback.

Proof. (a) Since any two pullbacks of the same diagram are isomorphic, we show that $X \times Y$ satisfies the universal property of the pullback. The product $X \times Y$ comes with morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Since Z is terminal, there exists a unique morphism from $X \times Y$ to Z. This implies $\pi_X \circ \phi_X = \pi_Y \circ \phi_Y$.

Moreover, if there exists an object W in $\mathcal{C}$ with morphisms $\psi_X : W \to X$ and $\psi_Y : W \to Y$ satisfying $\phi_X \circ \psi_X = \phi_Y \circ \psi_Y$, then by the universal property of the product $X \times Y$, there exists a unique morphism $\eta : W \to X \times Y$ so that $\psi_X = \pi_X \circ \eta$ and $\psi_Y = \pi_Y \circ \eta$. Hence $X \times Y$ satisfies the universal property of the pullback of ϕ_X and ϕ_Y.

(b) We claim $X \times_Z Y \cong X \cap Y$. Let both $\widetilde{\phi_X} : X \cap Y \to X$ and $\widetilde{\phi_Y} : X \cap Y \to Y$ be inclusion. Then $f(\widetilde{\phi_Y}(u)) = f(u) = u = g(u) = g(\widetilde{\phi_X}(u))$.

$$\begin{array}{ccc} X \cap Y & \xrightarrow{\widetilde{\phi_Y}} & X \\ \downarrow{\scriptstyle \widetilde{\phi_X}} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

If W is a set, with set maps $h : W \to X$ and $j : W \to Y$ satisfying $f \circ h = g \circ j$, then $h(w) = f(h(w)) = g(j(w)) = j(w)$, so $h = j$.

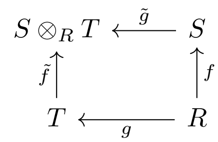
Then $k : W \to X \cap Y$, $w \mapsto j(w) = h(w)$ is such that $h = \widetilde{\phi_X} \circ k$ and $j = \widetilde{\phi_Y} \circ k$. The map k is unique with this property, since any map ℓ satisfying $h = \widetilde{\phi_X} \circ \ell$ and $j = \widetilde{\phi_Y} \circ \ell$ also satisfies $\ell(w) = h(w) = k(w)$.

(c) Since the category of affine schemes is equivalent to the opposite category of rings, we will show that the pushout of the two given R-algebras always exists.

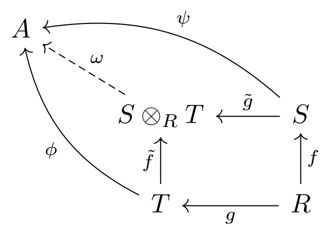
Starting with two R algebras $f : R \to S$ and $g : R \to T$, we claim that $S \otimes_R T$ together with the maps

$$\begin{aligned} \tilde{g} : S &\to S \otimes_R T \\ s &\mapsto s \otimes 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{f} : T &\to S \otimes_R T \\ t &\mapsto 1 \otimes t \end{aligned}$$

is the pushout of f and g. First, recall that the (standard) R-action on S and T is respectively given by $r.s := f(r)s$ and $r.t := g(r)s$. In particular, this means that the simple tensors $f(r) \otimes 1$ and $1 \otimes g(r)$ are equal, implying $\tilde{g}(f(r)) = \tilde{f}(g(r))$.



Now suppose A is an S-algebra and a T-algebra via ring homomorphisms $\psi : S \to A$ and $\phi : T \to A$, and suppose these morphisms satisfy $\psi \circ f = \phi \circ g$. Note that A is then an R-algebra by the action $r.a := \phi(g(r))a = \psi(f(r))a$.



Define a map $\omega : S \times T \to A$ by $(s,t) \mapsto \psi(s)\phi(t)$. The reader will have no difficulty checking that ω is bilinear... Just kidding. This is a final exam, so let's at least check linearity in the first component: Let $r_1, r_2 \in R, s_1, s_2 \in S$, and $t \in T$. Then

$$\begin{aligned}
\omega(r_2.s_1 + r_2.s_2, t) &= (\psi(r_1.s_1) + \psi(r_2.s_2))\phi(t) \\
&= (\psi(f(r_1)s_1) + \psi(f(r_2)s_2))\phi(t) \\
&= (\psi(f(r_1))\psi(s_1) + \psi(f(r_2))\psi(s_2))\phi(t) \\
&= (r_1.\psi(s_1) + r_2.\psi(s_2))\phi(t) \\
&= r_1.\psi(s_1)\phi(t) + r_2.\psi(s_2)\phi(t) \\
&= r_1.\omega(s_1, t) + r_2.\omega(s_2, t).
\end{aligned}$$

Linearity in the second component is similar. Hence there exists a unique, well-defined morphism (which we again denote by ω)

$$\begin{aligned}
\omega : S \otimes_R T &\to A \\
s \otimes t &\to \psi(s)\phi(t).
\end{aligned}$$

Then $\omega(\tilde{g}(s)) = \omega(s \otimes 1) = \psi(s)$ and $\omega(\tilde{f}(t)) = \omega(1 \otimes t) = \phi(t)$. If $\nu : S \otimes_R T \to A$ is a morphism satisfying $\nu\tilde{g} = \psi$ and $\nu\tilde{f} = \phi$, then

$$\nu(s \otimes t) = \nu(s \otimes 1)\nu(1 \otimes t) = \psi(s)\phi(t) = \omega(s \otimes t).$$

Since ω agrees with ν on simple tensors, then $\omega = \nu$. Hence $(S \otimes_R T, \tilde{f}, \tilde{g})$ is the pushout of f and g.

□

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: nicholas-camacho@uiowa.edu