Homework for
Smooth Manifolds

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Beware: Some solutions may be incorrect!

Exercise 1. Prove Cauchy's Mean Value Theorem: If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof. Define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then h is continuous on $[a, b]$ and differentiable on (a, b). Then

$$\begin{aligned} h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\ &= f(a)g(b) - g(a)f(b) \\ &= f(a)g(b) - g(a)f(b) + f(b)g(b) - g(b)f(b) \\ &= f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)] \\ &= h(b). \end{aligned}$$

By Rolle's Theorem, there exists $c \in (a, b)$ so that $h'(c) = 0$. That is,

$$0 = h'(c) = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)],$$

and so $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$. ☕

Exercise 2. Prove L'Hôpital's Rule: Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and $g' \neq 0$ on (a, b). If there exists a $c \in (a, b)$ for which $f(c) = g(c) = 0$ and $f', g' : (a, b) \to \mathbb{R}$ are continuous, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Proof. For x not equal to c but close to c, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}.$$

Applying the limit to both sides as x approaches c, and we get $f'(c)/g'(c)$. Since f' and g' are both continuous at c, then $f'(c) = \lim_{x \to c} f'(x)$ and $g'(c) = \lim_{x \to c} g'(x)$ and the result follows. ☕

Exercises from Spivak

Exercise 2-1. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it is continuous at a.

Proof. We begin by proving the following (Exercise 1-10): If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \leq M|h|$ for $h \in \mathbb{R}^m$.

First, let $[t_{ij}]$ be the matrix associated with T with i,j-entry t_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq m$. For $h \in \mathbb{R}^m$,

$$T(h) = \begin{pmatrix} t_{11} & \dots & t_{1m} \\ \vdots & \ddots & \vdots \\ t_{n1} & \dots & t_{nm} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m t_{1j}h_j \\ \vdots \\ \sum_{j=1}^m t_{nj}h_j \end{pmatrix}.$$

Let $\tilde{t_i}$ denote the i-th row in $[t_{ij}]$. Then by the Cauchy-Schwartz inequality, we have for fixed i

$$\langle \tilde{t_i}, h\rangle = \sum_{j=1}^m t_{ij}h_j \leq |\tilde{t_i}||h| = \left(\sum_{j=1}^m t_{ij}^2\right)^{1/2} \left(\sum_{j=1}^m h_j^2\right)^{1/2},$$

and so

$$\left(\sum_{j=1}^m t_{ij}h_j\right)^2 \leq \left(\sum_{j=1}^m t_{ij}^2 \sum_{j=1}^m h_j^2\right). \tag{$*$}$$

Now,

$$\begin{aligned}
|T(h)| \leq \left(\sum_{j=1}^m t_{1j}h_j\right)^2 + \cdots + \left(\sum_{j=1}^m t_{nj}h_j\right)^2 &= \sum_{i=1}^n \left(\sum_{j=1}^m t_{ij}h_j\right)^2 \\
&\leq \sum_{i=1}^n \left(\sum_{j=1}^m t_{ij}^2 \sum_{j=1}^m h_j^2\right) && \text{(by $(*)$)} \\
&= \sum_{i=1}^n \sum_{j=1}^m t_{ij}^2 |h|^2 \\
&\leq \left(\sum_{i=1}^n \sum_{j=1}^m t_{ij}^2 |h|^2\right)^{1/2} = \left(\sum_{i=1}^n \sum_{j=1}^m t_{ij}^2\right) |h|
\end{aligned}$$

Whew! Now, onto the proof.

Let M be the bound described above for the linear map $Df(a)$. For nonzero $h \in \mathbb{R}^m$ but close to 0,

$$\begin{aligned}
|f(a+h) - f(a)| &= |f(a+h) - f(a) - Df(a)(h) + Df(a)(h)| \\
&\leq |f(a+h) - f(a) - Df(a)(h)| + M|h| \\
&= |h| \left(\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|}\right) + M|h|.
\end{aligned}$$

Certainly $M|h| \to 0$ as $h \to 0$, and by hypothesis, $|h|^{-1}[|f(a+h) - f(a) - Df(a)(h)|] \to 0$ as $h \to 0$. Thus, $|f(a+h) - f(a)| \to 0$ as $h \to 0$ and so f is continuous at a. ☕

Exercise 2-5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f is a function of the kind considered in Problem 2-4 so that f is not differentiable at $(0,0)$.

Proof. Define a function g on the unit circle by $(a,b) \mapsto a|b|$. Then $g(0,1) = g(1,0) = 0$ and $g(-x,-y) = -x|-y| = -x|y| = -g(x,y)$. Moreover, for $(x,y) \neq (0,0)$

$$|(x,y)| \cdot g\left(\frac{(x,y)}{|(x,y)|}\right) = \sqrt{x^2+y^2} \cdot \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{|y|}{\sqrt{x^2+y^2}} = \frac{x|y|}{\sqrt{x^2+y^2}} = f(x,y),$$

ans so g satisfies the properties described in Exercise 2-4, which means f is not differentiable at $(0,0)$. ☕

Exercise 2-6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$. Show that f is not differentiable at $(0,0)$.

Proof. If f were differentiable at $(0,0)$, then its derivative would be 0. To see this, we use the hint given in Exercise 2-4 to compute the following:

$$\begin{aligned} 0 = \lim_{(h,0)\to(0,0)} \frac{|f(h,0) - f(0,0) - Df(0,0)(h,0)|}{|(h,0)|} &= \lim_{(h,0)\to(0,0)} \frac{|Df(0,0)(h,0)|}{\sqrt{h^2}} \\ &= \lim_{(h,0)\to(0,0)} \frac{|h||Df(0,0)(1,0)|}{|h|} \\ &= |Df(0,0)(1,0)|. \end{aligned}$$

Similarly,

$$0 = \lim_{(0,k)\to(0,0)} \frac{|f(0,k) - f(0,0) - Df(0,0)(0,k)|}{|(0,k)|} = |Df(0,0)(0,1)|.$$

So $Df(0,0)(1,0) = Df(0,0)(0,1) = 0$, and so for any $(a,b) \in \mathbb{R}^m$

$$Df(0,0)(a,b) = ab[Df(0,0)(1,1)] = ab[Df(0,0)(1,0) + Df(0,0)(0,1)] = 0.$$

However,

$$\lim_{(h,h)\to(0,0)} \frac{|f(h,h) - f(0,0) - 0|}{|(h,h)|} = \lim_{(h,h)\to(0,0)} \frac{\sqrt{|hh|}}{\sqrt{h^2+h^2}} = \lim_{(h,h)\to(0,0)} \frac{\sqrt{h^2}}{\sqrt{2}\sqrt{h^2}} = \frac{1}{\sqrt{2}}.$$

☕

Exercise 2-10. Find f'.

We give the matrix representation for Df in terms of the standard basis for $\mathbb{R}^n$.

(a) $f(x,y,z) = x^y$

$$Df(x,y,z) = \begin{pmatrix} yx^{y-1} & x^y \ln y & 0 \end{pmatrix}.$$

(b) $f(x,y,z) = (x^y, z)$

$$Df(x,y,z) = \begin{pmatrix} yx^{y-1} & x^y \ln y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) $f(x,y) = \sin(x \sin y)$

$$Df(x,y) = \begin{pmatrix} \cos(x\sin y)\sin y & \cos(x \sin y) x \cos^2 y \end{pmatrix}.$$

(d) $f(x,y,z) = \sin(x \sin(y \sin z))$

$$\begin{aligned} Df(x,y,z) = (\, &\cos(x\sin(y\sin z)) \cdot \sin(y \sin z), \\ &\cos(x \sin(y \sin z)) \cdot x\cos(y\sin z) \cdot \sin z, \\ &\cos(x\sin(y\sin z)) \cdot x \cos(y \sin z) \cdot y \cos z) \end{aligned}$$

Exercise 2-12.

(a) Prove that if f is bilinear, then

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} = 0.$$

Proof. We first prove the following Lemma:

Lemma. *Let f and E_ℓ be as described in Exercise 2-14 below; let $h_\ell \in E_\ell$ and fix $a_m \in E_m$ for all $m \neq \ell$. Then there exists a $\gamma \in E_\ell$ so that*

$$|f(a_1, \ldots, h_\ell, \ldots, a_k)| \leq |h_\ell||f(a_1, \ldots, \gamma, \ldots, \ldots, a_k)|$$

Proof of Lemma. Define

$$g_\ell(x) = \frac{|f(a_1, \ldots, x, \ldots, a_k)|}{|x|} = \left| f\left(a_1, \ldots, \frac{x}{|x|}, \ldots, a_k\right)\right|.$$

If S is the sphere in E_ℓ, then we let $\tilde{g}_\ell := g_\ell|_S$. Then $\tilde{g}_\ell$ is a continuous function on S and since S is compact, there exists $\gamma \in S$ so that $\tilde{g}_\ell(y) \leq \tilde{g}_\ell(\gamma)$ for all $y \in S$ by the Mean Value Theorem. So, $g_\ell(h_\ell) = \tilde{g}_\ell(h_\ell/|h_\ell|) \leq \tilde{g}_\ell(\gamma)$ and we get

$$\frac{|f(a_1, \ldots, h_\ell, \ldots, a_k)|}{|h_\ell|} \leq \frac{|f(a_1, \ldots, \gamma, \ldots, a_k)|}{|\gamma|} = |f(a_1, \ldots, \gamma, \ldots, a_k)|.$$

☕

Onto the proof of the exercise. Note that $|h| = \sqrt{|h|^2} \leq \sqrt{|h|^2 + |k|^2} = |(h,k)|$. Now it's just a simple application of the Lemma:

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} \leq \lim_{(h,k)\to(0,0)} \frac{|h||f(\gamma,k)|}{|h|} = \lim_{(h,k)\to(0,0)} |f(\gamma,k)| = |f(\gamma,0)| = 0.$$

(b) Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.

Proof.

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{|f(a+h,b+k) - f(a,b) - Df(a,b)(h,k)|}{|(h,k)|} \\
&\quad = \lim_{(h,k)\to(0,0)} \frac{|f(a,b) + f(a,k) + f(h,b) + f(h,k) - f(a,b) - f(a,y) + f(x,b)|}{|(h,k)|} \\
&\quad = \lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} = 0.
\end{aligned}$$

(c) Show that the formula for $Dp(a,b)$ in Theorem 2-3 is a special case of (b).

Proof. Notice that

$$\begin{aligned}
p(x+h,y) &= xy + hy = p(x,y) + p(h,y), \\
p(x,h+k) &= xh + xk = p(x,h) + p(x,y), \text{ and} \\
p(ax,y) &= axy = ap(x,y) = xay = p(x,ay)
\end{aligned}$$

So, p is bilinear. By (b), $Dp(a,b)(x,y) = p(x,b) + p(a,y) = bx + ay$, which is precisely what is shown in Theorem 2-3.

Exercise 2-14. Let E_i for $i = \overline{1,k}$ be euclidean spaces of various dimensions.

(a) If f is multilinear and $i \neq j$, show that for $h = (h_1, \ldots, h_k)$, with $h_\ell \in E_\ell$, we have

$$\lim_{h\to 0} \frac{|f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)|}{|h|} = 0$$

Proof. Notice that

$$|(h_i,h_j)| = \sqrt{|h_i|^2 + |h_j|^2} \leq \sqrt{|h_1|^2 + \cdots + |h_k|^2} = |(h_1, \ldots, h_k)| = |h|.$$

Let $g(h_i,h_j) = f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)$. Then g is bilinear and by Exercise 2-12(a)

$$\lim_{h\to 0} \frac{|f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)|}{|h|} \leq \lim_{(h_i,h_j)\to(0,0)} \frac{|g(h_i,h_j)|}{|(h_i,h_j)|} = 0.$$

(b) Prove that

$$Df(a_1,\dots,a_k)(x_1,\dots,x_k) = \sum_{i=1}^{k} f(a_1,\dots,a_{i-1},x_i,a_{i+1},\dots,a_k).$$

Proof. For notational convenience, let I_r denote $i_1 < \cdots < i_r$ for indices $i_1,\dots,i_r$. We have

$$\begin{aligned}
&f(a+h) = f(a_1+h_1,\dots,a_k+h_k)\\
&= f(a_1,\dots,a_k) + \sum_{1\le I_1\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_k)\\
&\qquad + \sum_{1\le I_2\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_2-1},h_{i_2},a_{i_2+1},\dots,a_k)\\
&+ \sum_{1\le I_3\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_2-1},h_{i_2},a_{i_2+1},\dots,a_{i_3-1},h_{i_3},a_{i_3+1},\dots,a_k)\\
&+ \dots\\
&+ \sum_{1\le I_{k-1}\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_{k-1}-1},h_{i_{k-1}},a_{i_{k-1}+1},\dots,a_k)\\
&+ f(h_1,\dots,h_k).
\end{aligned}$$

This gives

$$\begin{aligned}
&f(a+h) - f(a) - Df(a)(h)\\
&= \sum_{1\le I_2\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_2-1},h_{i_2},a_{i_2+1},\dots,a_k)\\
&+ \cdots +\\
&\sum_{1\le I_{k-1}\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_{k-1}-1},h_{i_{k-1}},a_{i_{k-1}+1},\dots,a_k)\\
&+ f(h_1,\dots,h_k).
\end{aligned}$$

Now for any I_r, by the Lemma in Exercise 2-12, there exists γ_r such that

$$\begin{aligned}
&|f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|\\
&\qquad \le |h_{i_1}||f(a_1,\dots,a_{i_1-1},\gamma_r,a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|.
\end{aligned}$$

Moreover, notice that $|h_{i_1}| = \sqrt{|h_{i_1}|^2} \le \sqrt{|h_{i_1}|^+ \cdots + |h_{i_r}|^2} = |(h_{i_1},\dots,h_{i_r})| = |h|$, and so

$$\begin{aligned}
&\frac{|f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|}{|h|}\\
&\qquad \le \frac{|h_{i_1}||f(a_1,\dots,a_{i_1-1},\gamma_r,a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|}{|h_{i_1}|}\\
&\qquad = |f(a_1,\dots,a_{i_1-1},\gamma_r,a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|.
\end{aligned}$$

If we let $h \to 0$, then the right-hand-side goes to

$$|f(a_1,\dots,a_{i_1-1},\gamma_r,a_{i_1+1},\dots,a_{i_r-1},0,a_{i_r+1},\dots,a_k)| = 0,$$

since the value of a multilinear function is zero whenever any input vector is 0. So,

$$\lim_{h\to 0}\frac{\left|\sum_{1\le I_r\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)\right|}{|h|}$$

$$\begin{aligned}
&\le \sum_{1\le I_r\le k}\lim_{h\to 0}\frac{|f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|}{|h|}\\
&\le \sum_{1\le I_r\le k}\lim_{h\to 0}|f(a_1,\dots,a_{i_1-1},\gamma_r,a_{i_1+1},\dots,a_{i_r-1},h_{i_r},a_{i_r+1},\dots,a_k)|\\
&= 0
\end{aligned}$$

Let γ be such that $|f(h_1,\dots,h_k)|\le |h_1||f(\gamma,h_2,\dots,h_k)|$. Finally, we compute the limit:

$$\begin{aligned}
&\lim_{h\to 0}\frac{|f(a+h)-f(a)-Df(a)(h)|}{|h|}\\
&=\lim_{h\to 0}\left[|h|^{-1}\right]\Bigg|\sum_{1\le I_2\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_2-1},h_{i_2},a_{i_2+1},\dots,a_k)\\
&\qquad +\dots\\
&\qquad +\sum_{1\le I_{k-1}\le k} f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_{k-1}-1},h_{i_{k-1}},a_{i_{k-1}+1},\dots,a_k)\\
&\qquad + f(h_1,\dots,h_k)\Bigg|\\
&\le \sum_{1\le I_2\le k}\lim_{h\to 0}\frac{|f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_2-1},h_{i_2},a_{i_2+1},\dots,a_k)|}{|h|}\\
&\qquad +\dots\\
&\qquad +\sum_{1\le I_{k-1}\le k}\lim_{h\to 0}\frac{|f(a_1,\dots,a_{i_1-1},h_{i_1},a_{i_1+1},\dots,a_{i_{k-1}-1},h_{i_{k-1}},a_{i_{k-1}+1},\dots,a_k)|}{|h|}\\
&\qquad +\lim_{h\to 0}\left[(|h_1||f(\gamma,\dots,h_k)|)\Big/|h_1|\right]\\
&\le \sum_{1\le I_2\le k}\lim_{h\to 0}|f(a_1,\dots,a_{i_1-1},\gamma_2,a_{i_1+1},\dots,a_{i_2-1},h_{i_2},a_{i_2+1},\dots,a_k)|\\
&\qquad +\dots\\
&\qquad +\sum_{1\le I_{k-1}\le k}\lim_{h\to 0}|f(a_1,\dots,a_{i_1-1},\gamma_{k-1},a_{i_1+1},\dots,a_{i_{k-1}-1},h_{i_k-1},a_{i_{k-1}+1},\dots,a_k)|\\
&= 0.
\end{aligned}$$

☕

Exercise 1. Recall that if $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions on $[a,b]$ that converges uniformly to $f : [a,b] \to \mathbb{R}$ then f is continuous and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Suppose that $f : U \to \mathbb{R}$ is in $C^1(U)$, where $U \subseteq \mathbb{R}^2$ is open, and $[a,b] \times [c,d] \subset U$. Let

$$F(y) = \int_a^b f(x,y)dx.$$

Prove that $\frac{d}{dy}F(y) = \int_a^b \frac{\partial}{\partial y} f(x,y)dx$.

Proof. Fix $x \in [a,b]$. Let $\{a_n\}$ be a sequence converging to 0 with $a_n \neq 0$ for all n and define $\varphi_n : [c,d] \to \mathbb{R}$ by

$$\varphi_n(y) = \frac{f(x,y+a_n) - f(x,y)}{a_n}.$$

For the sake of rigor, assume $|a_n| \leq 1$ for all n, and extend f to $[d, d+1]$ by $f(x,y) := f(x,d)$ for all $y \in [d, d+1]$.

Let $\epsilon > 0$. We wish to show that $\{\varphi_n\}$ converges uniformly to $\frac{\partial}{\partial y} f(x,y)$ on $[c,d]$. That is, we wish to find suitable N such that

$$\left|\varphi_n(y) - \frac{\partial}{\partial y} f(x,y)\right| < \epsilon.$$

for all $n \geq N$ and for all $y \in [c,d]$.

Since $f \in C^1(U)$, $\frac{\partial}{\partial y} f(x,y)$ is continuous on $[c,d]$, and therefore uniformly continuous on $[c,d]$ since $[c,d]$ is compact. So there exists $\delta > 0$ such that for all $w, z \in [c,d]$, if $|w - z| < \delta$ then

$$\left|\frac{\partial}{\partial y} f(x,w) - \frac{\partial}{\partial y} f(x,z)\right| < \epsilon. \tag{$*$}$$

Let $y \in [c,d]$ and choose N so that for all $n \geq N$, $|(y + a_n) - y| = |a_n| < \delta$. For all $n \geq N$, there exists c_n between y and $y + a_n$ so that

$$\varphi_n(y) = \frac{f(x,y+a_n) - f(x,y)}{a_n} = \frac{\partial}{\partial y} f(x,c_n)$$

by the Mean Value Theorem. Since c_n is between y and $y + a_n$, then $|c_n - y| < \delta$ for all $n \geq N$. Therefore, we can apply $(*)$ to the points c_n and y for all $n \geq N$ to obtain

$$\begin{aligned}\left|\varphi_n(y) - \frac{\partial}{\partial y} f(x,y)\right| &= \left|\frac{f(x,y+a_n) - f(x,y)}{a_n} - \frac{\partial}{\partial y} f(x,y)\right| \\ &= \left|\frac{\partial}{\partial y} f(x,c_n) - \frac{\partial}{\partial y} f(x,y)\right| < \epsilon.\end{aligned}$$

Therefore, $\{\varphi_n\} \to \frac{\partial}{\partial y} f$ uniformly on $[c,d]$. Then

$$\lim_{n\to\infty} \int_a^b \varphi_n dx = \int_a^b \frac{\partial}{\partial y} f(x,y)dx$$

Moreover, we have

$$\begin{aligned}
\frac{d}{dy}F(y) &= \lim_{n\to\infty} \frac{\int_a^b f(x, y+a_n) - \int_a^b f(x,y)}{a_n} \\
&= \lim_{n\to\infty} \int_a^b \frac{f(x,y+a_n) - f(x,y)}{a_n} \\
&= \lim_{n\to\infty} \int_a^b \varphi_n \\
&= \int_a^b \frac{\partial}{\partial y} f(x,y)dx.
\end{aligned}$$

Whew! ☕

Exercise 2. Let C be a commutative algebra with unity over $\mathbb{R}$. Since homomorphisms of commutative algebras are linear and send 1 to 1, it is easy to see that any homomorphism $\phi : C \to \mathbb{R}$ is onto, and the kernel of ϕ,

$$\mathfrak{m} = \{c \in C \mid \phi(c) = 0\}$$

is a maximal ideal. In particular, $\mathfrak{m}$ is a linear subspace of C as a vector space, and if $c \in C$ and $m \in \mathfrak{m}$, then $cm \in \mathfrak{m}$. Also if $\mathfrak{m} \subset I$ and I satisfies these two conditions given for $\mathfrak{m}$, then $I = C$. A homomorphism like ϕ is called a *place* of the commutative algebra. If C was an algebra of functions on a set X, then ϕ would be evaluation at a point in X. The places of a commutative algebra play the role of points of the algebra.

Remark: In general, for algebras that are not necessarily commutative, the points of the algebra correspond to onto homomorphisms where the image is $n \times n$ matrices with coefficients in $\mathbb{R}$. Such a homomorphism is called an *irreducible representation.*

Define $D : C \to \mathbb{R}$ to be a *derivation centered at the place* ϕ if D is $\mathbb{R}$-linear and for any $f, g \in C$,

$$D(fg) = \phi(f)D(g) + \phi(g)D(f).$$

We denote the set of derivations of C centered at the place ϕ by $T_\phi C$.

By $(\mathfrak{m}/\mathfrak{m}^2)^*$, we mean linear maps $L : \mathfrak{m} \to \mathbb{R}$ so that if $m_1, m_2 \in \mathfrak{m}$, then $L(m_1 m_2) = 0$. The goal of this exercise is to prove that

$$T_\phi C = (\mathfrak{m}/\mathfrak{m}^2)^*.$$

(a) Prove that if $D \in T_\phi C$ then the restriction of D to $\mathfrak{m}$ defines an element of $(\mathfrak{m}/\mathfrak{m}^2)^*$. Hence there is a map defined by restriction,

$$res : T_\phi C \to (\mathfrak{m}/\mathfrak{m}^2)^*.$$

Proof. The proof is straight forward. If $D \in T_\phi C$, and $f, g \in \mathfrak{m}$, then $\phi(f) = \phi(g) = 0$ and so

$$D(fg) = \phi(f)D(g) + \phi(g)D(f) = 0.$$

Thus $D|_\mathfrak{m} : \mathfrak{m} \to \mathbb{R}$ is a linear map so that $f, g \in \mathfrak{m}$ implies $D(fg) = 0$, and hence $D|_\mathfrak{m} \in (\mathfrak{m}/\mathfrak{m})^*$. ☕

(b) Prove that if $L \in (\mathfrak{m}/\mathfrak{m}^2)^*$, then the extension of L to $L : C \to \mathbb{R}$ given by $L(f) = L(f - \phi(f))$ is a derivation centered at ϕ. (This depends on the fact that if $f \in C$ then $f - \phi(f) \in \mathfrak{m}$).

Proof. Given $L \in (\mathfrak{m}/\mathfrak{m}^2)^*$, let $E : C \to \mathbb{R}$ be the extension of L to C given by $E(f) = L(f - \phi(f))$. We show that E is $\mathbb{R}$-linear, vanishes on $\mathfrak{m}^2$, and sends constants to 0. Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in C$. Then

$$\begin{aligned}
E(\alpha f + \beta g) &= L(\alpha f + \beta g - \phi(\alpha f + \beta g)) \\
&= L(\alpha f + \beta g - \alpha\phi(f) - \beta\phi(g)) \\
&= L(\alpha f - \alpha\phi(f)) + L(\beta g - \beta\phi(g)) \\
&= L(\alpha(f - \phi(f))) + L(\beta(g - \phi(g))) \\
&= \alpha L(f - \phi(f)) + \beta L(g - \phi(g)) \\
&= \alpha E(f) + \beta E(g).
\end{aligned}$$

For $h, k \in \mathfrak{m}$, $\phi(hk) = \phi(h)\phi(k) = 0$. So,

$$E(hk) = L(hk - \phi(hk)) = L(hk) - L(\phi(hk)) = 0 - L(0) = 0.$$

And finally

$$E(\alpha) = L(\alpha - \alpha\phi(1)) = L(0) = 0.$$

Therefore for $f, g \in C$, we get

$$\begin{aligned}
E(fg) &= E(fg) - E((f - \phi(f))(g - \phi(g))) \\
&= E(fg) - E(fg - f\phi(g) - g\phi(f) + \phi(g)\phi(f)) \\
&= E(fg - (fg - f\phi(g) - g\phi(f) + \phi(g)\phi(f))) \\
&= E(f\phi(g) + g\phi(f) - \phi(g)\phi(f))) \\
&= E(f\phi(g)) + E(g\phi(f)) - E(\phi(g)\phi(f))) \\
&= \phi(g)E(f) + \phi(f)E(g).
\end{aligned}$$

☕

(c) Put it all together to prove that the two linear spaces $T_\phi C$ and $(\mathfrak{m}/\mathfrak{m}^2)^*$ are isomorphic.

Proof. Given $L \in (\mathfrak{m}/\mathfrak{m}^2)^*$, use part (b) to extend L to a derivation E in $T_\phi C$. Then $res(E) = L$ and so res is surjective.

Suppose $res(D) = D|_{\mathfrak{m}} = 0$. Then if $f \in C$, $f - \phi(f) \in \mathfrak{m}$. Also, $D(\phi(f)) = \phi(f)D(1) = 0$ since derivations vanish on constants. Then

$$D(f) = D(f) - D(f - \phi(f)) = D(\phi(f)) = 0,$$

and so $D \equiv 0$ and res is injective. Moreover, res is a homomorphism since for $\alpha, \beta \in \mathbb{R}$ and $D_1, D_2 \in T_\phi C$

$$res(\alpha D_1 + \beta D_2) = (\alpha D_1 + \beta D_2)|_{\mathfrak{m}} = \alpha D_1|_{\mathfrak{m}} + \beta D_2|_{\mathfrak{m}} = \alpha res(D_1) + \beta res(D_2).$$

☕

Exercises from Spivak

Exercise 2-17. Find the partial derivatives of the following functions.

(a) $f(x,y,z) = x^y$.
$$\frac{\partial f}{\partial x} = yx^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \ln x, \quad \frac{\partial f}{\partial z} = 0.$$

(b) $f(x,y,z) = z$.
$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 1.$$

(c) $f(x,y) = \sin(x \sin y)$.
$$\frac{\partial f}{\partial x} = \cos(x \sin y)\sin y, \quad \frac{\partial f}{\partial y} = \cos(x \sin y) x \cos^2 y.$$

(d) $f(x,y,z) = \sin(x \sin(y \sin z))$.
$$\begin{aligned} \frac{\partial f}{\partial x} &= \cos(x \sin(y \sin z)) \cdot \sin(y \sin z), \\ \frac{\partial f}{\partial y} &= \cos(x \sin(y \sin z)) \cdot x \cos(y \sin z) \cdot \sin z, \\ \frac{\partial f}{\partial z} &= \cos(x \sin(y \sin z)) \cdot x \cos(y \sin z) \cdot y \cos z. \end{aligned}$$

Exercise 2-20. Find the partial derivatives of f in terms of the derivatives of g and h.[1]

(a) $f(x,y) = g(x)h(y)$
$$D_1 f(x,y) = h(y) Dg(x), \quad D_2 f(x,y) = g(x) Dh(y).$$

(b) $f(x,y) = g(x)^{h(y)}$
$$D_1 f(x,y) = h(y) g(x)^{h(y)-1} Dg(x), \quad D_2 f(x,y) = g(x)^{h(y)} \ln g(x) Dh(y)$$

(c) $f(x,y) = g(x)$
$$D_1 f(x,y) = Dg(x), \quad D_2 f(x,y) = 0.$$

(d) $f(x,y) = g(y)$
$$D_2 f(x,y) = 0, \quad D_1 f(x,y) = Dg(y).$$

(e) $f(x,y) = g(x+y)$
$$D_1 f(x,y) = Dg(x+y), \quad D_2 f(x,y) = Dg(x+y).$$

Exercise 2-22. If $f : \mathbb{R}^2 \to \mathbb{R}$ and $D_2 f = 0$, show that f is independent of the second variable. If $D_1 f = D_2 f = 0$, show that f is constant.

[1] I am aware that I am switching my notation from that in Exercise 2-17. I just want to make sure I am comfortable with both!

Proof. Fix $x_0 \in \mathbb{R}$ and let $g(y) = f(x_0, y)$. For $y_1, y_2 \in \mathbb{R}$, since g is differentiable, we have by the mean value theorem that there exists c between y_1 and y_2 such that

$$g(y_1) - g(y_2) = Dg(c) = 0,$$

and so $g(y_1) = g(y_2)$; that is, $f(x_0, y_1) = f(x_0, y_2)$ and so f is independent of the second variable and therefore constant in y. Similarly, we get $D_1 f = 0$ and so f is independent of the first variable and therefore constant in x, making f constant everywhere. ☕

Exercise 2-24. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that $D_2 f(x, 0) = x$ for all x and $D_1 f(0, y) = -y$ for all y.

Proof. We have

$$D_1 f(x,y) = \begin{cases} \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

and

$$D_2 f(x,y) = \begin{cases} \frac{x^5-4x^2y^2-xy^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

from which it follows that

$$D_2 f(x,0) = \frac{x^5}{(x^2)^2} = x \quad \text{and} \quad D_1 f(0,y) = \frac{-y^5}{(y^2)^2} = -y.$$

☕

(b) Show that $D_{1,2} f(0,0) \neq D_{2,1} f(0,0)$.

Proof. We have

$$D_{1,2} f(0,y) = D_2(D_1 f(0,y)) = D_2(-y) = -1$$

but

$$D_{2,1} f(x,0) = D_1(D_2(x,0)) = D_1(x) = 1.$$

☕

Exercise 2-28. Find expressions for the partial derivatives of the following functions:

In each of the following, let a be the argument of f. For example, $a := g(x)k(y), g(x)+h(y)$ in part (a). We use the formula of Theorem 2-9 to calculate $D_i F$:

(a) $F(x,y) = f(g(x)k(y), g(x) + h(y))$.

$$\begin{aligned} D_1 F(x,y) &= D_1 f(a) \cdot D_1(g(x)k(y)) + D_2 f(a) \cdot D_1(g(x) + h(y)) \\ &= D_1 f(a) \cdot g'(x)k(y)) + D_2 f(a) \cdot g'(x), \\ D_2 F(x,y) &= D_1 f(a) \cdot D_2(g(x)k(y)) + D_2 f(a) \cdot D_2(g(x) + h(y)) \\ &= D_1 f(a) \cdot g(x)k'(y) + D_2 f(a) \cdot h'(y). \end{aligned}$$

(b) $F(x, y, z) = f(g(x + y), h(y + z))$.

$$\begin{aligned} D_1F(x, y) &= D_1f(a) \cdot D_1(g(x + y)) + D_2f(a) \cdot D_1(h(y + z)) \\ &= D_1f(a) \cdot g'(x + y), \\ D_2F(x, y) &= D_1f(a) \cdot D_2(g(x + y)) + D_2f(a) \cdot D_2(h(y + z)) \\ &= D_1f(a) \cdot g'(x + y) + D_2f(a) \cdot h'(y + z), \\ D_3F(x, y) &= D_1f(a) \cdot D_3(g(x + y)) + D_2f(a) \cdot D_3(h(y + z)) \\ &= D_2f(a) \cdot h'(y + z). \end{aligned}$$

(c) $F(x, y, z) = f(x^y, y^z, z^x)$.

$$\begin{aligned} D_1F(x, y) &= D_1f(a) \cdot D_1(x^y) + D_2f(a) \cdot D_1(y^z) + D_3f(a) \cdot D_1(z^x) \\ &= D_1f(a) \cdot yx^{y-1} + D_3f(a) \cdot z^x \ln(z), \\ D_2F(x, y) &= D_1f(a) \cdot D_2(x^y) + D_2f(a) \cdot D_2(y^z) + D_3f(a) \cdot D_2(z^x) \\ &= D_1f(a) \cdot x^y \ln(x) + D_2f(a) \cdot zy^{z-1}, \\ D_3F(x, y) &= D_1f(a) \cdot D_3(x^y) + D_2f(a) \cdot D_3(y^z) + D_3f(a) \cdot D_3(z^x) \\ &= D_2f(a) \cdot y^z \ln(y) + D_3f(a) \cdot xz^{x-1}. \end{aligned}$$

(d) $F(x, y) = f(x, g(x), h(x, y))$.

$$\begin{aligned} D_1F(x, y) &= D_1f(a) \cdot D_1(x) + D_2f(a) \cdot D_1(g(x)) + D_3f(a) \cdot D_1(h(x, y)) \\ &= D_1f(a) + D_2f(a) \cdot g'(x) + D_3f(a) \cdot D_1(h(x, y)), \\ D_2F(x, y) &= D_1f(a) \cdot D_2(x) + D_2f(a) \cdot D_2(g(x))D_3f(a) \cdot D_2(h(x, y)) \\ &= D_3f(a) \cdot D_2(h(x, y)). \end{aligned}$$

Exercise 2-29. Let $f : \mathbb{R}^n \to \mathbb{R}$. For $x \in \mathbb{R}^n$, the limit

$$\lim_{t\to 0} \frac{f(a + tx) - f(a)}{t},$$

if it exists, is denoted $D_x f(a)$, and called the *directional derivative* of f at a, in the direction of x.

(a) Show that $D_{e_i} f(a) = D_i f(a)$.

Proof.

$$\lim_{t\to 0} \frac{f(a + te_i) - f(a)}{t} = \lim_{t\to 0} \frac{f(a_1, \dots, a_i + t, \dots, a_n) - f(a)}{t} = D_i f(a).$$

☕

(b) Show that $D_{tx} f(a) = tD_x f(a)$.

Proof.

$$D_{tx}f(a) = \lim_{h\to 0} \frac{f(a + h(tx)) - f(a)}{h} = \lim_{th\to 0} t\frac{f(a + (th)x) - f(a)}{th} = tD_x f(a).$$

☕

(c) If f is differentiable at a, show that $D_x f(a) = Df(a)(x)$ and therefore $D_{x+y}f(a) = D_x f(a) + D_y f(a)$.

Proof. If $x = 0$, the proof is trivial. For nonzero $x \in \mathbb{R}^n$, $tx \to 0$ as $t \to 0$. Since f is differentiable, we have

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{|f(a+tx) - f(a) - Df(a)(tx)|}{|tx|} \\ &= \lim_{t\to 0} \frac{|f(a+tx) - f(a) - tDf(a)(x)|}{|t|} \frac{1}{|x|} \\ &= \lim_{t\to 0} \left| \frac{f(a+tx) - f(a) - tDf(a)(x)}{t} \right| \frac{1}{|x|} \\ &= \lim_{t\to 0} \left| \frac{f(a+tx) - f(a)}{t} - Df(a)(x) \right| \frac{1}{|x|} \end{aligned}$$

which implies

$$D_x f(a) = \lim_{t\to 0} \frac{f(a+tx) - f(a)}{t} = Df(a)(x).$$

Therefore,

$$D_{x+y}f(a) = Df(a)(x+y) = Df(a)(x) + Df(a)(y) = D_x f(a) + D_y f(a).$$

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Exercise 2-34. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous* of degree m if $f(tx) = t^m f(x)$ for all x. If f is also differentiable, show that

$$\sum_{i=1}^{n} x^i D_i f(x) = mf(x).$$

Hint: If $g(t) = f(tx)$, find $g'(1)$.

Proof. Using the hint, let $g(t) = f(tx) = f(tx^1, tx^2, \ldots, tx^n)$. By Theorem 2-9, we have

$$g'(t) = D_1 g(t) = \sum_{j=1}^{n} D_j f(tx) \cdot D_1(tx^j) = \sum_{j=1}^{n} D_j f(tx) \cdot x^j.$$

Since $g(t) = f(tx) = t^m f(x)$, then we also have $g'(t) = mt^{m-1} f(x)$. So $g'(1) = mf(x)$ and

$$g'(1) = \sum_{j=1}^{n} D_j f(x) \cdot x^j.$$

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Intro to Manifolds, Tu – *Exercises within the Text*

Exercise 3.13. (Symmetrizing operator) Show that the k-linear function Sf is symmetric.

Proof. For $\tau \in S_k$,

$$\begin{aligned}
\tau(Sf) &= \tau\left(\sum_{\sigma \in S_k} \sigma f\right) && \text{(definition of } Sf\text{)} \\
&= \sum_{\sigma \in S_k} \tau(\sigma f) && (\tau \text{ is linear}) \\
&= \sum_{\sigma \in S_k} (\tau\sigma f) && \text{(Lemma 3.11)} \\
&= \sum_{\mu \in S_k} \mu f && (S_k \text{ is a group } \implies \{\tau\sigma\}_{\sigma \in S_k} = S_k) \\
&= Sf && \text{(definition of } Sf\text{)}
\end{aligned}$$

Exercise 3.15. (Alternating operator) If f is a 3-linear function on a vector space V and $v_1, v_2, v_3 \in V$ what is $(Af)(v_1, v_2, v_3)$?

Proof. We have $S_3 = \{(), (12), (13), (23), (123), (132)\}$ with respective signs $\{1, -1, -1, -1, 1, 1\}$. So

$$\begin{aligned}
(Af)(v_1, v_2, v_3) &= \sum_{\sigma \in S_3} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) \\
&= f(v_1, v_2, v_3) - f(v_2, v_1, v_3) - f(v_3, v_2, v_1) \\
&\qquad - f(v_1, v_3, v_2) + f(v_3, v_1, v_2) + f(v_2, v_3, v_1)
\end{aligned}$$

Exercise 3.17. (Associativity of the tensor product) Check that the tensor product of multilinear functions is associative: If f, g and h are multilinear functions on V, then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Proof. Let f, g and h be k, ℓ and m -linear on V, respectively. Then

$$\begin{aligned}
((f \otimes g) \otimes h)(v_1, \ldots, v_{k+\ell+m}) &= (f \otimes g)(v_1, \ldots, v_{k+\ell}) g(v_{k+\ell+1}, \ldots, v_{k+\ell+m}) \\
&= (f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+\ell})) h(v_{k+\ell+1}, \ldots, v_{k+\ell+m}) \\
&= f(v_1, \ldots, v_k)(g(v_{k+1}, \ldots, v_{k+\ell}) h(v_{k+\ell+1}, \ldots, v_{k+\ell+m})) \\
&= f(v_1, \ldots, v_k)((g \otimes h)(v_{k+1}, \ldots, v_{k+\ell+m})) \\
&= (f \otimes (g \otimes h))(v_1, \ldots, v_{k+\ell+m}).
\end{aligned}$$

Exercise 3.20. (Wedge product of two 2-covectors) For $f, g \in A_2(V)$, write out the definition of $f \wedge g$ using $(2,2)$-shuffles.

Proof. The $(2,2)$ shuffles of S_4 are:

$$\begin{bmatrix}1&2&3&4\\1&2&3&4\end{bmatrix}, \begin{bmatrix}1&2&3&4\\1&3&2&4\end{bmatrix}, \begin{bmatrix}1&2&3&4\\1&4&2&3\end{bmatrix}, \begin{bmatrix}1&2&3&4\\2&3&1&4\end{bmatrix}, \begin{bmatrix}1&2&3&4\\2&4&1&3\end{bmatrix}, \begin{bmatrix}1&2&3&4\\3&4&1&2\end{bmatrix},$$

with respective signs: 1, -1, 1, 1, -1, 1. Let $v_1, v_2, v_3, v_4 \in V$. Then

$$\begin{aligned}
f \wedge g &= \frac{1}{2!2!} A(f \otimes g)(v_1, v_2, v_3, v_4) \\
&= \frac{1}{2!2!} \sum_{\sigma \in S_4} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, v_{\sigma(2)}) g(v_{\sigma(3)}, v_{\sigma(4)}) \\
&= \sum_{\substack{(2,2)\text{-shuffles} \\ \sigma \in S_4}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, v_{\sigma(2)}) g(v_{\sigma(3)}, v_{\sigma(4)}) \\
&= f(v_1, v_2) g(v_3, v_4) - f(v_1, v_3) g(v_2, v_4) + f(v_1, v_4) g(v_2, v_3) \\
&\quad + f(v_2, v_3) g(v_1, v_4) - f(v_2, v_4) g(v_1, v_3) + f(v_3, v_4) g(v_1, v_2).
\end{aligned}$$

☕

Exercise 3.22. (Sign of a permutation) Let $\tau \in S_{k+\ell}$ be given by

$$\tau = \begin{bmatrix} 1 & \dots & \ell & \ell+1 & \dots & \ell+k \\ k+1 & \dots & k+\ell & 1 & \dots & k \end{bmatrix}$$

Show that $\operatorname{sgn} \tau = (-1)^{k\ell}$.

Proof. To determine the sign of τ, we need to determine how many transpositions to compose with τ to obtain the identity permutation. First, we need to perform exactly ℓ transpositions to move 1 to the first position. In particular,

$$\begin{bmatrix} 1 & k+1 \\ k+1 & 1 \end{bmatrix} \begin{bmatrix} 1 & k+2 \\ k+2 & 1 \end{bmatrix} \begin{bmatrix} 1 & k+3 \\ k+3 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & k+\ell-1 \\ k+\ell-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & k+\ell \\ k+\ell & 1 \end{bmatrix} \tau$$

will result in the permutation

$$\begin{bmatrix} 1 & 2 & \dots & \ell & \ell+1 & \dots & \ell+k \\ 1 & k+1 & \dots & k+\ell-1 & k+\ell & \dots & k \end{bmatrix}.$$

This permutation has sign equal to $(-1)^{\ell}$. To obtain the identity, we need to perform this same process for all numbers 1 through k, resulting in a sign of $((-1)^{\ell})^k = (-1)^{k\ell}$. ☕

Exercise 4.3. (A basis for 3-covectors) Let x^1, x^2, x^3, x^4 be the coordinates on $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

Proof. Using the standard basis

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \frac{\partial}{\partial x^2}\bigg|_p, \frac{\partial}{\partial x^3}\bigg|_p, \frac{\partial}{\partial x^4}\bigg|_p \right\}$$

of $T_p(\mathbb{R}^4)$, we have by Proposition 4.1 the dual basis

$$\{(dx^1)|_p, (dx^2)|_p, (dx^3)|_p, (dx^4)|_p\}$$

for the cotangent space $T_p^*(\mathbb{R}^n)$. By Proposition 3.29, we need to consider all strictly increasing sets of indices of length 3 from the set $\{1, 2, 3, 4\}$. We get

$$I_1 = (1 < 2 < 3),\ I_2 = (1 < 2 < 4),\ I_3 = (1 < 3 < 4),\ \text{and } I_4 = (2 < 3 < 4).$$

So we have as a basis for $A_3(T_p(\mathbb{R}^4))$

$$\begin{aligned} dx_p^{I_1} &= dx_p^1 \wedge dx_p^2 \wedge dx_p^3, \\ dx_p^{I_2} &= dx_p^1 \wedge dx_p^2 \wedge dx_p^4, \\ dx_p^{I_3} &= dx_p^1 \wedge dx_p^3 \wedge dx_p^4, \\ dx_p^{I_4} &= dx_p^2 \wedge dx_p^3 \wedge dx_p^4. \end{aligned}$$

☕

Exercise 4.4. (Wedge product of a 2-form with a 1-form). Let ω be a 2-form and τ a 1- form on $\mathbb{R}^3$. If X, Y, Z are vector fields on M, find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z.

Proof. Fix a point $p \in M$. We consider the $(2, 1)$-shuffles of S_3:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix},$$

These have respective signs $1, -1, 1$. So,

$$\begin{aligned} (\omega \wedge \tau)_p(X_p, Y_p, Z_p) &= (\omega_p \wedge \tau_p)(X_p, Y_p, Z_p) \\ &= \frac{1}{2!1!} \sum_{\sigma \in S_3} (\operatorname{sgn} \sigma) \sigma(\omega_p(X_p, Y_p)\tau_p(Z_p)) \\ &= \sum_{\substack{(2,1)\text{-shuffles} \\ \sigma \in S_3}} (\operatorname{sgn} \sigma) \sigma\omega_p(X_p, Y_p) \sigma\tau_p(Z_p) \\ &= \omega_p(X_p, Y_p)\tau_p(Z_p) - \omega_p(X_p, Z_p)\tau_p(Y_p) + \omega_p(Y_p, Z_p)\tau_p(X_p). \end{aligned}$$

As p varies over all of M, we get

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X).$$

☕

Exercise 4.9. (A closed 1-form on the punctured plane). Define a 1-form ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-ydx + xdy).$$

Show that ω is closed.

Proof. Let's do this computation! To make it a bit cleaner, let

$$f(x,y) = \frac{-y}{x^2+y^2} \quad \text{and} \quad g(x,y) = \frac{x}{x^2+y^2}.$$

Then we have [1]

$$\frac{\partial f}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial f}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2},$$

and

$$\frac{\partial g}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \qquad \frac{\partial g}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}.$$

Here we go:

$$\begin{aligned}
d\omega &= d\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) \\
&= d\left(fdx + gdy\right) \\
&= df \wedge dx + dg \wedge dy \\
&= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \wedge dy \\
&= \frac{\partial f}{\partial x}dx \wedge dx + \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dy + \frac{\partial g}{\partial y}dy \wedge dy \\
&= \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dy \\
&= -\frac{\partial f}{\partial y}dx \wedge dy + \frac{\partial g}{\partial x}dx \wedge dy \\
&= \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx \wedge dy \\
&= (0)dx \wedge dy \\
&= 0
\end{aligned}$$

☕

[1] Notice that these two functions satisfy the Cauchy-Riemann Equations!

Intro to Manifolds, Tu – *End of Section Exercises*

Exercise 3.3. A basis for k-tensors
Let V be a vector space of dimension n with basis $e_1, \ldots e_n$. let $\alpha^1, \ldots, \alpha^n$ be the dual basis for V^*. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \ldots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \ldots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(V)$). In particular, this shows that $\dim L_k(V) = n^k$.

Proof. Let $\alpha^I = \alpha^{i_1} \otimes \ldots \otimes \alpha^{i_k}$ and $e_J = (e_{j_1}, \ldots, e_{j_k})$. Let $f \in L_k(V)$. We claim $f = \sum_I f(e_I)\alpha^I$, where I ranges over all multi-indices $\{(i_1, \ldots, i_k)\}$ of length k. Let $g = \sum_I f(e_I)\alpha^I$. Then

$$g(e_J) = \sum_I f(e_I)\alpha^I(e_J) = \sum_I f(e_I)\delta^I_J = f(e_J),$$

and thus $g = f$ since multi-linear functions are determined by their action on basis elements. Hence $\{\alpha^I\}_I$ spans $L_k(V)$.

Suppose $0 = \sum_I c_I \alpha^I$ for some scalars c_I for all I. Then applying both sides to e_J gives

$$0 = \sum_I c_I \alpha^I(e_J) = \sum_I c_I \delta^I_J = c_J.$$

And so $c_I = 0$ for all I, and hence the α^I are linearly independent. ☕

Exercise 3.4. A characterization of alternating k-tensors
Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged:

$$f(\ldots, v_{i+1}, v_i, \ldots) = -f(\ldots, v_i, v_{i+1}, \ldots) \qquad (*)$$

for $i = 1, \ldots, k-1$.

Proof. ($\Rightarrow$) We have $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$. Given i, let $\sigma = (i, i+1) \in S_k$. Then

$$\begin{aligned} f(\ldots, v_{i+1}, v_i, \ldots) &= f(\ldots, v_{\sigma(i)}, v_{\sigma(1+1)}, \ldots) \\ &= \sigma f(\ldots, v_i, v_{i+1}, \ldots) \\ &= (\operatorname{sgn} \sigma) f(\ldots, v_i, v_{i+1}, \ldots) \\ &= -f(\ldots, v_i, v_{i+1}, \ldots). \end{aligned}$$

($\Leftarrow$) Suppose $(*)$ holds and let $\sigma \in S_k$. Since $S_k = \left\langle \{(i, i+1)\}_{i=1}^{n-1} \right\rangle$, then

$$\sigma = (i_1, i_1+1)(i_2, i_2+1)\ldots(i_m, i_m+1)$$

for some $m \in \mathbb{Z}^+$ and $(i_j, i_j+1) \in S_k$ all $j = 1, \ldots, m$. Now,

$$\begin{aligned} \sigma f(v_1, \ldots, v_k) &= [(i_1, i_1+1)\ldots(i_m, i_m+1)] f(v_1, \ldots, v_k) \\ &= [(i_1, i_1+1)\ldots(i_{m-1}, i_{m-1}+1)] f(\ldots, v_{m+1}, v_m, \ldots) \\ &= [(i_1, i_1+1)\ldots(i_{m-1}, i_{m-1}+1)](-1) f(\ldots, v_m, v_{m+1}, \ldots) \\ &\;\;\vdots \\ &= (-1)^m f(v_1, \ldots, v_k). \end{aligned}$$

If σ is even, then so is m, and

$$\sigma f(v_1,\ldots,v_k) = (-1)^m f(v_1,\ldots,v_k) = f(v_1,\ldots,v_k) = (\operatorname{sgn}\sigma) f(v_1,\ldots,v_k).$$

Similarly, if σ is odd, then so is m, and

$$\sigma f(v_1,\ldots,v_k) = (-1)^m f(v_1,\ldots,v_k) = -f(v_1,\ldots,v_k) = (\operatorname{sgn}\sigma) f(v_1,\ldots,v_k).$$

☕

Exercise 3.5. Another characterization of alternating k-tensors
Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if $f(v_1,\ldots,v_k) = 0$ whenever two of the vectors $v_1,\ldots,v_k$ are equal.

Proof. ($\Rightarrow$) Suppose f is alternating and that $v_m = v_\ell$ for $1 \le m, \ell \le k$. Let $\sigma = (m\ell) \in S_k$ and suppose without loss of generality that $m < \ell$. Then

$$\begin{aligned} f(v_1,\ldots,v_\ell,\ldots,v_m,\ldots,v_k) &= f(v_{\sigma(1)},\ldots,v_{\sigma(m)},\ldots,v_{\sigma(\ell)},\ldots,v_{\sigma(k)}) \\ &= \sigma f(\ldots,v_\ell,\ldots,v_m,\ldots,) \\ &= -f(\ldots,v_\ell,\ldots,v_m,\ldots) && (f \text{ is alternating}) \\ &= -f(\ldots,v_m,\ldots,v_\ell,\ldots). && (v_m = v_\ell) \end{aligned}$$

So

$$f(\ldots,v_\ell,\ldots,v_m,\ldots) = -f(v_1,\ldots,v_m,\ldots,v_\ell,\ldots,v_k)$$

implies

$$f(v_1,\ldots,v_\ell,\ldots,v_m,\ldots,v_k) = 0.$$

($\Leftarrow$) Notice that

$$\begin{aligned} f(\ldots,v_i,v_{i+1},\ldots) + f(\ldots,v_{i+1},v_i,\ldots) &= f(\ldots,v_i,v_{i+1},\ldots) + f(\ldots,v_i,v_i,\ldots) \\ &\quad + f(\ldots,v_{i+1},v_{i+1},\ldots) + f(\ldots,v_{i+1},v_i,\ldots) \\ &= f(\ldots,v_i,v_i+v_{i+1},\ldots) \\ &\quad + f(\ldots,v_{i+1},v_i+v_{i+1},\ldots) \\ &= f(\ldots,v_i+v_{i+1},v_i+v_{i+1},\ldots) \\ &= 0. \end{aligned}$$

So

$$f(\ldots,v_i,v_{i+1},\ldots) = -f(\ldots,v_{i+1},v_i,\ldots),$$

and by Exercise 3.4, f is alternating. ☕

Exercise 3.7. Transformation rule for a wedge product of covectors
Suppose two sets of covectors on a vector space V, $\beta^1, \dots, \beta^k$ and $\gamma^1, \dots, \gamma^k$, are related by

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j, \quad i = 1, \dots, k,$$

for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k.$$

Proof. Since $\wedge$ is distributive, we obtain

$$\beta^1 \wedge \cdots \wedge \beta^k = \left(\sum_{j_1=1}^{k} a^1_{j_1}\gamma^{j_1}\right) \wedge \cdots \wedge \left(\sum_{j_k=1}^{k} a^k_{j_k}\gamma^{j_k}\right) = \sum_{j_1}^{k} \cdots \sum_{j_k}^{k} a^1_{j_1} \cdots a^k_{j_k} \gamma^{j_1} \wedge \cdots \wedge \gamma^{j_k}$$

Since $\gamma^{j_1} \wedge \cdots \wedge \gamma^{j_k} = 0$ if any of the indices j_i are repeated, each set of indices $j_{i_1} \cdots j_{i_k}$ which have no repetition correspond to a bijection between the set $\{1, \dots, k\}$ with itself; that is, they correspond to a permutation in S_k. So the above multi-sum becomes

$$\sum_{\sigma \in S_k} a^1_{\sigma(1)} \cdots a^k_{\sigma(k)} \gamma^{\sigma(1)} \wedge \cdots \wedge \gamma^{\sigma(k)}$$

Now since the wedge product is anticommutative, we obtain the desired formula:

$$\beta^1 \wedge \cdots \wedge \beta^k = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a^1_{\sigma(1)} \cdots a^k_{\sigma(k)} \gamma^1 \wedge \cdots \wedge \gamma^k \ = (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k.$$

☕

Exercise 4.1. A 1-form on $\mathbb{R}^3$
Let ω be the 1-form $zdx - dz$ and let X be the vector field $y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

Proof.

$$\begin{aligned} \omega(X) = (zdx - dz)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) &= zdx\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - dz\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\ &= z\left(y\frac{\partial x}{\partial x} + x\frac{\partial x}{\partial y}\right) - \left(y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right) \\ &= zy \end{aligned}$$

$$\begin{aligned} d\omega = d(zdx - dz) &= dz \wedge dx - d(1) \wedge dz \\ &= dz \wedge dx - 0 \wedge dz \\ &= dz \wedge dx. \end{aligned}$$

☕

Exercise 4.2. A 2-form on $\mathbb{R}^3$
At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\boldsymbol{a}, \boldsymbol{b}) = \omega_p\left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}\right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

for tangent vectors $\boldsymbol{a}, \boldsymbol{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

Proof. Since ω is a 2-form on $\mathbb{R}^3$, we have

$$\omega = \alpha dx^1 \wedge dx^2 + \beta dx^1 \wedge dx^3 + \gamma dx^2 \wedge dx^3$$

for some C^∞ functions α, β, γ. For $p \in \mathbb{R}^3$ and $\boldsymbol{a}, \boldsymbol{b} \in T_p(\mathbb{R}^3)$,

$$\begin{aligned}
\omega_p(\boldsymbol{a}, \boldsymbol{b}) &= \alpha(p) dx^1 \wedge dx^2(\boldsymbol{a}, \boldsymbol{b}) + \beta(p) dx^1 \wedge dx^3(\boldsymbol{a}, \boldsymbol{b}) + \gamma(p) dx^2 \wedge dx^3(\boldsymbol{a}, \boldsymbol{b}) \\
&= \alpha(p)[dx^1(\boldsymbol{a}) dx^2(\boldsymbol{b}) - dx^1(\boldsymbol{b}) dx^2(\boldsymbol{a})] \\
&\quad + \beta(p)[dx^1(\boldsymbol{a}) dx^3(\boldsymbol{b}) - dx^1(\boldsymbol{b}) dx^3(\boldsymbol{a})] \\
&\quad\quad + \gamma(p)[dx^2(\boldsymbol{a}) dx^3(\boldsymbol{b}) - dx^2(\boldsymbol{b}) dx^3(\boldsymbol{a})] \\
&= \alpha(p)[a^1 b^2 - b^1 a^2] + \beta(p)[a^1 b^3 - b^1 a^3] + \gamma(p)[a^2 b^3 - b^2 a^3].
\end{aligned}$$

Since

$$\det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix} = a^1 b^2 - b^1 a^2,$$

we must have $\alpha(p) = p^3$ and $\beta = \gamma \equiv 0$, and so $\omega = \alpha dx^1 \wedge dx^2$. ☕

Exercise 4.3. Exterior calculus
Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r,θ)-plane, not the (x,y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

Proof. We have maps $x, y : \mathbb{R}^2 \to \mathbb{R}$ given by $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$. By Proposition 4.2, we can write dx and dy as

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \quad \text{and} \quad dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta.$$

Then

$$dx = \cos\theta dr - r\sin\theta d\theta \quad \text{and} \quad dy = \sin\theta dr + r\cos\theta d\theta.$$

Now, we compute $dx \wedge dy$:

$$\begin{aligned}
dx \wedge dy &= \left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) \wedge \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right) \\
&= \left(\frac{\partial x}{\partial r}dr \wedge \frac{\partial y}{\partial r}dr\right) + \left(\frac{\partial x}{\partial r}dr \wedge \frac{\partial y}{\partial \theta}d\theta\right) + \left(\frac{\partial x}{\partial \theta}d\theta \wedge \frac{\partial y}{\partial r}dr\right) + \left(\frac{\partial x}{\partial \theta}d\theta \wedge \frac{\partial y}{\partial \theta}d\theta\right) \\
&= \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial r}dr \wedge dr\right) + \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta}dr \wedge d\theta\right) + \left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}d\theta \wedge dr\right) + \left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta}d\theta \wedge d\theta\right) \\
&= 0 + \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta}dr \wedge d\theta\right) + \left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}d\theta \wedge dr\right) + 0 \\
&= \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) dr \wedge d\theta \\
&= (\cos\theta r\cos\theta - (-r\sin\theta)\sin\theta)dr \wedge d\theta \\
&= r(\cos^2\theta + \sin^2\theta)dr \wedge d\theta \\
&= r dr \wedge d\theta.
\end{aligned}$$

☕

Exercise 4.4. Exterior calculus
Suppose the standard coordinates on R^3 are called ρ, ϕ, and θ. If $x = \rho \sin\theta, \cos\theta, y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

Proof. We have maps $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ given by

$$x(\rho,\phi,\theta) = \rho\sin\phi\cos\theta, \qquad y(\rho,\phi,\theta) = \rho\sin\phi\sin\theta, \quad \text{and} \quad z(\rho,\phi,\theta) = \rho\cos\phi$$

So by Proposition 4.2, we get

$$\begin{aligned}
dx &= \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \theta} d\theta \\
&= \sin\phi\cos\theta d\rho + \rho\cos\theta\cos\phi d\phi - \rho\sin\phi\sin\theta d\theta, \\
dy &= \frac{\partial y}{\partial \rho} d\rho + \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \theta} d\theta \\
&= \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta \\
dz &= \frac{\partial z}{\partial \rho} d\rho + \frac{\partial z}{\partial \phi} d\phi + \frac{\partial z}{\partial \theta} d\theta \\
&= \cos\phi d\rho - \rho\sin\phi d\phi.
\end{aligned}$$

Since $\partial z/\partial\theta = 0$, we remove it from the following computation. Here. We. Go.

$$\begin{aligned}
dx \wedge dy \wedge dz &= \left(\frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \theta} d\theta\right) \wedge \left(\frac{\partial y}{\partial \rho} d\rho + \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \theta} d\theta\right) \\
&\qquad \wedge \left(\frac{\partial z}{\partial \rho} d\rho + \frac{\partial z}{\partial \phi} d\phi\right) \\
&= \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \rho} d\phi \wedge d\theta \wedge d\rho + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\frac{\partial z}{\partial \rho} d\theta \wedge d\phi \wedge d\rho \\
&\qquad + \frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} d\rho \wedge d\theta \wedge d\phi + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \rho}\frac{\partial z}{\partial \phi} d\theta \wedge d\rho \wedge d\phi \\
&= \left[\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \rho} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\frac{\partial z}{\partial \rho} - \frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \rho}\frac{\partial z}{\partial \phi}\right] d\rho \wedge d\phi \wedge d\theta \\
&= \left[\left(\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\right)\frac{\partial z}{\partial \rho} + \left(-\frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \theta} + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \rho}\right)\frac{\partial z}{\partial \phi}\right] d\rho \wedge d\phi \wedge d\theta \\
&= [(\rho\cos\theta\cos\phi\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta\rho\cos\phi\sin\theta)\cos\phi \\
&\qquad + (-\sin\phi\cos\theta\rho\sin\phi\cos\theta - \rho\sin\phi\sin\theta\sin\phi\sin\theta)(-\rho\sin\phi)] d\rho \wedge d\phi \wedge d\theta \\
&= [(\rho^2\cos^2\theta\cos^2\phi\sin\phi + \rho^2\sin^2\theta\cos^2\phi\sin\phi) \\
&\qquad + (\rho^2\sin^3\phi\cos^\theta + \rho^2\sin^2\theta\sin^3\phi)] d\rho \wedge d\phi \wedge d\theta \\
&= \rho^2[\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) + \sin^3\phi(\cos^2\theta + \sin^2\theta)] d\rho \wedge d\phi \wedge d\theta \\
&= \rho^2[\cos^2\phi\sin\phi + \sin^3\phi] d\rho \wedge d\phi \wedge d\theta \\
&= \rho^2[\sin\phi(\cos^2\phi\sin^2\phi)] d\rho \wedge d\phi \wedge d\theta \\
&= \rho^2\sin\phi d\rho \wedge d\phi \wedge d\theta
\end{aligned}$$

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Exercise 4.5. Wedge product
Let α be a 1-form and β be a 2-form on $\mathbb{R}^3$. Then

$$\begin{aligned}\alpha &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3\\ \beta &= b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2.\end{aligned}$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

Proof. As we distribute across the wedge product $\alpha \wedge \beta$, we disregard terms which would give $dx^i \wedge dx^i = 0$.

$$\begin{aligned}\alpha \wedge \beta &= (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2)\\ &= a_1 b_1 dx^1 \wedge dx^2 \wedge dx^3 + a_2 b_2 dx^2 \wedge dx^3 \wedge dx^1 + a_3 b_3 dx^3 \wedge dx^1 \wedge dx^2\\ &= a_1 b_1 dx^1 \wedge dx^2 \wedge dx^3 + a_2 b_2 dx^1 \wedge dx^2 \wedge dx^3 + a_3 b_3 dx^1 \wedge dx^2 \wedge dx^3\\ &= (a_1 b_1 + a_2 b_2 + a_3 b_3) dx^1 \wedge dx^2 \wedge dx^3.\end{aligned}$$

Getting to know $SU(2)$ and $SO(3)$

The quaternions are just $\mathbb{R}^4$ equipped with a multiplication. In order to make working with quaternionic multiplication tractable, we denote the elements of $\mathbb{R}^4$ as

$$\mathbb{H} = \{a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \ \mid a, x, y, z \in \mathbb{R}\}.$$

If $q = a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then the real part of q, denoted $\mathrm{Re}(q)$ is a and the imaginary part of q, denoted, $\mathrm{Im}(q)$ is the vector $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Suppose that $q = a + \vec{v}$ and $q' = b + \vec{w}$, where $a, b \in \mathbb{R}$ and $\vec{v}, \vec{w} \in \mathbb{R}^3$. Define the product qq' by

$$qq' = ab - \vec{v} \cdot \vec{w} + a\vec{w} + b\vec{v} + \vec{v} \times \vec{w}.$$

Quaternionic multiplication is associative and bilinear. Define the complex conjugate of q by $\overline{q} = a - \vec{v}$.

Exercise 1. Prove that $\mathrm{Re}(q\overline{q'}) = q \cdot q'$. Prove that $||q|| = \sqrt{q\overline{q}}$. Compute $\boldsymbol{ij}, \boldsymbol{jk}, \boldsymbol{ki}$, and $\boldsymbol{i}^2, \boldsymbol{j}^2, \boldsymbol{k}^2$.

Proof. Let $q = a + \vec{v}$ and $q' = b + \vec{w}$. Then

$$q\overline{q'} = (a + \vec{v})(b - \vec{w}) = ab + \vec{v} \cdot \vec{w} - a\vec{w} + b\vec{v} - \vec{v} \times \vec{w},$$

and so $\mathrm{Re}(q\overline{q'}) = ab + \vec{v} \cdot \vec{w}$. On the other hand, $q \cdot q' = (a + \vec{v}) \cdot (b + \vec{w}) = ab + \vec{v} \cdot \vec{w}$.

Now,

$$||q|| = \sqrt{a^2 + ||v||^2} = \sqrt{a^2 + ||v||^2 - a\vec{v} + a\vec{v} + \vec{v} \times \vec{v}} = \sqrt{q\overline{q}}$$

And finally, we have $\boldsymbol{i} = 0 + (1,0,0)$, $\boldsymbol{j} = 0 + (0,1,0)$, and $\boldsymbol{k} = 0 + (0,0,1)$. So,

$$\boldsymbol{ij} = (0 + (1,0,0))(0 + (0,1,0)) = (1,0,0) \times (0,1,0) = (0,0,1) = \boldsymbol{k}.$$

Similarly we get $\boldsymbol{jk} = \boldsymbol{i}$ and $\boldsymbol{ki} = \boldsymbol{j}$. Moreover,

$$\boldsymbol{i}^2 = (0 + (1,0,0))(0 + (1,0,0)) = -((1,0,0) \cdot (1,0,0)) = -1.$$

Similarly we get $\boldsymbol{j}^2 = -1$ and $\boldsymbol{k}^2 = -1$. ☕

Exercise 2. Prove that if $q \neq 0 + \vec{0}$, then q has a multiplicative inverse given by $\overline{q}/q\overline{q}$.

Proof. Let $q = a + \vec{v}$. Then

$$q\left(\frac{\overline{q}}{q\overline{q}}\right) = (a + \vec{v})\left(\frac{a - \vec{v}}{a + ||v^2||}\right) = \frac{a + ||v^2||}{a + ||v^2||} = 1,$$

and similarly $(\overline{q}/q\overline{q})q = 1$. ☕

Exercise 3. Prove that $q, q' \in \mathbb{H}$ commute if and only if their imaginary parts are linearly dependent.

Proof. Recall that $\times$ is anticommutative. So

$$\begin{aligned} qq' = q'q &\iff ab - \vec{v} \cdot \vec{w} + a\vec{w} + b\vec{v} + \vec{v} \times \vec{w} = ba - \vec{w} \cdot \vec{v} + b\vec{v} + a\vec{w} + \vec{w} \times \vec{v} \\ &\iff \vec{v} \times \vec{w} = \vec{w} \times \vec{v} \\ &\iff \vec{v} \times \vec{w} = -\vec{v} \times \vec{w} \\ &\iff \vec{v} \times \vec{w} = 0 \\ &\iff \vec{v}, \vec{w} \text{ are linearly dependent.} \end{aligned}$$

☕

Exercise 4. Define $SU(2) = \{q \in \mathbb{H} \mid q\overline{q} = 1\}$.

- Prove that $SU(2)$ is a Lie group.

  Proof. Let $F : \mathbb{R}^4 \to \mathbb{R}$ be given by $q \mapsto ||q||$. Then

  $$F_* = \left(\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}, \frac{\partial F}{\partial x^3}, \frac{\partial F}{\partial x^4}\right) = \left(\frac{x_1}{||x||}, \frac{x_2}{||x||}, \frac{x_3}{||x||}, \frac{x_4}{||x||}\right).$$

  Then F_* is never zero, and so all values of $\mathbb{R}$ are regular values of F. In particular, 1 is a regular value of F and so by the Regular Level Set Theorem,

  $$F^{-1}(1) = SU(2)$$

  is a regular submanifold of $\mathbb{R}^4$ of dimension $4 - 1 = 3$.

  To see that $SU(2)$ is a group, we note that $1 \in SU(2)$ and so $SU(2) \neq \emptyset$. Moreover, for $p, q \in SU(2)$,

  $$||pq^{-1}|| = ||p\overline{q}|| = p\overline{q}\overline{p\overline{q}} = p\overline{q}\ \overline{p}q = p\overline{q}q\overline{p} = 1, \quad \text{(See the claim proven in Exercise 5!)}$$

  and so $pq^{-1} \in SU(2)$, and hence $SU(2)$ is a group.

  Now, the multiplication map $\mu : SU(2) \times SU(2) \to SU(2) \to SU(2)$ is simply a restriction of the multiplication map $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$, which is smooth since each of the component functions of multiplication in $\mathbb{H}$ is a polynomial.

  Similarly, the inversion map $\iota : SU(2) \to SU(2)$ given by $q \mapsto \overline{q}/q\overline{q}$ is the restriction of the inversion map on $\mathbb{H}$ to $SU(2)$. The inversion map on $\mathbb{H}$ is a rational expression defined for all $q \neq 0$, which is smooth, and since $0 \notin SU(2)$, then ι is smooth.

  ☕

- Prove $T_1 SU(2) = \mathbb{R}^3$. Let $c : (-\epsilon, \epsilon) \to SU(2)$ be a smooth curve starting at 1. Write

  $$c(t) = \alpha(t) + i\beta(t) + j\gamma(t) + k\delta(t).$$

  Then in the "calculus sense", $c'(t) = \alpha'(t) + i\beta'(t) + j\gamma'(t) + k\delta'(t)$, which we can also think of as an element of the quaternions. Prove that $c'(0)$ is purely imaginary, i.e., $\alpha'(0) = 0$.

  Proof. Since $c(t) \in SU(2)$ for all $t \in (-\epsilon, \epsilon)$, then

  $$1 = c(t)\overline{c(t)} = ||c(t)||^2 = (\alpha(t))^2 + (\beta(t))^2 + (\gamma(t))^2 + (\delta(t))^2.$$

  The right hand side is now a function from $\mathbb{R}$ to $\mathbb{R}$, and so taking the derivative of both sides,

  $$0 = 2\left(\alpha(t)\alpha'(t) + \beta(t)\beta'(t) + \gamma(t)\gamma'(t) + \delta(t)\delta'(t)\right).$$

  Now, $c(0) = 1$, and so $\beta(0) = \gamma(0) = \delta(0) = 0$, and so

  $$0 = 2\alpha(0)\alpha'(0) = 2\alpha'(0),$$

  and hence $\alpha'(0) = 0$. ☕

- Show that the commutator of two vectors at the identity is twice the cross product of those vectors.

Proof. Let $\vec{v} = (0 + \vec{v}), \vec{w} = (0 + \vec{w}) \in T_1 SU(2) = \mathbb{R}^3$. Then

$$\begin{aligned}
[\vec{v}, \vec{w}] &= \vec{v}\vec{w} - \vec{w}\vec{v} \\
&= (0 + \vec{v})(0 + \vec{w}) - (0 + \vec{w})(0 + \vec{v}) \\
&= (-\vec{v} \cdot \vec{w} + \vec{v} \times \vec{w}) - (-\vec{w} \cdot \vec{v} + \vec{w} \times \vec{v}) \\
&= (-\vec{v} \cdot \vec{w} + \vec{v} \times \vec{w}) \\
&= \vec{v} \times \vec{w} - (-\vec{v} \times \vec{w}) \\
&= 2(\vec{v} \times \vec{w})
\end{aligned}$$

☕

Exercise 5. Any Lie group acts on its tangent space at the identity by conjugation. This is called the *adjoint representation*. Given $q \in SU(2)$ and $\vec{w} \in \mathbb{R}^3$ define

$$Ad(q)(\vec{w}) = q\vec{w}\overline{q}$$

Prove $Ad(q)$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Prove that $Ad(q) : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the dot product.

Proof. Let $\alpha \in \mathbb{R}$ and $\vec{w}, \vec{u} \in \mathbb{R}^3$. Then

$$Ad(q)(\vec{w} + \alpha\vec{u}) = q(\vec{w} + \alpha\vec{u})\overline{q} = q\vec{w}\overline{q} + \alpha q\vec{u}\overline{q} = Ad(q)(\vec{w}) + \alpha Ad(q)(\vec{u}).$$

and hence $Ad(q)$ is linear.

Now, we claim that for all $s, t \in \mathbb{H}$, we have $\overline{st} = \overline{t}\ \overline{s}$: If $s = a + \vec{u}$ and $t = b + \vec{w}$, then

$$\overline{st} = ab - \vec{u} \cdot \vec{w} - (a\vec{w} + b\vec{u} + \vec{u} \times \vec{w}).$$

On the other hand,

$$\begin{aligned}
\overline{t}\ \overline{s} &= (b - \vec{w})(a - \vec{u}) \\
&= ab - \vec{u} \cdot \vec{w} - b\vec{u} - a\vec{w} + \vec{w} \times \vec{u} \\
&= ab - \vec{u} \cdot \vec{w} - b\vec{u} - a\vec{w} - \vec{u} \times \vec{w} \\
&= ab - \vec{u} \cdot \vec{w} - (b\vec{u} + a\vec{w} + \vec{u} \times \vec{w}) \\
&= \overline{st},
\end{aligned}$$

which gives the claim. Now, let $\vec{u}, \vec{w} \in \mathbb{R}^3$. From Exercise 1, we have

$$Ad(q)(\vec{u}) \cdot Ad(q)(\vec{w}) = \operatorname{Re}\left(Ad(q)(\vec{u})\overline{Ad(q)(\vec{w})}\right).$$

So,

$$\begin{aligned}
\operatorname{Re}\left(Ad(q)(\vec{u})\overline{Ad(q)(\vec{w})}\right) &= \operatorname{Re}\left(q\vec{u}\overline{q}\ \overline{(q\vec{w})\overline{q}}\right) \\
&= \operatorname{Re}\left(q\vec{u}\overline{q}\ \overline{\overline{q}}\overline{(q\vec{w})}\right) && \text{(by the claim)} \\
&= \operatorname{Re}\left(q\vec{u}\overline{q}\ q\overline{(q\vec{w})}\right) \\
&= \operatorname{Re}\left(q\vec{u}\overline{(q\vec{w})}\right) \\
&= \operatorname{Re}\left(q(\vec{u}\ \overline{\vec{w}}\ \overline{q})\right) && \text{(by the claim)} \\
&= \operatorname{Re}\left((\vec{u}\ \overline{\vec{w}}\ \overline{q})q\right) && \text{(since } \operatorname{Re}(st) = \operatorname{Re}(ts)\text{)} \\
&= \operatorname{Re}(\vec{u}\overline{\vec{w}}) \\
&= \vec{u} \cdot \vec{w}.
\end{aligned}$$

Exercise 6. Recall that $SU(3)$ is the Lie group of all linear maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ to that preserve the dot product, and have determinant 1. Prove that the adjoint representation of $SU(2)$ defines a homomorphism,

$$\theta : SU(2) \to SO(3)$$

and that homomorphism is a smooth mapping. (Here you could just write out a formula for it and see it is smooth as both manifolds are submanifolds of Euclidean spaces.)

Proof. Since $Ad(q)$ preserves the dot product, the matrix corresponding to $Ad(q)$ is orthogonal by definition[1], i.e., $Ad(q) \in O(3)$. So, we define $\theta : SU(2) \to O(3)$ by $q \mapsto Ad(q)$. Since $1 \in SU(2)$ and $\theta(1) = Ad(1) = I_3$, then $\theta(1) \in SO(3)$.

Next, we show that θ is a homomorphism and is smooth. Once we show this, we can conclude that $\text{Im}(\theta) \subset SO(3)$ since $O(3)$ has two connected components: matrices with determinant 1, and matrices with determinant -1. Since $\theta(1) \in SO(3)$ and θ is smooth, the image of θ must be connected, and hence lie completely in $SO(3)$.

$\underline{\theta \text{ is a homomorphism}}$: Let $q, p \in SU(2)$. Note that the multiplication in $SU(2)$ is quaternion multiplication given above and the multiplication in $O(3)$ is composition (viewing the matrices in $O(3)$ as linear maps.)

$$\begin{aligned}
\theta(qp)(\vec{v}) &= Ad(qp)(\vec{v}) \\
&= qp\vec{v}\overline{pq} \\
&= qp\vec{v}\overline{q}\ \overline{p} && \text{(by the claim in Exercise 5)} \\
&= q\left(Ad(p)(\vec{v})\right)\overline{q} \\
&= Ad(q)\left(Ad(p)(\vec{v})\right) \\
&= \left(\theta(q) \circ \theta(p)\right)(\vec{v})
\end{aligned}$$

$\underline{\theta \text{ is smooth}}$: Per the hint, we give a formula for θ as a map between $\mathbb{R}^4$ and $\mathbb{R}^9$. This amounts to giving a formula for $Ad(q)(\vec{w})$ and showing that it is smooth. Let $q = a + \vec{u}$. We find that

$$Ad(q)(\vec{w}) = a^2\vec{w} + 2a(\vec{u} \times \vec{w}) - \vec{w}||\vec{u}||^2.$$

The component functions of this map consist of smooth operations: multiplication, squaring, adding, subtracting. Hence θ is a smooth map between $\mathbb{R}^4$ and $\mathbb{R}^9$, and hence is a smooth map between the submanifolds $SU(2)$ of $\mathbb{R}^4$ and $SO(3)$ of $\mathbb{R}^9$. ☕

[1] per Wikipedia

Exercise 7. Recall that a roation of $\mathbb{R}^2$ by an angle of φ is given by

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

They are characterized by the fact that they are orthogonal and they have determinant $+1$. A rotation by an angle φ about an axis$\vec{v} \in R^3$ has $\vec{v}$ as an eigenvector of the eigenvalue $+1$ and acts as a rotation of angle φ on the copy of R^2 that is orthogonal to $\vec{v}$.

Prove Euler's theorem, that every element of $SO(3)$ is a rotation about some axis. How can you compute the angle of rotation without changing bases?

Proof. Let $A \in SO(3)$. Then

$$\begin{aligned} \det(I - A) &= \det(A^T)\det(I - A) \\ &= \det(A^T - I) \\ &= \det((A - I)^T) \\ &= \det(A - I) \\ &= -\det(I - A), \end{aligned}$$

and so $\det(I - A) = 0$, i.e., 1 is an eigenvalue of A and hence A is a rotation. ☕

*** Was not able to get to these problems. Sorry Jesse! ***

Exercise 8. Without great injury to yourself or those around you, prove that if $q = cos\phi + \sin\phi\vec{v} \in SU(2)$ then $Ad(q)$ is rotation by an angle of 2ϕ radians about the axis $\vec{v}$. Use this to conclude that the kernel of $\theta : SU(2) \to SO(3)$ is just ± 1.

Exercise 9. We say that $p, q \in SU(2)$ are conjugate if and only if there is $r \in SU(2)$ with $rp\overline{r} = q$.

- Prove that p and q are conjugate if and only if $\mathrm{Re}(p) = \mathrm{Re}(q)$. To do this with as little pain as possible, figure out where the axis of $rp\overline{r}$ is in terms of the axis of p and the action of $Ad(r)$ on $\mathbb{R}^3$.
- Describe the conjugacy classes of $SU(2)$ as geometric objects. What are the different conjugacy classes diffeomorphic to?

Exercise 10. Prove that $\theta : SU(2) \to SO(3)$ is onto, and the inverse image of each element of $SO(3)$ is two antipodal points on $S^3 = SU(2)$. Use this to construct a homeomorphism between $\mathbb{R}P(3)$ and $SO(3)$.

Intro to Manifolds, Tu – *End of Section Exercises*

Exercise 8.1. Differential of a map
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$, and $\frac{\partial}{\partial w}$ at $F(p)$.

Proof. We have

$$x = u \circ F, \quad y = v \circ F, \quad \text{and} \quad xy = w \circ F.$$

Then

$$F_*\left(\frac{\partial}{\partial x}\Big|_p\right)(u) = \frac{\partial}{\partial x}(u \circ F) = \frac{\partial x}{\partial x} = 1$$
$$F_*\left(\frac{\partial}{\partial x}\Big|_p\right)(v) = \frac{\partial}{\partial x}(v \circ F) = \frac{\partial y}{\partial x} = 0$$
$$F_*\left(\frac{\partial}{\partial x}\Big|_p\right)(w) = \frac{\partial}{\partial x}(w \circ F) = \frac{\partial xy}{\partial x} = y,$$

and so

$$F_*\left(\frac{\partial}{\partial x}\Big|_p\right) = \frac{\partial}{\partial u} + y\frac{\partial}{\partial w}.$$

☕

Exercise 8.2. Differential of a linear map
Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$ there is a canonical identification $T_p(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}|_p \mapsto \boldsymbol{a} = \left\langle a^1, \ldots, a_n \right\rangle.$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \to T_{L(p)}(\mathbb{R}^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Proof. Let

$$X_p = \sum a^i \frac{\partial}{\partial x^i}\Big|_p \in T_p(\mathbb{R}^n).$$

Then

$$\begin{aligned} L_*(X_p)x^i = X_p(x^i \circ L) &= X_p L^i \\ &= \lim_{t \to 0} \frac{L^i(p + t\boldsymbol{a}) - L^i(p)}{t} \\ &= \lim_{t \to 0} \frac{L^i(p) + tL(\boldsymbol{a}) - L^i(p)}{t} \\ &= L^i(\boldsymbol{a}). \end{aligned}$$

Since this is true for all coordinates x^i, then $L_*(X_p) = L(\boldsymbol{a})$. ☕

Exercise 8.3. Differential of a map
Fix a real number α and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v})|_p$, find a and b in terms of x, y, and α.

Proof. Note that

$$\begin{aligned} (u \circ F)(x, y) &= F^1(x, y) = x\cos\alpha - y\sin\alpha \\ (v \circ F)(x, y) &= F^2(x, y) = x\sin\alpha + y\cos\alpha. \end{aligned}$$

Then

$$\begin{aligned} a = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)(u) = F_*(X_p)(u) &= X_p(u \circ F) \\ &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)(x\cos\alpha - y\sin\alpha) \\ &= -y\frac{\partial(x\cos\alpha - y\sin\alpha)}{\partial x} + x\frac{\partial(x\cos\alpha - y\sin\alpha)}{\partial y} \\ &= -y\cos\alpha - x\sin\alpha. \end{aligned}$$

Similarly,

$$b = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)(v) = F_*(X_p)(v) = X_p(u \circ F) = -y\sin\alpha + x\cos\alpha.$$

☕

Exercise 8.6. Velocity vector
Let $p = (x, y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, t \in R,$$

is a curve with initial point p in $\mathbb{R}^2$. Compute the velocity vector $c_p'(0)$.

Proof. We have

$$c_p = (c_p^1, c_p^2) = (x\cos 2t - y\sin 2t, x\sin 2t + y\cos 2t),$$

and so

$$c_{*,p}(t) = \begin{bmatrix} -2x\sin 2t - 2y\cos 2t \\ 2t\cos 2t - 2y\sin 2t \end{bmatrix}.$$

This gives

$$c_{*,p}(0) = \begin{bmatrix} -2x\sin 0 - 2y\cos 0 \\ 2t\cos 0 - 2y\sin 0 \end{bmatrix} = \begin{bmatrix} -2y \\ 2x \end{bmatrix}$$

☕

Exercise 8.7. Tangent space to a product
If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$\pi_{1*}, \pi_{2*} : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

Proof. *** Copied from the back of the book. I need some help understanding this. See you in office hours! ***

If $(U, \phi) = (U, x^1, \ldots, x^m)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ are charts about p in M and q in N respectively, then by Proposition 5.18, a chart about (p, q) in $M \times N$ is

$$(U \times V, \phi \times \psi) = (U \times V, (\pi_1^*\phi, \pi_2^*\psi)) = (U \times V, \overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n),$$

where $\overline{x}^i = \pi_1^* x^i$ and $\pi_2^* y^i$. Let $\pi_{1_*}(\partial/\partial\overline{x}^j) = \sum a_j^i \partial/\partial x^i$. Then

$$a_j^i = \pi_{1_*}\left(\frac{\partial}{\partial \overline{x}}^j\right) x^i = \frac{\partial}{\partial \overline{x}^j}\left(x^i \circ \pi_1\right) = \frac{\partial \overline{x}^i}{\partial \overline{x}^j}\delta_j^i.$$

This really means that

$$\pi_{1_*}\left(\left.\frac{\partial}{\partial \overline{x}^j}\right|_{(p,q)}\right) = \left.\frac{\partial}{\partial x^j}\right|_p. \tag{1}$$

Similarly,

$$\pi_{1_*}\left(\frac{\partial}{\partial \overline{y}^j}\right) = 0, \quad \pi_{2_*}\left(\frac{\partial}{\partial \overline{x}^j}\right) = 0, \quad \pi_{2_*}\left(\frac{\partial}{\partial \overline{y}^j}\right) = \frac{\partial}{\partial y^j}. \tag{2}$$

A basis for $T_{(p,q)}(M \times N)$ is

$$\left.\frac{\partial}{\partial \overline{x}^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial \overline{x}^m}\right|_{(p,q)}, \left.\frac{\partial}{\partial \overline{y}^1}\right|_{(p,q)}, \ldots \left.\frac{\partial}{\partial \overline{y}^n}\right|_{(p,q)}.$$

A basis for $T_pM \times T_qN$ is

$$\left(\left.\frac{\partial}{\partial x^1}\right|_p, 0\right), \ldots, \left(\left.\frac{\partial}{\partial x^m}\right|_p, 0\right), \left(0, \left.\frac{\partial}{\partial y^1}\right|_p\right), \ldots \left(0, \left.\frac{\partial}{\partial y^n}\right|_p, \right).$$

By (1) and (2), the linear map (π_{1_*}, π_{2_*}) maps a basis of $T_{(p,q)}(M \times N)$ to a basis of $T_pM \times T_qN$, and is therefore an isomorphism. ☕

Exercise 8.10. Local maxima
A real valued function $f : M \to \mathbb{R}$ on a manifold M is said to have a *local maximum* at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.

(a) Prove that if a differentiable function $f : I \to \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$, then $f'(p) = 0$.

Proof. Let $\{q_n\}, \{r_n\} \subset U$ be such that $q_n < p$ and $p < r_n$ for all n and $\{q_n\}$ and $\{r_n\}$ converge to p. Then

$$f'(p) = \lim_{n\to\infty} \frac{f(q_n) - f(p)}{q_n - p} \geq 0,$$

and

$$f'(p) = \lim_{n\to\infty} \frac{f(r_n) - f(p)}{r_n - p} \leq 0.$$

Hence $f'(p) = 0$. ☕

(b) Prove that a local maximum of a C^∞ function $f : M \to \mathbb{R}$ is a critical point of f . (Hint: Let X_p be a tangent vector in T_pM and let $c(t)$ be a curve in M starting at p with initial vector X_p. Then $f \circ c$ is a real-valued function with a local maximum at 0. Apply (a).)

Proof. Using the hint, we have $(f \circ c)(0) = f(c(0)) = f(p)$. Then

$$0 = (f \circ c)'(0) = (f \circ c)_{*,0} = f_{*,c(0)} \circ c_{*,0} = f_{*,p}c'(0) = f_{*,p}(X_p).$$

Since X_p was an arbitrary tangent vector in T_pM, $f_{*,p} \equiv 0$, and hence it is not surjective. So p is a critical point of f. ☕

Exercise 1. Suppose that $S \subset M$ is a compact regular submanifold, and $F : M \to N$ is smooth, so that $F|_S$ is one-to-one, and for every $p \in S$, $F_{*p} : T_pM \to T_{F(p)}N$ is a linear isomorphism. Prove that there is U open with $S \subset U$ so that $F : U \to N$ is a diffeomorphism onto its image.

Proof. Thank you for your help on this one Jesse!

By the Inverse Function Theorem, F is a local diffeomorphism on S. So at every $p \in S$, there exists a coordinate chart (V_p, ϕ_p) on which F is a diffeomorphism. For all p, replace V_p with $\phi_p^{-1}(B(\phi(p), \epsilon_p))$ for some small ϵ_p. Notice that $\bigcup_{p \in S} V_p$ covers S, and since S is compact, there exists $V_{p_1}, \ldots, V_{p_n}$ such that $S \subset \bigcup_{i=1}^n V_{p_i} =: V$. Then $\overline{V} = \bigcup_{i=1}^n \overline{V_{p_i}}$ is compact since it is a closed subset of a compact space.

Now for all $p \in S$, choose ϵ'_p small enough so that $V'_p := \phi_p^{-1}(B(\phi_p(p), \epsilon'_p)) \subseteq V$. Define $V' := \cup_{p \in S} V'_p \subseteq V$. Note that $\overline{V'} \subset \overline{V}$, and hence $\overline{V'}$ is compact.

We want to show that there exists some open set U containing S and contained in V on which F is injective. Then, by the inverse function theorem, F will be a diffeomorphism on U. To that end, suppose such a U does not exist. Then for all U containing S and contained in V, there exists distinct $x, y \in U$ such that $F(x) = F(y)$. In light of this, we define

$$U_n = \bigcup_{p \in S} \phi_p^{-1}\left(B\left(\phi_p(p), \frac{\epsilon_p}{n}\right)\right)$$

for all $n \in \mathbb{N}$, where the ϵ'_ps are the same ones from before where $V'_p := \phi_p^{-1}(B(\phi_p(p), \epsilon'_p))$. Notice that we have $U_n \subset \overline{U_n} \subset \overline{V'}$ and $U_n \subset V$. For all n, pick distinct $x_n, y_n \in U_n$ such that $F(x_n) = F(y_n)$. So $\{(x_n, y_n)\}$ is a sequence in the compact space[1] $\overline{V'} \times \overline{V'}$, and so there exists a convergent subsequence $\{(x_{n_k}, y_{n_k})\}$ converging to some (x_0, y_0). Notice that by construction of the U_n's, we have $(x_0, y_0) \in S \times S$. Then since F is continuous,

$$F(x_0) = \lim_{k \to \infty} F(x_{n_k}) = \lim_{k \to \infty} F(y_{n_k}) = F(y_0),$$

and since F is injective on S, we have $x_0 = y_0$. Now, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $x_{n_k}, y_{n_k} \in V'_{x_0}$. But $F|_{V'_{x_0}}$ is a diffeomorphism, and in particular, injective, and so $F(x_{n_k}) = F(y_{n_k})$ for all $k \geq K$, a contradiction. ☕

[1] Tychonoff's Theorem!

Exercise 2. Suppose that $F : M \to N$ is smooth. M is compact, N is connected, and at every $p \in M$, $F_{*,p} : T_pM \to T_{F(p)}N$ is an isomorphism. Prove that F is a covering projection.

Proof. Since M is compact and nonempty, N is connected and Hausdorff, and F is continuous and open, then F is surjective.

Since $F_{*,p}$ is an isomorphism for all $p \in M$, then $\dim M = \dim N$; say $\dim M = n$. Since F is a submersion on all of M, then in particular, no point of N is a critical value of F. Hence every point of N is a regular value of F. So if $b \in N$, then by the Regular Level Set Theorem, $S := F^{-1}(\{b\}) \subset M$ is a regular submanifold of dimension $n - n = 0$, i.e., S is a collection of points. Since N is Hausdorff, the set $\{b\}$ is closed, and since F is continuous, S is closed. As a closed subset of a compact space, S is compact, and therefore S is a finite collection of points.

For all $p \in S$, since $F_{*,p}$ is an isomorphism, then F is locally invertible at p by the Inverse Function Theorem, i.e., there exists a neighborhood V_p of p so that $F : V_p \to F(V_p)$ is a diffeomorphism. Shrink each open set in the collection $\{V_p\}_{p\in S}$ if necessary so that they are disjoint to obtain $\{U_p\}_{p\in S}$. Then $S = \bigsqcup_{p\in S} U_p$.

Define $U := \bigsqcup_{p\in S} F(U_p)$. Then U is a neighborhood of b and $F^{-1}(U) = \bigsqcup_{p\in S} U_p$. Moreover, $F|_{U_p}$ is a diffeomorphism for all p. Therefore F is a covering map.

☕

Exercise 3. Suppose that $p : E \to B$ and $q : E' \to B$ are n-dimensional vector bundles, where E, E', B, p, q are all smooth. Suppose that $F : E \to E'$ is a smooth bundle map. That is, $q \circ F = p$ and for any $b \in B$, $F : p^{-1}(b) \to q^{-1}(b)$ is a linear isomorphism. Prove that F is a homeomorphism. *Hint: You need to prove that the inverse is continuous, it suffices to do this in a trivialization.*

Proof. We first show that F is bijective. Every $e \in E'$ is in some fiber $q^{-1}(b)$, and since $F(p^{-1}(b)) = q^{-1}(b)$, then F is surjective. If $F(a) = F(b)$, then this point in E' is in some fiber $q^{-1}(b)$ on which F is injective since F is an isomorphism, and hence $a = b$.

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "p"'] & & E' \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Now by the hint, we need to show that F^{-1} is continuous, i.e., that F is an open map in a trivialization. Let $b \in B$, and let W and V be respective trivializing open sets at b for p and q. Then let $U := W \cap V$ and let $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ and $\psi : q^{-1}(U) \to U \times \mathbb{R}^n$ be the trivializations.

Define $G : U \times \mathbb{R}^n \to \times\mathbb{R}^n$ by $\psi \circ F \circ \varphi^{-1}$. Then $G^{-1} = \varphi \circ F^{-1} \circ \psi^{-1}$, and since φ and ψ^{-1} are continuous, it follows that F^{-1} is continuous precisely when G^{-1} is continuous.

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[r, "\varphi^{-1}"] \arrow[rrr, bend left, "G"] \arrow[drr, bend right, "\pi_U"'] & p^{-1}(U) \arrow[r, "F"] \arrow[dr, "p"] & q^{-1}(U) \arrow[r, "\psi"] \arrow[d, "q"] & U \times \mathbb{R}^n \arrow[dl, bend left, "\pi_U"] \\ & & U & \end{tikzcd}$$

Let $(x, v) \in U \times \mathbb{R}^n$. Then $G(x, v) = (x, A(x) \cdot v)$ for some matrix $A(x)$. We get that G is the identity on the first component since φ^{-1}, F, and ψ are all linear isomorphisms on the fibers and hence preserve base points. So G is also fiber-preserving.

Since φ^{-1}, F, and ψ are all smooth, then so is $A : U \to GL_n(\mathbb{R})$, which means $A^{-1} : U \to GL_n(\mathbb{R})$ is also smooth. Then $G^{-1}(y, w) = (y, (A(y))^{-1} \cdot w)$. Since A^{-1} is smooth, G^{-1} is smooth and so *a fortiori*, G^{-1} continuous.

☕

Exercise 4. Recall that a frame of a vector bundle $p : E \to B$ is a collection of smooth sections $s_1, \dots s_n : B \to E$, that is $p \circ s_i = Id_B$, so that for every $b \in B$, $\{s_1(b), \dots, s_n(b)\}$ is a basis for $p^{-1}(b)$. Prove that $p : E \to B$ is trivial if and only if it admits a smooth frame.

Proof. ($\Rightarrow$) Suppose $p : E \to B$ is trivial; that is, there exists a smooth map $f : E \to B \times \mathbb{R}^n$ such that the following diagram commutes, ($p = \pi_B \circ f$)

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & B \times \mathbb{R}^n \arrow[dl, "\pi_B"] \\ & B & \end{tikzcd}$$

and such that f is a linear isomorphism on each fiber; i.e., for each $b \in B$, $p^{-1}(b) \xrightarrow{\sim} \{b\} \times \mathbb{R}^n$. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$. For each $i \in \{1, \dots, n\}$ define

$$\overline{e}_i : B \to B \times \mathbb{R}^n \quad \text{by} \quad b \mapsto (b, e_i).$$

Then $\{\overline{e}_i\}_i$ defines a smooth frame over the vector bundle $\pi_B : B \times \mathbb{R}^n \to B$ because: $\pi_B \circ \overline{e}_i = \mathbb{1}_B$, and, for all $b \in B$, the set $\{\overline{e}_i(b)\}_i = \{(b, e_i)\}_i$ forms a basis for the vector space $\{b\} \times \mathbb{R}^n$.

Define $s_i : B \to E$ by $s_i = f^{-1} \circ \overline{e}_i$ for all $i \in \{1, \dots, n\}$. Then s_i is smooth since both f^{-1} and $\overline{e}_i$ are smooth, and

$$p \circ s_i = p \circ f^{-1} \circ \overline{e}_i = \pi_B \circ \overline{e}_i = \mathbb{1}_B,$$

which means the s_i are sections of the vector bundle $p : E \to B$. Moreover, for all $b \in B$ and for all $i \in \{1, \dots, n\}$,

$$s_i(b) = f^{-1}(\overline{e}_i) = f^{-1}(b, e_i).$$

Since f is a linear isomorphism, it takes bases to bases, and hence $\{f^{-1}(b, e_i)\}_i = \{s_i(b)\}_i$ is a basis for $p^{-1}(b)$. Hence $\{s_i\}_i$ constitutes a smooth frame for the vector bundle $p : E \to B$.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"] & & B \times \mathbb{R}^n \arrow[dl, "\pi_B"'] \\ & B \arrow[ul, bend left, "s_i"] \arrow[ur, bend right, "\overline{e}_i"'] & \end{tikzcd}$$

($\Leftarrow$) Now suppose $p : E \to B$ admits a smooth frame, say $\{s_i\}_i$. Let $e \in E$ and $p(e) = b$. Then $\{s_i(b)\}_i$ is a basis for the vector space $p^{-1}(b)$ and so e can be written uniquely as $e = \sum a_i s_i(b)$ for some $a_i \in \mathbb{R}$. Define

$$f : E \to B \times \mathbb{R}^n \quad \text{by} \quad e \mapsto (p(e), a_1, \dots, a_n).$$

We first show that f is linear on the fibers of p. Let $b \in B$ and $e, e' \in p^{-1}(b)$ with $e = \sum a_i s_i(b)$ and $e' = \sum c_i s_i(b)$. For $\lambda \in \mathbb{R}$,

$$\begin{aligned} f(e + \lambda e') &= f\left(\sum (a_i + \lambda c_i) s_i(b)\right) \\ &= (b, a_1 + \lambda c_1, \dots, a_n + \lambda c_n) \\ &= (b, a_1, \dots, a_n) + \lambda(b, c_1, \dots, c_n) \\ &= f(e) + \lambda f(e'). \end{aligned}$$

Now if $(b, a_1, \ldots, a_n) \in B \times \mathbb{R}^n$, then

$$f\left(\sum a_i s_i(b)\right) = (b, a_1, \ldots, a_n),$$

and so f is surjective. If $f(e) = (b, 0, \ldots, 0)$, then $e = \sum 0 s_i(b) = 0$, hence f is injective.

Finally, f is fiber-preserving because

$$\pi_B(f(e)) = \pi_B(p(e), a_1, \ldots, a_n) = p(e),$$

i.e., $p = \pi_B \circ f$.

Finally, we show that f is smooth by showing that $\det(F_*) \neq 0$ in a trivializing open set U. Then applying the Inverse Function Theorem, f is smooth on U. So, let ϕ be the trivialization of U and let $\{t_i\}_i$ be the frame of ϕ. That is, for $e \in p^{-1}(U)$, $e = \sum a_i t_i(p(e))$. Now, relative to the char ϕ, we get

$$f_* = \left(\begin{array}{c|c} I & 0 \\ \hline & \\ * & \text{TM} \end{array}\right)$$

where TM stands for "transition matrix", which is the matrix which is the change of basis matrix from $U \times \mathbb{R}^n$ to $U \times \mathbb{R}^n$. Then $\det(F_*) \neq 0$. Therefore, f makes $p : E \to B$ a trivial bundle. ☕

Exercise 5. There is an action of the group $\mathbb{Z}$ on $\mathbb{R}^2$ given by $n.(x,y) = (x+n, (-1)^n y)$. Denote the quotient space $Mob = \mathbb{R}^2/\sim$.

1. Prove that $p : Mob \to S^1$ given by $p([x,y]) = e^{2\pi \boldsymbol{i} x}$ is well defined and continuous.

   Proof. Let $\sim$ be the relation $(x,y) \sim (x',y')$ if and only if there exists $n \in \mathbb{Z}$ such that $(x+n, (-1)^n y) = (x', y')$. Let $q : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ be the quotient map induced by this relation. Define $f : \mathbb{R}^2 \to S^1$ by $(x,y) \mapsto e^{2\pi i x}$. Suppose $(x,y) \sim (x',y')$ with $(x+n, (-1)^n y) = (x', y')$. Then

   $$f(x',y') = e^{2\pi i x'} = e^{2\pi i (x+n)} = e^{2\pi i x} e^{2\pi i n} = e^{2\pi i x} = f(x,y),$$

   which means f is constant on the fibers above $\mathbb{R}^2/\sim$. Hence f descends to a map $\tilde{f} : \mathbb{R}^2/\sim \to S^1$ which makes the following diagram commute:

   $$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[d, "q"'] & S^1 \\ \mathbb{R}^2/\sim \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

   So $f = \tilde{f} \circ q$, and for all $[x,y] \in \mathbb{R}^2/\sim$, we have

   $$\tilde{f}([x,y]) = f(q^{-1}([x,y])) = e^{2\pi i x}.$$

   Notice that $p = \tilde{f}$, which is well defined. Moreover, because f is continuous, then so is p.

2. Prove that $p : Mob \to S^1$ is a line bundle by supplying local trivializations.

3. Prove that $p : Mob \to S^1$ is not trivial by proving it has no nonvanishing sections. *Use the intermediate value theorem on* $[0,1]$ *where we map* $[0,1]$ *to* S^1 *using* $exp(2\pi \mathbf{i} x)$.

Exercise 1. Let $p : E \to B$ be a smooth n-plane bundle over the smooth manifold B. The fibers $p^{-1}(b)$ are all n-dimensional vector spaces over $\mathbb{R}$. A **metric** on E is an assignment of an innerproduct

$$<\ ,\ >_b : p^{-1}(b) \times p^{-1}(b) \to \mathbb{R}. \tag{1}$$

This means the pairings $<\ ,\ >_b$ are bilinear, symmetric and positive definite. Also the pairings are required to vary smoothly. If $U \subset B$, a **local frame** is a collection of smooth sections $s_1, \dots, s_n : U \to E$, so that at each $b \in U$, $s_1(b), \dots, s_n(b)$ are a basis for $p^{-1}(b)$. We form an $n \times n$ matrix valued function $g : U \to M_n(\mathbb{R})$ whose entries are

$$g_{ij}(c) = < s_i(c), s_j(c) >_c .$$

We say the metric is **smooth** if for every $b \in B$ there is U open with $b \in U$ and a local frame $s_1, \dots, s_n$ so that the entries of the matrix g are smooth functions on U.

Prove that if $p : E \to B$ is a smooth n-plane bundle then it always admits a smooth metric. *Hint: Partition of unity. You need to check the convex sum of inner products is an inner product.*

Proof. We first argue that the vector bundle $p : E \to B$ admits local frames, for the purpose of defining local inner products, which will then help us to define a metric on all of E.

If (U, φ) is a trivialization of E over B, then the map $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ (by definition) causes the bundle $p : p^{-1}(U) \to U$ to become trivial over U; that is, the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times \mathbb{R}^n \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

and φ is a linear isomorphism on each fiber; i.e., for each $b \in U$, $p^{-1}(b) \xrightarrow{\sim} \{b\} \times \mathbb{R}^n$. We saw in the previous homework that a trivial bundle always admits a smooth frame $\{s_i\}$. In particular, we showed that $s_i : U \to p^{-1}(U)$ is defined by $s_i = \varphi^{-1} \circ \overline{e}_i$, where $\overline{e}_i : U \to U \times \mathbb{R}^n$ is given by $b \mapsto (b, e_i)$ and where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$.

For $b \in U$, we want to define an inner product on $p^{-1}(b)$. Well, we already have a standard inner product (the dot product) on elements of $\mathbb{R}^n$ and an isomorphism $p^{-1}(b) \xrightarrow{\sim} \{b\} \times \mathbb{R}^n$. It therefore seems reasonable to define an inner product $<\ ,\ >_U$ on a fibers above points in U in terms of the dot product in $\mathbb{R}^n$. To that end, if $e, e' \in p^{-1}(b)$, then we can write $e = \sum a^i s_i(b)$ and $e' = \sum c^i s_i(b)$ for unique $a_i, c_i \in \mathbb{R}$. Then

$$(\pi_{\mathbb{R}^n} \circ \varphi)(e) = \pi_{\mathbb{R}^n}(\varphi(e)) = \pi_{\mathbb{R}^n}(b, c^1, \dots, c^n) = (c^1, \dots, c^n),$$

where $\pi_{\mathbb{R}^n}$ is projection onto $\mathbb{R}^n$. So we define $<\ ,\ >_{U,b}$ by

$$< e, e' >_{U,b} = (\pi_{\mathbb{R}^n} \circ \varphi)(e) \cdot (\pi_{\mathbb{R}^n} \circ \varphi)(e') = \sum a^i c^i,$$

where $\cdot$ denotes the usual dot product in $\mathbb{R}^n$. Given this definition, we want to check that the entries $g_{ij}(b) =< s_i(b), s_j(b) >_{U,b}$ are smooth functions on U:

$$\begin{aligned}
< s_i(b), s_j(b) >_{U,b} &= (\pi_{\mathbb{R}^n} \circ \varphi)(s_i(b)) \cdot (\pi_{\mathbb{R}^n} \circ \varphi)(s_j(b)) \\
&= (\pi_{\mathbb{R}^n} \circ \varphi \circ s_i)(b) \cdot (\pi_{\mathbb{R}^n} \circ \varphi \circ s_j)(b) \\
&= (\pi_{\mathbb{R}^n} \circ \varphi \circ \varphi^{-1} \circ \overline{e}_i)(b) \cdot (\pi_{\mathbb{R}^n} \circ \varphi \circ \varphi^{-1} \circ \overline{e}_j)(b) \\
&= (\pi_{\mathbb{R}^n} \circ \overline{e}_i)(b) \cdot (\pi_{\mathbb{R}^n} \circ \overline{e}_j)(b) \\
&= \pi_{\mathbb{R}^n}(b, e_i) \cdot \pi_{\mathbb{R}^n}(b, e_j) \\
&= \delta_i^j.
\end{aligned}$$

Now that we've defined a metric locally on fibers above points in a trivializing open set, we want to extend our definition to a global metric on all of E. First, cover B by trivializing open sets $\{U_\alpha\}$, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Define $< \, , \, >_b \colon p^{-1}(b) \times p^{-1}(b) \to \mathbb{R}$ on E by

$$< \, , \, >_b := \sum_\alpha \rho_\alpha(b) < \, , \, >_{U_\alpha} = \sum_{\alpha,\ b\in U_\alpha} \rho_\alpha(b) < \, , \, >_{U_\alpha,b} = \sum_{\alpha,\ b\in U_\alpha} < \, , \, >_{U_\alpha,b}$$

Then define a metric on E which assigns to each $b \in B$ the inner product $< \, , \, >_b$. This metric is positive definite since the sum of nonnegative scalings of positive definite inner products is positive definite. This metric is symmetric since

$$< e, e' >_b = \sum_{\alpha,\ b\in U_\alpha} < e, e' >_{U_\alpha,b} = \sum_{\alpha,\ b\in U_\alpha} < e', e >_{U_\alpha,b} =< e', e >_b \, .$$

Finally, the metric is bilinear since

$$\begin{aligned}
< e + \lambda e'', e' >_b &= \sum_{\alpha,\ b\in U_\alpha} < e + \lambda e'', e' >_{U_\alpha,b} \\
&= \sum_{\alpha,\ b\in U_\alpha} < e, e' >_{U_\alpha,b} + < \lambda e'', e' >_{U_\alpha,b} \\
&=< e, e' >_b + \lambda < e'', e' > .
\end{aligned}$$

(And similarly for the second factor of the pairing). ☕

Exercise 2. Suppose that $F : M \to N$ is smooth and $p : E \to N$ is a smooth bundle define

$$F^*E = \{(m, \vec{v}) \in M \times E \mid F(m) = p(\vec{v})\}$$

Define $p_1 : F^*E \to M$ by $p_1(m, \vec{v}) = m$. Use local trivializations $U \subset N$ of $p : E \to N$ to build local trivializations on $p_1 : F^*E \to M$ so that the coordinate changes are smooth.

Proof. Let (U, φ) be a trivialization of E over N. Since F is continuous, $F^{-1}(U) \subseteq M$ is open. We want to define a homeomorphism

$$\psi : p_1^{-1}(F^{-1}(U)) \to F^{-1}(U) \times \mathbb{R}^n.$$

so that $(F^{-1}(U), \psi)$ can serve as a local trivialization on $p_1 : F^*E \to M$. To that end, we make the following observation: If $(m, \vec{v}) \in p_1^{-1}(F^{-1}(U))$, then

$$m = p_1(m, \vec{v}) \in F^{-1}(U),$$

i.e., $F(m) \in U$. Hence $p^{-1}(F(m)) \subseteq p^{-1}(U)$ and since $p(\vec{v}) = F(m)$, we have $\vec{v} \in p^{-1}(p(v)) \subseteq p^{-1}(U)$. Since φ is defined on $p^{-1}(U)$, the above observation motives the following definition for ψ:

$$\psi(m, \vec{v}) = (m, \pi_{\mathbb{R}^n}(\varphi(\vec{v})).$$

Let π_M, π_E be projection onto M and E. Notice that $\psi = (\pi_M, \pi_{\mathbb{R}^n} \circ \varphi \circ \pi_E)$, and since all of the component functions of ψ are continuous, so is ψ. Now, we define ψ^{-1} and show that it is continuous and that it is indeed an inverse for ψ. Let

$$\psi^{-1} : F^{-1}(U) \times \mathbb{R}^n \to p_1^{-1}(F^{-1}(U))$$

be given by $\psi^{-1}(k, \vec{u}) = (k, \varphi^{-1}(F(k), \vec{u}))$. Then as before, ψ^{-1} is continuous since all of its component functions are continuous.

Interlude: We want to check that $(k, \varphi^{-1}(F(k), \vec{u}))$ is actually an element of $p_1^{-1}(F^{-1}(U)) \subseteq F^*E$. Note that since $\varphi|_{p^{-1}(\{F(k)\})} : p^{-1}(\{F(k)\}) \to \{F(k)\} \times \mathbb{R}^n$ is a linear map, then

$$\varphi^{-1}(F(k), \vec{u}) \in p^{-1}(\{F(k)\}),$$

and hence $p(\varphi^{-1}(F(k), \vec{u})) = F(k)$. In other words, the element $(k, \varphi^{-1}(F(k), \vec{u}))$ is indeed in F^*E. Moreover,

$$p_1(k, \varphi^{-1}(F(k), \vec{u})) = k \in F^{-1}(U),$$

and so $(k, \varphi^{-1}(F(k), \vec{u}))$ is an element of $p_1^{-1}(F^{-1}(U))$.

Then we have:

$$(\psi \circ \psi^{-1})(n, \vec{u}) = \psi(n, \varphi^{-1}(F(n), \vec{u})) = \Big(n, \pi_{\mathbb{R}^n}(F(n), \vec{u})\Big) = (n, \vec{u}).$$

Note that $\varphi(\vec{v}) = (p(\vec{v}), \pi_{\mathbb{R}^n}(\varphi(\vec{v}))$. So,

$$\begin{aligned}
(\psi^{-1} \circ \psi)(m, \vec{v}) &= \psi^{-1}\Big(m, \pi_{\mathbb{R}^n}(\varphi(\vec{v}))\Big) \\
&= \Big(m, \varphi^{-1}\Big(F(m), \pi_{\mathbb{R}^n}(\varphi(\vec{v}))\Big)\Big) \\
&= \Big(m, \varphi^{-1}\Big(p(\vec{v}), \pi_{\mathbb{R}^n}(\varphi(\vec{v}))\Big)\Big) && (F(m) = p(\vec{v})) \\
&= \Big(m, \varphi^{-1}(\varphi(\vec{v}))\Big) \\
&= (m, \vec{v}).
\end{aligned}$$

Hence ψ is a homeomorphism. We now show that ψ is a linear isomorphism on the fibers above $F^{-1}(U)$; that is, if $m \in F^{-1}(U)$, then the map

$$\psi|_{p_1^{-1}(\{m\})} : p_1^{-1}(\{m\}) \to \{m\} \times \mathbb{R}^n$$

is a linear isomorphism. If $(m, \vec{v}) \in p_1^{-1}(\{m\})$, then in particular $p(\vec{v}) = F(m)$. So when we consider the definition of ψ, we get that

$$\psi|_{p_1^{-1}(\{m\})} = \Big(\pi_M, (\pi_{\mathbb{R}^n} \circ \varphi)|_{p^{-1}(\{F(m)\})}\Big),$$

and hence $\psi|_{p_1^{-1}(\{m\})}$ is a linear isomorphism since $\varphi|_{p^{-1}(\{F(m)\})}$ is a linear isomorphism.

Now, if $\{(U_\alpha, \varphi_\alpha)\}$ is a collection of trivializations of E over N, let $\{(F^{-1}(U_\alpha), \psi_\alpha)\}$ be a collection of homeomorphisms, where ψ_α corresponds to φ_α and is defined as in our construction above. If $F^{-1}(U_\alpha), F^{-1}(U_\beta)$ are two overlapping open sets in the collection $\{(F^{-1}(U_\alpha), \psi_\alpha)\}$, we want to show that the coordinate change

$$\psi_\alpha \circ \psi_\beta^{-1} : F^{-1}(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to F^{-1}(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is smooth. Notice that

$$\psi_\alpha \circ \psi_\beta^{-1} = \Big(\pi_1, \pi_{\mathbb{R}^n} \circ \varphi_\alpha \circ \varphi_\beta(F \circ \pi_1 . \pi_{\mathbb{R}^n})\Big),$$

We know that $\varphi_\alpha \circ \varphi_\beta$ is smooth. This shows that the component functions of $\psi_\alpha \circ \psi_\beta^{-1}$ are smooth, and hence so is $\psi_\alpha \circ \psi_\beta^{-1}$.

Exercise 3. Suppose that $p : E \to B$ is a vector bundle. We say $E' \subset E$ is a subbundle of dimension k if for every $b \in B$, $E' \cap p^{-1}(b)$ is a vector subspace of dimension k and for each $b \in B$ there is U open with $b \in U$ and k smooth sections $s_1, \dots, s_k : U \to E$ so that for each $c \in U$, $s_1(c), \dots, s_k(c)$ form a basis of $E' \cap p^{-1}(c)$. Prove that if E' is a subbundle of E there is a subbundle E'' of E so that $E' \oplus E'' = E$ in the sense at each $b \in B$, $(E' \cap p^{-1}(b)) \oplus (E'' \cap p^{-1}(b)) = p^{-1}(b)$. *Hint: Use a metric, and orthogonal projection into the perpendicular.*

Notice that the restriction of the projection map to a subbundle is a bundle in its own right.

Proof. Let $b \in B$. By the same argument employed at the outset of Exercise 1, the vector bundle $p : E \to B$ admits local frames; that is, there exists a neighborhood U of b so that there exists n smooth sections $s_i : U \to p^{-1}(U)$, $1 \leq i \leq n$, such that $s_1(b), \dots, s_n(b)$ is a basis for the vector space $p^{-1}(b)$. Applying Gram-Schmidt orthogonalization, we assume without loss of generality that $s_1(b), \dots, s_n(b)$ is an orthonormal basis for $p^{-1}(b)$ and that $s_1(b), \dots, s_k(b)$ form a basis of $E' \cap p^{-1}(b)$.

By Exercise 1, we have a smooth metric defined on E. So, we define the orthogonal complement of E' in E:

$$E'' := \{e'' \in E \ | < e'', e' >_{p(e'')} = 0 \text{ for all } e' \in E' \cap (p^{-1}(p(e'')))\}.$$

Certainly we have $(E' \cap p^{-1}(b)) + (E'' \cap p^{-1}(b)) \subseteq p^{-1}(b)$. Now if $e \in p^{-1}(b)$, then there exists real numbers λ_i, $1 \leq i \leq n$ so that $e = \lambda_1 s_1(b) + \cdots + \lambda_n s_n(b)$. Define

$$e' := \sum_{i=1}^{k} \lambda_i s_i(b) \in E' \cap p^{-1}(b),$$

and $e'' := e - e'$. For notational brevity, let s_j denote $s_j(b)$ in the next computation. Recall that since $s_1, \dots, s_n$ is an orthonormal basis, $< s_i, s_j >= \delta_i^j$. For any $j \in \{1, \dots, k\}$

$$\begin{aligned} < e'', s_j >_b &= < e, s_j >_b - < e', s_j >_b \\ &= < e, s_j >_b - \lambda_1 < s_1, s_j >_b - \cdots - \lambda_j < s_j, s_j >_b - \cdots - \lambda_k < s_k, s_j >_b \\ &= < e, s_j >_b - \lambda_j \\ &= 0, \end{aligned}$$

i.e., $e'' \in E'' \cap p^{-1}(b)$ and so $e = e' + e'' \in (E' \cap p^{-1}(b)) + (E'' \cap p^{-1}(b))$, which gives $(E' \cap p^{-1}(b)) + (E'' \cap p^{-1}(b)) = p^{-1}(b)$.

Finally, it follows from the definition of E'' that $(E' \cap p^{-1}(b)) \cap (E'' \cap p^{-1}(b)) = \{0\}$. Hence we've shown $(E' \cap p^{-1}(b)) \oplus (E'' \cap p^{-1}(b)) = p^{-1}(b)$.

Exercise 4. Suppose that $i : M \to N$ is the embedding of a smooth regular submanifold of codimension k. Suppose further than TN has a smooth metric on it. Define the normal space ν_p to $T_{i(p)}(i(M))$ to be the linear subspace of $T_{i(p)}N$ that is perpendicular to $T_{i(p)}(i(M))$. The normal bundle ν is the subbundle of i^*TN whose fiber over p is ν_p. Prove that $i^*TN = (i^*T(i(M))) \oplus \nu$. **Hint: Use the last problem.**

Proof. We have the following diagram:

$$\begin{tikzcd} i^*TN \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "i"'] & N \end{tikzcd}$$

Let $p \in M$. Since $i^*Ti(M)$ and ν are subbundles of i^*TN, we want to use Exercise 3 to show

$$\pi_1^{-1}(p) = \Big(\pi_1^{-1}(p) \cap i^*Ti(M)\Big) \oplus \Big(\pi_1^{-1}(p) \cap \nu\Big).$$

First, we consider the set

$$\pi_1^{-1}(p) = \{(p, \vec{v}) \in i * TN \ \mid \vec{v} \in TN, i(p) = \pi(\vec{v})\}. \qquad (🦇)$$

By definition of the map π, we have that $\vec{v} \in T_{\pi(\vec{v})}N$ for all $\vec{v} \in TN$. So if $(p, \vec{v}) \in \pi_1^{-1}(p)$, then $\vec{v} \in T_{i(p)}N$. Now since $T_{i(p)}i(M)$ is a subspace of $T_{i(p)}N$, then we can write

$$T_{i(p)}N = T_{i(p)}i(M) \oplus (T_{i(p)}i(M))^\perp = T_{i(p)}i(M) \oplus \nu_p.$$

In particular, if $\vec{v} \in T_{i(p)}N$, there exists unique elements $v_1 \in T_{i(p)}i(M)$ and $v_2 \in \nu_p$ such that $\vec{v} = v_1 + v_2$. So then (🦇) becomes

$$\begin{aligned} \pi_1^{-1}(p) &= \{(p, v_1 + v_2) \in i^*TN \ \mid v_1 \in T_{i(p)}i(M), v_2 \in \nu_p, i(p) = \pi(\vec{v})\} \\ &= \{(p, v_1) + (p, v_2) \in i^*TN \ \mid \text{“}\rule{6cm}{0.4pt}\text{”}\} \\ &= \Big(\{p\} \times T_{i(p)}i(M)\Big) \oplus \Big(\{p\} \times \nu_p\Big) \\ &= \Big(\left(\pi_1|_{i^*Ti(M)}\right)^{-1}(p)\Big) \oplus \Big(\left(\pi_1|_\nu\right)^{-1}(p)\Big) \\ &= \Big(\pi_1^{-1}(p) \cap i^*Ti(M)\Big) \oplus \Big(\pi_1^{-1}(p) \cap \nu\Big). \end{aligned}$$

☕

Exercise 5. Assume that $M \subset \mathbb{R}^N$ is a smooth regular submanifold, and that M is compact. Let $p : \nu \to M$ be the normal bundle. The elements of ν consist or ordered pairs $(m, \vec{v})$ where $m \in M$ and $\vec{v} \in T_m M^{\perp}$. Define $F : \nu \to \mathbb{R}^N$ given by $F(m, \vec{v}) = m + \vec{v}$. Show that F can be restricted to an open subset $U \subset \nu$ containing M as the zero section, such that $F|_U$ is a diffeomorphism onto an open neighborhood of M in $\mathbb{R}^N$. *Hint: Your last homework.*

The last exercise is the **Regular Neighborhood Theorem** Here we can make a tube lemma type argument to show we can choose U so that in each fiber of ν it is an open ϵ ball.

Proof. Define $M_0 := \{(m, 0) \mid m \in M\} = s_0(M) \subset \nu$, i.e., M_0 is a copy of M in ν as the zero section. Notice that since the smooth section s_0 is continuous and M is compact, then so is M_0. We want to show that the map F has the following properties: (1) F is smooth on ν; (2) $F|_{m_0}$ is injective; (3) For $m \in M$, $F_{*,(m,0)} : T_{m,0}\nu \to T_m\mathbb{R}^N$ is an isomorphism. Then using the first exercise of the previous homework, the result follows.

(1) follows from the fact that vector addition in $\mathbb{R}^N$ is smooth. (2) follows trivially from the fact that $m + 0 = n + 0 \iff m = n$. For (3), it suffices to show that locally in a trivialization say φ, we have $(F \circ \varphi)_{*,(m,0)}$ is an isomorphism. To that end, we show $\det((F \circ \varphi)_{*,(m,0)})$ is nonzero. Then the inverse function theorem gives that $(F \circ \varphi)_{*,(m,0)}$ is an isomorphism. We have

$$F \circ \varphi^{-1} : M \times \mathbb{R}^{N-k} \to \mathbb{R}^N \quad \text{where} \quad (m, w) \mapsto m + \Big((\pi_2 \circ \varphi)|_{p^{-1}(m)}\Big)^{-1}(w).$$

Now, defining $\varphi_2 := \pi_2 \circ \varphi$, we have

$$(F \circ \varphi)_{*,(m,0)} = \left(\begin{array}{c|c} I_k & 0 \\ \hline * & (\varphi_2|_{p^{-1}(m)})^{-1} \end{array}\right),$$

and hence $\det(F \circ \varphi)_{*,(m,0)})$ is nonzero. ☕

Intro to Manifolds, Tu – *End of Section Exercises*

Exercise 15.9. Structure of a general linear group

(a) For $r \in \mathbb{R}^\times := \mathbb{R} - \{0\}$, let M_r be the matrix

$$\begin{bmatrix} r & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = [r\ e_1\ e_2 \dots\ e_n],$$

where $e_1, e_2, \dots, e_n$ is the standard basis for $\mathbb{R}^n$. Prove that the map

$$\begin{aligned} f : GL(n,\mathbb{R}) &\to SL(n,\mathbb{R}) \times \mathbb{R}^\times, \\ A &\mapsto (AM_{1/\det A}, \det A) \end{aligned}$$

is a diffeomorphism.

Proof. Since the component functions of f are matrix multiplication and the determinant function, which are smooth, then f is smooth. Define a map

$$\begin{aligned} f^{-1} : SL(n,\mathbb{R}) \times \mathbb{R}^\times &\to GL(n,\mathbb{R}), \\ (B, r) &\mapsto (M_r B). \end{aligned}$$

Then f^{-1} is smooth since it is defined by matrix multiplication and in an inverse for f since

$$\begin{aligned} (f^{-1} \circ f)(A) = f^{-1}(AM_{1/\det A}, \det A) &= M_{\det A} A M_{1/\det A} \\ &= [\det A e_1\ e_2 \dots\ e_n][(A/\det A)e_1\ e_2 \dots\ e_n] \\ &= [Ae_1\ Ae_2 \dots\ Ae_n] \\ &= A, \\ (f \circ f^{-1})(B, r) = f(M_r B) &= ((M_r B) M_{1/\det M_r B}, \det M_r B) \\ &= ((M_r B) M_{1/r \det B}, \det M_r B) \\ &= ([re_1\ e_2 \dots\ e_n] B [(1/r)e_1\ e_2 \dots\ e_n], r) \\ &= ([rB_1 e_1\ B_2 e_2 \dots\ B_2 e_n][(1/r)e_1\ e_2 \dots\ e_n], r) \\ &= (B, r). \end{aligned}$$

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(b) Show that the center of $GL(2,\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$, corresponding to the subgroup of scalar matrices, and that the center of $SL(2,\mathbb{R})\times\mathbb{R}^\times$ is isomorphic to $\{\pm 1\}\times\mathbb{R}^\times$. The group $\mathbb{R}^\times$ has two elements of order 2, while the group $\{\pm1\}\times\mathbb{R}^\times$ has four elements of order 2. Since their centers are not isomorphic, $GL(2,\mathbb{R})$ and $SL(2,\mathbb{R})\times\mathbb{R}^\times$ are not isomorphic as groups.

Proof. An element of the center of $GL(2,\mathbb{R})$ is a scalar multiple of the identity and hence we define a map $\varphi : \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mapsto a$, which is clearly bijective and is linear since

$$\varphi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = ab = \varphi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\varphi\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

Hence we have an isomorphism. Now, the center of $SL(2,\mathbb{R})\times\mathbb{R}^\times$ are elements of the form $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, r\right)$ and $\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, r\right)$ for $r\in R^\times$, which is clearly isomorphic to $\{\pm1\}\times\mathbb{R}^\times$.

(c) Show that $h : GL(3,\mathbb{R}) \to SL(3,\mathbb{R})\times\mathbb{R}^\times$ given by $A \mapsto \big((\det A)^{-1/3}A, \det A\big)$ is a Lie group isomorphism.

Proof. We have

$$\begin{aligned} h(AB) &= \Big((\det(A)\det(B))^{-1/3}AB, \det(A)\det(B)\Big) \\ &= \Big(\det(A)^{-1/3}A, \det(A)\Big)\Big(\det(B)^{-1/3}B, \det(B)\Big) \\ &= h(A)h(B), \end{aligned}$$

and hence h is a homomorphism. If $\big((\det A)^{-1/3}A, \det A\big) = \big((\det B)^{-1/3}B, \det B\big)$, then $\det A = \det B$ and $\det A)^{-1/3}A = \det B)^{-1/3}B$, which together give $A = B$ and hence h is injective. Let $(B, r) \in SL(3,\mathbb{R})\times\mathbb{R}^\times$. Then let $A := r^{1/3}B$. Then $\det(A) = r$ and

$$h(A) = (\det(A)^{-1/3}r^{1/3}B, \det A) = (r^{-1/3}r^{1/3}B, r) = (B, r).$$

Exercise 15.10. Orthogonal group

Show that the orthogonal group $O(n)$ is compact by proving that it is a closed and bounded subset of $R^{n\times n}$.

Proof. We have that $O(n)$ is closed since it is the preimage of the closed set $\{I\}$ of under the continuous map $A \mapsto AA^t$. If $A = (a_{ij}) \in O(n)$, then $(AA^T)_{jj} = \sum_{k=1}^n a_{jk}a_{kj} = \sum_{k=1}^n a_{kj}^2$, and since $AA^T = I$, then $\sum_{k=1}^n a_{kj}^2 = 1$. Hence

$$||A|| = \left(\sum_{i=1}^n\sum_{k=1}^n a_{kj}^2\right)^{1/2} = \left(\sum_{i=1}^n 1)\right)^{1/2} = n^{1/2},$$

and hence $||A||$ is bounded.

Exercise 15.11. Special orthogonal group $SO(2)$

The special orthogonal group $SO(n)$ is defined to be the subgroup of $O(n)$ consisting of matrices of determinant 1. Show that every matrix $A \in SO(2)$ can be written in the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ. Then prove that $SO(2)$ is diffeomorphic to the circle S^1.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(3)$. Then

$$I = AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ca+db & c^2+d^2 \end{bmatrix},$$

which gives $a^2+b^2 = 1 = c^2+d^2$. So there exists $\theta, \phi \in [0, 2\pi)$ so that $\cos\theta = a$, $-\sin\theta = b$, $\sin\phi = c$, and $d = \cos\phi$. Since $ac+bd = 0$, we get

$$0 = \cos\theta\sin\phi - \sin\theta\cos\phi = \sin(\phi - \theta),$$

which gives $\phi - \theta = 0$, i.e., $\phi = \theta$. So A becomes

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Conversely, we have

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^\theta + \cos^2\theta \end{bmatrix} = I,$$

and hence every matrix in $SO(3)$ has the desired form. Now since every point in S^1 can be written in the form $\cos^2\theta + \sin^2\theta$, we define a map

$$\begin{aligned} f : SO(2) &\to S^1 \\ (\cos\theta, \sin\theta, -\sin\theta, \cos\theta) &\mapsto (\cos\theta, \sin\theta), \end{aligned}$$

where we are identifying the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with the 4-tuple $(\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$, and a point $p = \cos^2\theta + \sin^2\theta \in S^1$ with the pair $(\cos\theta, \sin\theta)$. Then f is simply the restriction of the projection map on the first two factors, which is smooth. Then if we define

$$\begin{aligned} f^{-1} : S^1 &\to SO(2) \\ (\cos\theta, \sin\theta) &\mapsto (\cos\theta, \sin\theta, -\sin\theta, \cos\theta), \end{aligned}$$

then f^{-1} is an inverse map for f. Then $f^{-1} = (\pi_1, \pi_2, \ell_{-1} \circ \pi_2, \pi_1)$, where $\ell_{-1} : \mathbb{R} \to \mathbb{R}$ is left multiplication by -1. So f^{-1} is smooth since all of its component functions are smooth. ☕

Exercise 15.12. Unitary Group

The unitary group $U(n)$ is defined to be

$$U(n) = \{A \in GL(n, \mathbb{C}) \mid \overline{A}^T A = I\},$$

where A denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of $A : (\overline{A})_{ij} = \overline{a_{ij}}$. Show that $U(n)$ is a regular submanifold of $GL(n, \mathbb{C})$ and that $\dim U(n) = n^2$.

Proof. Using the map $f : GL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ given by $A \mapsto \overline{A}^T A$, we have that $f^{-1}(I) = U(n)$. Since $U(n)$ is a subgroup of $GL(n, \mathbb{C})$, f is continuous, and $\{I\}$ is closed, then $U(n)$ is a closed subgroup, and hence an embedded Lie subgroup .

To find the dimension of $U(n)$, we find the dimension of the tangent space at the identity. Let $X \in T_I U(n)$, and choose a curve $c : (-\epsilon, \epsilon) \to U(n)$ starting at I with $c'(0) = X$. Then $\overline{c(t)}^T c(t) = I$ for all t, and so applying the matrix product rule we have

$$0 = \overline{c'(t)}^T c(t) + \overline{c(t)}^T c'(t).$$

At $t = 0$ we have $\overline{X}^T = -X$. Thus X is skew-Hermitian. Since skew-Hermitian matrices are completely determined by their entries in the upper triangle and their diagonal entries, we have

$$\frac{\#(\text{matrix entries}) - \#(\text{diagonal entries})}{2} = \frac{n^2 - n}{2}$$

complex numbers to choose for in the upper triangle, which is the same as choosing $n^2 - n$ real numbers. Now since we have $\overline{X}^T = -X$ for all skew-Hermitian X, this means that the diagonal entries must be purely imaginary. Hence we have $n^2 - n + n = n^2$ choices to determine X. Hence the dimension of $T_I U(n)$ is n^2 and hence $\dim U(n) = n^2$. ☕

Exercise 15.15. Symplectic Group

Let $\mathbb{H}$ be the skew field of quaternions. The symplectic group $Sp(n)$ is defined to be

$$Sp(n) = \{A \in GL(n, \mathbb{H}) \mid \overline{A}^T A = I\},$$

where $\overline{A}$ denotes the quaternionic conjugate of A. Show that $Sp(n)$ is a regular submanifold of $GL(n, \mathbb{H})$ and compute its dimension.

Proof. This proof is essentially the same as the one in Exercise 15.12, except when we calculate the dimension of $T_I Sp(n)$:

Since skew-$\mathbb{H}$ matrices are completely determined by their entries in the upper triangle and their diagonal entries, we have

$$\frac{\#(\text{matrix entries}) - \#(\text{diagonal entries})}{2} = \frac{n^2 - n}{2}$$

quaternion numbers to choose for in the upper triangle, which is the same as choosing $4((n^2 - n)/2) = 2(n^2 - n)$ real numbers. Now since we have $\overline{X}^T = -X$ for all skew-$\mathbb{H}$ X, this means that the diagonal entries must be purely imaginary. The imaginary component of elements in $\mathbb{H}$ consist of three components. Hence we have $2(n^2-n)+3n = 2n^2+n$ choices to determine X. Hence the dimension of $T_I Sp(n)$ is $2n^2+n$ and hence $\dim Sp(n) = 2n^2+n$. ☕

Exercise 16.1. Skew-Hermitian matrices

A complex matrix $X \in C^{n\times n}$ is said to be *skew-Hermitian* if its conjugate transpose $\overline{X}^T$ is equal to $-X$. Let V be the vector space of $n \times n$ skew-Hermitian matrices. Show that $\dim V = n^2$.

Proof. See Exercise 15.12. ☕

Exercise 16.2. Lie algebra of a unitary group

Show that the tangent space at the identity I of the unitary group $U(n)$ is the vector space of $n \times n$ skew-Hermitian matrices.

Proof. See Exercise 15.12. ☕

Exercise 16.3. Lie algebra of a symplectic group

Show that the tangent space at the identity I of the symplectic group $Sp(n) subset GL(n, \mathbb{H})$ is the vector space of all $n \times n$ quaternionic matrices X such that $\overline{X}^T = -X$.

Proof. See Exercise 15.15. ☕

Intro to Manifolds, Tu – *End of Section Exercises*

Exercise 17.1. A 1-form on $\mathbb{R}^2 - \{(0,0)\}$

Let the standard coordinates on $\mathbb{R}^2$ by x, y and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad \text{and} \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 - \{(0,0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.
Solution:
Let $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}$. Then

$$\omega(X) = \left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = \frac{(-y)^2}{x^2+y^2} + \frac{x^2}{x^2+y^2} = 1$$
$$\omega(Y) = \left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \frac{-yx}{x^2+y^2} + \frac{xy}{x^2+y^2} = 0.$$

Exercise 17.4. Liouville form on the cotangent bundle

(a) Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold M, and let

$$(\pi^{-1}U, \tilde{\phi}) = (\pi^{-1}U, \overline{x}^1, \ldots, \overline{x}^n, c_1, \ldots, c_n)$$

be the induced chart on the cotangent bundle T^*M. Find a formula for the Liouville form λ on $\pi^{-1}U$ in terms of the coordinates $\overline{x}^1, \ldots, \overline{x}^n, c_1, \ldots, c_n$.
Solution:
Let π^* denote the dual of the differential of the projection $\pi : T^*M \to M$, i.e., $\pi^* = (\pi_*)^\vee$. Let $p \in U$ and $\omega \in T_p^*U$. Then $\lambda_{\omega(p)} = \omega_p \circ \pi_* = \pi^*(\omega_p)$. Let $\omega = \sum c_i dx^i$. Now, using the fact that $\pi^*(dx^i) = d(\pi^* x^i) = d(x^i \circ \pi) = d\overline{x}^i$, we have

$$\lambda_{\omega(p)} = \pi^*(\omega_p) = \pi^*\left(\sum c_i dx^i\right) = \sum c_i \pi^*(dx^i) = \sum c_i d\overline{x}^i.$$

(b) Prove that the Liouville form λ on T^*M is C^∞. (*Hint:* Use (a) and Proposition 17.6)
Solution:
We can write $\lambda_{\omega(p)} = \sum c_i d\overline{x}^i + \sum 0 dc^i$. As coordinate functions, the coefficients c^i are all smooth relative to the frame $\{d\overline{x}^i, dc_i\}$ and so λ is smooth.

Exercise 18.2. Linearity of the pullback

Prove Proposition 18.9

Proposition 1. *If $F : M \to N$ is a C^∞ map. If ω, τ are k-forms on M and a is a real number, then*

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$;

(ii) $F^*(a\omega) = aF^*\omega$.

Proof. Let $p \in M$ and $X_1, \dots, X_k \in T_pM$. Then

$$\begin{aligned}
(F^*(\omega+\tau))_p(X_1,\dots X_k) &= (\omega+\tau)_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_k)\\
&= (\omega_{F(p)}+\tau_{F(p)})(F_{*,p}X_1,\dots,F_{*,p}X_k)\\
&= \omega_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_k)+\tau_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_k)\\
&= (F^*\omega)_p(X_1,\dots X_k)+(F^*\tau)_p(X_1,\dots X_k).\\
(F^*(a\omega))_p(X_1,\dots X_k) &= (a\omega)_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_k)\\
&= a(\omega_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_k))\\
&= a(F^*\omega)_p(X_1,\dots X_k).
\end{aligned}$$

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Exercise 18.3. Pullback of a wedge product

Prove Proposition 18.11

Proposition 2. *If $F : M \to N$ is a C^∞ map of manifolds and ω and τ are differential forms on M, then*

$$F^*(\omega\wedge\tau) = F^*\omega \wedge F^*\tau.$$

Proof. Let $\omega \in \Lambda^k(T^*N)$ and $\tau \in \Lambda^\ell(T^*N)$. Let $p \in M$ and let $X_1, \dots, X_k, X_{k+1}, \dots, X_{k+\ell}$ be in T_pM. Then

$$\begin{aligned}
F^*(\omega\wedge\tau)_p(X_1,\dots,X_{k+\ell}) &= (\omega\wedge\tau)_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_{k+\ell})\\
&= (\omega_{F(p)}\wedge\tau_{F(p)})(F_{*,p}X_1,\dots,F_{*,p}X_{k+\ell})\\
&= \sum_{\sigma\in S_{k+\ell}}(\operatorname{sgn}\sigma)\ \omega_{F(p)}(F_{*,p}X_1,\dots,F_{*,p}X_k)\tau_{F(p)}(F_{*,p}X_{k+1},\dots,F_{*,p}X_{k+\ell})\\
&= \sum_{\sigma\in S_{k+\ell}}(\operatorname{sgn}\sigma)\ (F^*\omega)_p(X_1,\dots,X_k)(F^*\tau)_p(X_{k+1},\dots,X_{k+\ell})\\
&= ((F^*\omega)_p\wedge(F^*\tau)_p))(X_1,\dots,X_k,X_{k+1},\dots,X_{k+\ell})\\
&= (F^*\omega\wedge F^*\tau)_p(X_1,\dots,X_k,X_{k+1},\dots,X_{k+\ell}).
\end{aligned}$$

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Exercise 18.8. Pullback by a surjective submersion

In Subsection 19.5, we will show that the pullback of a C^∞ form is C^∞. Assuming this fact for now, prove that if $\pi : \tilde{M} \to M$ is a surjective submersion, then the pullback map $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

Proof. Exercises 18.2 and 18.3 show that π^* is an algebra homomorphism. Now, suppose $\pi^*(\omega) = 0$ for $\omega \in \Omega^*(M)$. We want to show that $\omega \equiv 0$. That is, if $q \in M$ and $Y_1, \ldots, Y_k \in T_qM$, then $\omega_q(Y_1, \ldots, Y_k) = 0$. Since π is surjective, there exists $p \in \tilde{M}$ such that $\pi(p) = q$. Since $\pi_{*,p}$ is surjective, there exists $X_i \in T_p\tilde{M}$ such that $F_{*,p}X_i = Y_i$ for all $1 \le i \le k$. So

$$0 = (\pi^*\omega)_p(X_1, \ldots, X_k) = \omega_{\pi(p)}(\pi_{*,p}X_1, \ldots, \pi_{*,p}X_k) = \omega_q(Y_1, \ldots, Y_k),$$

as desired. Hence π^* is an injective algebra homomorphism. ☕

Exercise 19.1. Pullback of a differential form

Let U be the open set $]0,\infty[\times]0,\pi[\times]0,2\pi[$ in the (ρ,ϕ,θ)-space $\mathbb{R}^3$. Define $F : U \to \mathbb{R}^3$ by

$$F(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

If x, y, z are the standard coordinates on the target $\mathbb{R}^3$, show that

$$F^*(dx\wedge dy\wedge dz) = \rho^2\sin\phi \; d\rho\wedge d\phi\wedge d\theta.$$

Solution:
We have

$$\begin{aligned}
F^*(dx) &= d(F^*x) = d(x\circ F) = d(\rho\sin\phi\cos\theta) = \sin\phi\cos\theta \; d\rho + \rho\cos\phi\cos\theta \; d\phi - \rho\sin\phi\sin\theta \; d\theta,\\
F^*(dy) &= d(F^*y) = d(y\circ F) = d(\rho\sin\phi\sin\theta) = \sin\phi\sin\theta \; d\rho + \rho\cos\phi\sin\theta \; d\phi + \rho\sin\phi\cos\theta \; d\theta,\\
F^*(dz) &= d(F^*z) = d(z\circ F) = d(\rho\cos\phi) = \cos\phi \; d\rho - \rho\sin\phi \; d\phi.
\end{aligned}$$

Let's do this:

$$\begin{aligned}
F^*(dx\wedge dy\wedge dz) &= F^*dx\wedge F^*dy\wedge F^*dz\\
&= (\sin\phi\cos\theta \; d\rho + \rho\cos\phi\cos\theta \; d\phi - \rho\sin\phi\sin\theta \; d\theta)\\
&\quad\wedge(\sin\phi\sin\theta \; d\rho + \rho\cos\phi\sin\theta \; d\phi + \rho\sin\phi\cos\theta \; d\theta)\\
&\quad\quad\wedge(\cos\phi \; d\rho - \rho\sin\phi \; d\phi)\\
&= (\sin\phi\cos\theta)(\rho\sin\phi\cos\theta)(-\rho\sin\phi) \; d\rho\wedge d\theta\wedge d\phi\\
&\quad + (\rho\cos\phi\cos\theta)(\rho\sin\phi\cos\theta)(\cos\phi) \; d\phi\wedge d\theta\wedge d\rho\\
&\quad\quad + (-\rho\sin\phi\sin\theta)(\sin\phi\sin\theta)(-\rho\sin\phi) \; d\theta\wedge d\rho\wedge d\phi\\
&\quad\quad\quad + (-\rho\sin\phi\sin\theta)(\rho\cos\phi\sin\theta)(\cos\phi) \; d\theta\wedge d\phi\wedge d\rho\\
&= -\rho^2(\sin^3\phi\cos^2\theta) \; d\rho\wedge d\theta\wedge d\phi\\
&\quad + (\rho^2\cos^2\phi\cos^2\theta\sin\phi) \; d\phi\wedge d\theta\wedge d\rho\\
&\quad\quad + \rho^2\sin^3\phi\sin^2\theta \; d\theta\wedge d\rho\wedge d\phi\\
&\quad\quad\quad + (-\rho^2\sin\phi\sin^2\theta\cos^2\phi \; d\theta\wedge d\phi\wedge d\rho\\
&= [\rho^2(\sin^3\phi\cos^2\theta) + (\rho^2\cos^2\phi\cos^2\theta\sin\phi)\\
&\quad + (\rho^2\sin^3\phi\sin^2\theta) + (\rho^2\sin\phi\sin^2\theta\cos^2\phi)] \; d\rho\wedge d\phi\wedge d\theta\\
&= [\rho^2\sin^3\phi(\cos^2\theta+\sin^2\theta) + \rho^2\sin\phi\cos^2\phi(\cos^2\theta+\sin^2\theta)] \; d\rho\wedge d\phi\wedge d\theta\\
&= [\rho^2\sin\phi(\sin^2\phi+\cos^2\phi)] \; d\rho\wedge d\phi\wedge d\theta\\
&= \rho^2\sin\phi \; d\rho\wedge d\phi\wedge d\theta
\end{aligned}$$

WHEW! And there you have it!

Exercise 19.2. Pullback of a differential form

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$F(x, y) = (x^2 + y^2, xy).$$
If u, v are the standard coordinates on the target $\mathbb{R}^2$, compute $F^*(udu + vdv)$.
Solution:

$$\begin{aligned}
F^*(udu + vdv) &= (F^*u)F^*du + (F^*v)F^*dv \\
&= (F^*u)dF^*u + (F^*v)dF^*v \\
&= (x^2 + y^2)d(x^2 + y^2) + (xy)d(xy) \\
&= (x^2 + y^2)(2xdx + 2ydy) + (xy)(ydx + xdy) \\
&= (2x^3 + 3xy^2)dx + (2x^2y + 2y^3)dy.
\end{aligned}$$

Exercise 19.3. Pullback of a differential form by a curve

Let τ be the 1-form $\tau = (-ydx+xdy)/(x^2+y^2)$ on $\mathbb{R}^2-\{\mathbf{0}\}$ by $\gamma(t) = (\cos t, \sin t)$. Compute $\gamma^*\tau$. (This problem is related to Example 17.16 in that if $i : S^1 \hookrightarrow \mathbb{R}^2 - \{\mathbf{0}\}$ is inclusion, then $\gamma = i \circ c$ and $\omega = i^*\tau$).
Solution:
Let $a = -y/(x^2 + y^2)$ and $b = x/(x^2 + y^2)$. Then

$$\begin{aligned}
\gamma^*\tau &= \gamma^*(adx + bdy) \\
&= (\gamma^*a)(d\gamma^*x) + (\gamma^*b)(d\gamma^*y) \\
&= (a \circ \gamma)(d(x \circ \gamma)) + (b \circ \gamma)(d(y \circ \gamma)) \\
&= \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}\right) d(\cos t) + \left(\frac{\cos t}{\cos^2 t + \sin^2 t}\right) d(\sin t) \\
&= (-\sin t)(-\sin t)dt + (\cos t)\cos t dt \\
&= dt
\end{aligned}$$

Exercise 19.5. Coordinate functions and differential forms

Let $f^1, \dots, f^n$ be C^∞ functions on a neighborhood U of a point p in a manifold of dimension n. Show that there is a neighborhood W of p on which $f^1, \dots, f^n$ form a coordinate system if and only if $(df^1 \wedge \cdots \wedge df^n)_p \neq 0$.

Proof. Define $f := (f^1, \dots, f^n)$ and let $(U, x^1, \dots, x^n)$ be a chart at p. Suppose there exists an open set W with $p \in W \subset U$ so that $(W, f^1, \dots, f^n)$ is a coordinate system at p. Then

$$\begin{aligned}
(W, f^1, \dots, f^n) \text{ is a coordinate system at } p &\iff \exists W \subseteq U, p \in W, f|_W \text{ is a diffeomorphism} \\
&\iff f : U \to \mathbb{R}^n \text{ is a locally invertible at } p \\
&\iff \det[\partial f^i/\partial x^j(p)] \neq 0 \\
&\iff (df^1 \wedge \cdots \wedge df^n)_p \\
&\qquad = \det[\partial f^i/\partial x^j(p)]dx^1 \wedge \cdots \wedge dx^n \neq 0
\end{aligned}$$

☕

Exercise 19.8. Nondegenerate 2-forms

A 2-covetor α on a $2n$-dimensional vector space V is said to be *nondegenerate* if $\alpha^n := \alpha \wedge \cdots \wedge \alpha$ (n times) is not the zero $2n$-covector. A 2 form ω on a $2n$-dimensional manifold M is said to be *nondegenerate* if at every point $p \in M$, the 2-covector ω_p is nondegenerate on the tangent space T_pM.

(a) Prove that on $\mathbb{C}^n$ with real coordinates $x^1, y^1, \dots, x^n y^n$, the 2-form $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ is nondegenerate.

Proof. We have $\omega^n = \left(\sum_{i=1}^n dx^i \wedge dy^i \right) \wedge \cdots \wedge \left(\sum_{i=1}^n dx^i \wedge dy^i \right)$.

The expanded product of ω^n will be a sum of wedge products. A single summand in ω^n will be a wedge product corresponding to n choices: Each being a choice of $dx^i \wedge dy^i$. So a generic summand in ω^n has the form

$$(dx^{i_1} \wedge dy^{i_1}) \wedge (dx^{i_2} \wedge dy^{i_2}) \cdots \wedge (dx^{i_n} \wedge dy^{i_n}),$$

for some choice of $i_1, i_2, \dots, i_n$. So ω^n be will be a sum of n^n terms. However, a summand in ω^n is nonzero if and only if all of the $dx^{i_j} \wedge dy^{i_j}$ are distinct. This corresponds to choosing $i_1, i_2, \dots, i_n$ by a permutation in S_n. So, a generic nonzero summand has the form

$$\bigwedge_{i=1}^n dx^{\sigma(i)} \wedge dy^{\sigma(i)} \tag{ʘ}$$

for some $\sigma \in S_n$. Hence

$$\omega^n = \sum_{\sigma \in S_n} \bigwedge_{i=1}^n dx^{\sigma(i)} \wedge dy^{\sigma(i)}. \tag{Ↄ}$$

Since each term in (ʘ) is a 2-form, we may commute terms without introducing a change in sign. Hence we can reorder each summand in ω^n so that the indices are in increasing order, and rewrite (ʘ) as

$$\bigwedge_{i=1}^n dx^i \wedge dy^i.$$

Hence (Ↄ) becomes

$$\omega^n = \sum_{\sigma \in S_n} \bigwedge_{i=1}^n dx^i \wedge dy^i = n! \bigwedge_{i=1}^n dx^i \wedge dy^i. \tag{$\star$}$$

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(b) Prove that if λ is the Liouville form on the total space T^*M of the cotangent bundle of an n-dimensional manifold M, then $d\lambda$ is a nondegenerate 2-form on T^*M.
Solution:
Using the formula we found in Exercise 17.4(a) and using insight gained from Exercise 19.8(a), (in particular the formula in ($\star$)), we have

$$(d\lambda)^n = d\left(\sum c_i d\overline{x}^i\right)^n = \left(\sum dc_i \wedge d\overline{x}^i\right)^n = n! \bigwedge_{i=1}^n dc_i \wedge d\overline{x}^i.$$

Exercise 20.1. The limit of a family of vector fields

Let I be an open interval, M a manifold, and $\{X_t\}$ a 1-parameter family of vector fields on M defines for all $t \neq t_0 \in I$. Show that the definition of $\lim_{t\to t_0} X_t$ in (20.1), if the limit exists is independent of coordinate charts.

Proof. Let $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ be two overlapping charts in M about p. Then for any t, we have

$$X_t|_p = \sum_i a^i(t,p) \frac{\partial}{\partial x^i}\bigg|_p = \sum_j b^j(t,p) \frac{\partial}{\partial y^j}\bigg|_p .$$

Then for any k, applying both sides to x^k gives

$$a^k(t,p) = \left(\sum_i a^i(t,p)\frac{\partial}{\partial x^i}\right) x^k = \left(\sum_j b^j(t,p)\frac{\partial}{\partial y^j}\right) x^k = \sum_j b^j(t,p)\frac{\partial x^k}{\partial y^j} \qquad (☎)$$

Moreover, for any j, since $\partial/\partial y^j \in T_pU$ we have

$$\frac{\partial}{\partial y^j} = \sum_i c^i \frac{\partial}{\partial x^i}.$$

Then for any k, applying both sides to x^k gives

$$\frac{\partial x^k}{\partial y^j} = \sum_i c^i \frac{\partial x^k}{\partial x^i} = c^k,$$

and so

$$\frac{\partial}{\partial y^j} = \sum_i \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}. \qquad (✂)$$

Hence we have

$$\begin{aligned}
\lim_{t\to t_0} X_t|_p &= \sum_i \lim_{t\to t_0} a^i(t,p) \frac{\partial}{\partial x^i}\bigg|_p \\
&= \sum_i \lim_{t\to t_0} \left(\sum_j b^j(t,p)\frac{\partial x^i}{\partial y^j}\bigg|_p\right)\frac{\partial}{\partial x^i}\bigg|_p \\
&= \lim_{t\to t_0} \sum_j b^j(t,p) \sum_i \frac{\partial x^i}{\partial y^j}\bigg|_p \frac{\partial}{\partial x^i}\bigg|_p \\
&= \lim_{t\to t_0} \sum_j b^j(t,p) \frac{\partial}{\partial y^j}\bigg|_p .
\end{aligned}$$

Therefore the limit, if it exists, is independent of coordinate charts. ☕

Exercise 20.3. Derivative of a smooth family of vector fields

Shew that the definition (20.3) of the derivative of a smooth family of vector fields on M is independent of the chart $(U, x^1, \ldots, x^n)$ containing p.

Proof. Using the same charts as in Exercise 20.1, by differentiating both sides of (☎) with respect to t (and then evaluating at t_0) we get

$$\frac{\partial a^k}{\partial t}(t_0, p) = \sum_j \frac{\partial b^j}{\partial t}(t_0, p)\frac{\partial x^k}{\partial y^j}.$$

(Note that we can do this since $\partial x^k/\partial y^j$ does not depend on t). Hence we have by (✂)

$$\begin{aligned}\left(\left.\frac{d}{dt}\right|_{t=t_0} X_t\right)_p &= \sum_i \frac{\partial a^i}{\partial t}(t_0,p)\left.\frac{\partial}{\partial x^i}\right|_p \\ &= \sum_i \left(\sum_j \frac{\partial b^j}{\partial t}(t_0,p)\left.\frac{\partial x^k}{\partial y^j}\right|_p\right)\left.\frac{\partial}{\partial x^i}\right|_p \\ &= \sum_j \frac{\partial b^j}{\partial t}(t_0,p)\sum_i \left.\frac{\partial x^k}{\partial y^j}\right|_p \left.\frac{\partial}{\partial x^i}\right|_p \\ &= \sum_j \frac{\partial b^j}{\partial t}(t_0,p)\left.\frac{\partial}{\partial y^j}\right|_p.\end{aligned}$$

Therefore the definition of the derivative of a smooth family of vector fields on M is independent of coordinate charts. ☕

Exercise 20.7. $\mathcal{F}$-Linearity and the Lie Derivative

Let ω be a differential form, X a vector field, and f a smooth function on a manifold. The Lie derivative $\mathcal{L}_X\omega$ is not $\mathcal{F}$-linear in either variable, but prove that is satisfies the following identity:

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega.$$

Proof. Starting with Cartan's Magic Formula, we have

$$\begin{aligned}\mathcal{L}_{fX}\omega &= d(\iota_{fX}\omega) + \iota_{fX}(d\omega) \\ &= d(f\iota_X\omega) + f\iota_X(d\omega) && (\iota_{fX} = f\iota_X) \\ &= df \wedge \iota_X\omega + fd(\iota_X\omega) + f\iota_X(d\omega) \\ &= df \wedge \iota_X\omega + f(d(\iota_X\omega) + \iota_X(d\omega)) \\ &= df \wedge \iota_X\omega + f\mathcal{L}_X\omega. && \text{(By Cartan's Magic Formula)}\end{aligned}$$

☕

Exercise 20.9. Interior multiplication on $\mathbb{R}^n$

Let $\omega = dx^1 \wedge \cdots \wedge dx^n$ be the volume of a form and $X = \sum x^i \partial/\partial x^i$ the radial vector field on $\mathbb{R}^n$. Compute the contraction $\iota_x \omega$.
Solution:
For any j,

$$\iota_X dx^j = dx^i(X) = dx^j \left(\sum x^i \partial/\partial x^i \right) = x^j.$$

Then by the formula in Proposition 20.7,

$$\begin{aligned}\iota_X \omega = \iota_X(dx^1 \wedge \cdots \wedge dx^n) &= \sum_{i=1}^n (-1)^{i-1} dx^i(X) dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n.\end{aligned}$$

Exercise 20.10. The Lie derivative on the 2-sphere

Let $\omega = xdy \wedge dz - ydx \wedge dy + zdx \wedge dy$ and $X = -y\partial/\partial x + x\partial/\partial y$ on the unit 2-sphere S^2 in $\mathbb{R}^3$.
Solution:
First we compute $\mathcal{L}_X x, \mathcal{L}_X y$, and $\mathcal{L}_X(z-y)$:

$$\begin{aligned}\mathcal{L}_X x = Xx &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) x = -y, \\ \mathcal{L}_X y = Xy &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) y = x, \\ \mathcal{L}_X z = Xy &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) z = 0, \\ \mathcal{L}_X(z-y) = X(z-y) &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) (z-y) = -x.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{L}_X\omega &= \mathcal{L}_X(xdy \wedge dz + (z-y)dx \wedge dy) \\ &= \mathcal{L}_X(xdy \wedge dz) + \mathcal{L}_X((z-y)dx \wedge dy) \\ &= \Big[(\mathcal{L}_X x)dy \wedge dz + x\mathcal{L}_X(dy \wedge dz)\Big] + \Big[(\mathcal{L}_X(z-y))dx \wedge dy + (z-y)\mathcal{L}_X(dx \wedge dy)\Big] \\ &= \Big[(\mathcal{L}_X x)dy \wedge dz\Big] + x\Big[(\mathcal{L}_X dy) \wedge dz + dy \wedge (\mathcal{L}_X dz)\Big] \\ &\qquad + \Big[(\mathcal{L}_X(z-y))dx \wedge dy\Big] + (z-y)\Big[(\mathcal{L}_X dx) \wedge dy + dx \wedge (\mathcal{L}_X dy)\Big] \\ &= \Big[(\mathcal{L}_X x)dy \wedge dz\Big] + x\Big[d(\mathcal{L}_X y) \wedge dz + dy \wedge d(\mathcal{L}_X z)\Big] \\ &\qquad + \Big[(\mathcal{L}_X(z-y))dx \wedge dy\Big] + (z-y)\Big[d(\mathcal{L}_X x) \wedge dy + dx \wedge d(\mathcal{L}_X y)\Big] \\ &= \Big[(-y)dy \wedge dz\Big] + x\Big[d(x) \wedge dz + dy \wedge d(0)\Big] \\ &\qquad + \Big[(-x)dx \wedge dy\Big] + (z-y)\Big[d(-y) \wedge dy + dx \wedge d(x)\Big] \\ &= -ydy \wedge dz + xdx \wedge dz - xdx \wedge dy.\end{aligned}$$

Intro to Manifolds, Tu – *End of Section Exercises*

Exercise 21.3. Equivalence of oriented atlases

Show that the relation in Definition 21.11 is an equivalence relation.

Definition 1. Two oriented atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ on a manifold M are said to be *equivalent* if the transition functions

$$\phi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \to \phi_\alpha(U_\alpha \cap V_\beta)$$

have positive Jacobian determinant for all α, β.

Proof. Reflexivity: If $\{(U_\alpha, \phi_\alpha)\}$ is an oriented atlas, then for any two overlapping charts $(U_\alpha, \phi_\alpha) = (U_\alpha, x^1, \dots, x^n)$ and $(U_\beta, \phi_\beta) = (U_\beta, y^1, \dots, y^n)$, we have by definition of oriented atlas that

$$\det\left[\frac{\partial x^i}{\partial y^j}\right] > 0$$

everywhere on $U_\alpha \cap U_\beta$. Let $r^1, \dots, r^n$ be the standard coordinates in $\mathbb{R}^n$. Then $(\phi_\alpha \circ \phi_\beta^{-1})^i = r^i \circ \phi_\alpha \circ \phi_\beta^{-1} = x^i \circ \phi_\beta$. So for any point $p \in U_\alpha \cap U_\beta$ we have

$$\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})^i}{\partial r^j}(\phi_\beta(p)) = \frac{\partial(r^i \circ \phi_\alpha \circ \phi_\beta^{-1})}{\partial r^j}(\phi_\beta(p)) = \frac{\partial(x^i \circ \phi_\beta^{-1})}{\partial r^j}(\phi_\beta(p)) = \frac{\partial x^i}{\partial y^j}(p),$$

where the last equality follows by definition of the partial derivative of the coordinate functions $x^i : U_\beta \to \mathbb{R}$ with respect to the coordinates y^j. Hence we have

$$\det\left[\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})^i}{\partial r^j}\right] = \det\left[\frac{\partial x^i}{\partial y^j}\right] > 0$$

everywhere on $U_\alpha \cap U_\beta$.

Symmetry: Suppose $\{(U_\alpha, \phi_\alpha)\} \sim \{(V_\alpha, \psi_\beta)\}$. If $(U_\alpha, x^1, \dots, x^n)$ and $(V_\beta, y^1, \dots, y^n)$ are two overlapping charts, then

$$\det\left[\frac{\partial(\phi_\alpha \circ \psi_\beta^{-1})^i}{\partial r^j}\right] = \det\left[\frac{\partial x^i}{\partial y^j}\right] > 0$$

everywhere on $U_\alpha \cap V_\beta$. But then

$$\det\left[\frac{\partial(\psi_\beta \circ \phi_\alpha^{-1})^i}{\partial r^j}\right] = \det\left[\frac{\partial y^i}{\partial x^j}\right] = \det\left[\frac{\partial x^i}{\partial y^j}\right]^T > 0,$$

and hence $\{(V_\alpha, \psi_\beta)\} \sim \{(U_\alpha, \phi_\alpha)\}$.

Transitivity: Suppose $\{(U_\alpha, \phi_\alpha)\} \sim \{(V_\alpha, \psi_\beta)\}$ and $\{(V_\beta, \psi_\beta)\} \sim \{(W_\gamma, \theta_\gamma)\}$. If $(U_\alpha, x^1, \dots, x^n)$, $(V_\beta, y^1, \dots, y^n)$, and $(W_\gamma, z^1, \dots, z^n)$ are overlapping charts, then

$$\det\left[\frac{\partial(\phi_\alpha \circ \psi_\beta^{-1})^i}{\partial r^j}\right] = \det\left[\frac{\partial x^i}{\partial y^j}\right] > 0 \quad \text{and} \quad \det\left[\frac{\partial(\psi_\beta \circ \theta_\gamma^{-1})^i}{\partial r^j}\right] = \det\left[\frac{\partial y^i}{\partial z^j}\right] > 0.$$

Since $\phi_\alpha \circ \theta_\gamma^{-1} = (\phi_\alpha \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ \theta_\gamma^{-1})$, then

$$\det\left[\frac{\partial(\phi_\alpha \circ \theta_\gamma^{-1})}{\partial r^i}\right] = \det\left[\frac{\partial x^i}{\partial y^j}\middle|\frac{\partial y^i}{\partial z^j}\right] > 0,$$

and so $\{(U_\alpha, \phi_\alpha)\} \sim \{(W_\gamma, \theta_\gamma)\}$. ☕

Exercise 21.4. Orientation-preserving diffeomorphisms

Let $F : (N, [\omega_N]) \to (M, [\omega_M])$ be an orientation-preserving diffeomorphism. If $\{(V_\alpha, \psi_\alpha)\} = \{(V_\alpha, y^1_\alpha, \dots, y^n_\alpha)\}$ is an oriented atlas on M that specifies the orientation on M, show that $\{(F^{-1}V_\alpha, F^*\psi_\alpha)\} = \{(F^{-1}V_\alpha, F^1_\alpha, \dots, F^n_\alpha)\}$ is an oriented atlas on N that specifies the orientation of N, where $F^i_\alpha = y^i_\alpha \circ F$.

Proof. We first show that the atlas $\{(F^{-1}V_\alpha, F^*\psi_\alpha)\}$ is orientable. Suppose $(F^{-1}V_\alpha, F^*\psi_\alpha) = (F^{-1}V_\alpha, (x^1_\alpha \circ F), \dots, (x^n_\alpha \circ F)$ and $(F^{-1}V_\beta, F^*\psi_\beta) = (F^{-1}V_\beta, (y^1_\beta \circ F), \dots, (y^n_\beta \circ F))$ are two overlapping charts in N. Then

$$\frac{\partial (F^*\psi_\alpha)^i}{\partial (F^*\psi_\beta)^j} = \frac{\partial ((F^*\psi_\alpha) \circ (F^*\psi_\beta)^{-1})^i}{\partial r^j} = \frac{\partial x^i_\alpha \circ F \circ F^{-1} \circ y^i_\beta}{\partial r^j} = \frac{\partial (x^i_\alpha \circ y^i_\beta)}{\partial r^j} = \frac{\partial x^i_\alpha}{\partial y^j_\beta},$$

and so

$$\det \left[\frac{\partial (F^*\psi_\alpha)^i}{\partial (F^*\psi_\beta)^j} \right] = \det \left[\frac{\partial x^i_\alpha}{\partial y^j_\beta} \right] > 0.$$

Now, the oriented atlas $\{(V_\alpha, \psi_\alpha)\} = \{(V_\alpha, y^1_\alpha, \dots, y^n_\alpha)\}$ determines an orientation on M, given by $\mu_M := [(\partial/\partial y^1_\alpha, \dots, \partial/\partial y^n_\alpha)]$. The orientation μ_M is associated to the equivalence class of the nowhere vanishing n-form ω_M on M. Hence we have

$$\omega_M(\partial/\partial y^1_\alpha, \dots, \partial/\partial y^n_\alpha) > 0.$$

We need to show that the atlas $\{(F^{-1}V_\alpha, F^*\psi_\alpha)\} = \{(F^{-1}V_\alpha, (y^1_\alpha \circ F), \dots, (y^n_\alpha \circ F))\}$ on N specifies the orientation of N. That is, we need to show

$$\omega_N \left(\frac{\partial}{\partial (y^1_\alpha \circ F)}, \dots, \frac{\partial}{\partial (y^n_\alpha \circ F)} \right) > 0. \tag{$\mathfrak{Y}$}$$

Let $(V_\alpha, y^1_\alpha, \dots, y^n_\alpha)$ be a chart about a point $q \in M$. Then the chart $(F^{-1}V_\alpha, (y^1_\alpha \circ F), \dots, (y^n_\alpha \circ F))$ contains $p := F^{-1}(q)$. Recall that $\{\partial/\partial y^i_\alpha|_q\}^n_{i=1}$ is a basis for T_qM and $\{\partial/\partial (y^i_\alpha \circ F)|_p\}^n_{i=1}$ is a basis for T_pN. So for some real numbers a^k_j, we have

$$F_{*,p} \left(\left. \frac{\partial}{\partial (y^j_\alpha \circ F)} \right|_p \right) = \sum_k a^k_j \left. \frac{\partial}{\partial y^k_\alpha} \right|_q. \tag{Ψ}$$

Applying both sides to y^i_α, we find that

$$a^i_j = \left(\sum_k a^k_j \left. \frac{\partial}{\partial y^i_\alpha} \right|_q \right) y^i_\alpha = F_{*,p} \left(\left. \frac{\partial}{\partial (y^j_\alpha \circ F)} \right|_p \right) y^i_\alpha = \left. \frac{\partial}{\partial (y^j_\alpha \circ F)} \right|_p (y^i_\alpha \circ F) = \delta^i_j.$$

So our equation in (Ψ) becomes

$$F_{*,p} \left(\left. \frac{\partial}{\partial (y^j_\alpha \circ F)} \right|_p \right) = \sum_i \delta^i_j \left. \frac{\partial}{\partial y^i_\alpha} \right|_q = \left. \frac{\partial}{\partial y^j_\alpha} \right|_q.$$

Since F is orientation preserving, then $[F^*\omega_M] = [\omega_N]$. So there exists a smooth function $f > 0$ on M such that $fF^*\omega_M = \omega_N$. Now, we have the following:

$$\begin{aligned}
\omega_N\left(\left.\frac{\partial}{\partial(y^1_\alpha \circ F)}\right|_p, \dots, \left.\frac{\partial}{\partial(y^n_\alpha \circ F)}\right|_p\right) &= f(F^*\omega_M)_p\left(\left.\frac{\partial}{\partial(y^1_\alpha \circ F)}\right|_p, \dots, \left.\frac{\partial}{\partial(y^n_\alpha \circ F)}\right|_p\right) \\
&= f\omega_M\left(F_{*,p}\left(\left.\frac{\partial}{\partial(y^1_\alpha \circ F)}\right|_p\right), \dots, F_{*,p}\left(\left.\frac{\partial}{\partial(y^n_\alpha \circ F)}\right|_p\right)\right) \\
&= f\omega_M\left(\left.\frac{\partial}{\partial y^1_\alpha}\right|_q, \dots, \left.\frac{\partial}{\partial y^j_\alpha}\right|_q\right) > 0,
\end{aligned}$$

which gives $(\mathfrak{D})$.

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Exercise 21.5. Orientation-preserving or orientation-reversing diffeomorphisms

Let U be the open set $(0,\infty) \times (0, 2\pi)$ in the (r,θ)-plane $\mathbb{R}^2$. We define $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ by $F(r,\theta) = (r\cos\theta, r\sin\theta)$. Decide whether F is orientation-preserving or orientation-reversing as a diffeomorphism onto its image.

Solution:

Using Proposition 21.8, F is orientation preserving if and only if

$$\det\begin{bmatrix} \dfrac{\partial F^1}{\partial dr} & \dfrac{\partial F^1}{\partial d\theta} \\ \dfrac{\partial F^2}{\partial dr} & \dfrac{\partial F^2}{\partial d\theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

is everywhere positive on U. Since $r \in (0,\infty)$, F is orientation preserving.

Exercise 21.6. Orientability of a regular level set in $\mathbb{R}^{n+1}$

Suppose $f(x^1, \dots, x^{n+1})$ is a C^∞ function on $\mathbb{R}^{n+1}$ with 0 as a regular value. Show that the zero set of f is an orientable submanifold of $\mathbb{R}^{n+1}$. In particular, the unit n-sphere S^n in $\mathbb{R}^{n+1}$ is orientable.

Proof. By Theorem 21.5, the zero set of f is an orientable submanifold if and only if there exists a C^∞ nowehere-vanishing n-form on $M := f^{-1}(\{0\})$. For each i, define the set

$$U_i = \{p \in \mathbb{R}^{n+1} \mid \partial f/\partial x^i(p) \neq 0\}.$$

Since 0 is a regular value of f, every point $p \in \mathbb{R}^{n+1}$ satisfies $\partial f/\partial x^i(p) \neq 0$ for some i. So $\{U_i\}_{i=1}^{n+1}$ is a cover of M. Define a top form ω on U_i by

$$\omega = (-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}}{\partial f/\partial x^i}.$$

By the implicit function theorem, in a neighborhood of a point $p \in U_i$, x^i is a function of $x^1, \dots, \widehat{x^i}, \dots, x^n$. It follows that $x^1, \dots, \widehat{x^i}, \dots, x^n$ can be used as local coordinates, and the n-form

$$(-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}}{\partial f/\partial x^i}$$

is C^∞ at p. Thus, ω is C^∞ on U_i and nowhere vanishing on M. ☕

Exercise 21.7. Orientability of a Lie group

Show that every Lie group G is orientable by constructing a nowhere-vanishing top form on G.

Proof. *** This proof belongs to Alex Bates ***

Let $e \in G$ be the identity and let $\{X_e^1, \dots, X_e^n\}$ be a basis for T_eG. Since for any $g \in G$, left multiplication ℓ_g is a diffeomorphism, we have an isomorphism $\ell_{g,*} : T_eG \to T_gG$. Hence we have a left-invariant vector fields $\{X^1, \dots, X^n\}$ on G (and therefore smooth, by Prop 16.8) given by $X^i(e) = X_e^i$ for all $1 \leq i \leq n$. Let $\{\alpha_e^1, \dots, \alpha_e^n\}$ be dual to $\{X_e^1, \dots, X_e^n\}$. Define a top form $\omega_e = \alpha_e^1 \wedge \cdots \wedge \alpha_e^n$. Then for any $g \in G$, we can define

$$\omega_g := \ell_{g^{-1}}^*(\omega_e) = \ell_{g^{-1}}^*(\alpha_e^1 \wedge \cdots \wedge \alpha_e^n).$$

Hence ω is a top form on G. To see that ω is nowhere-vanishing, first note that $\ell_{g,*} \circ \ell_{g^{-1},*} = \mathbb{1}_{T_eG}$. Now, for any $g \in G$, we have

$$\begin{aligned}
\omega_g(X_g^1, \dots, X_g^n) &= (\ell_{g^{-1}}^* \omega_e)(\ell_{g,*}X_e^1, \dots, \ell_{g,*}X_e^n) \\
&= \omega_e(\ell_{g^{-1},*}\ell_{g,*}X_e^1, \dots, \ell_{g^{-1},*}\ell_{g,*}X_e^n) \\
&= \omega_e(X_e^1, \dots, X_e^n) \\
&= \alpha_e^1 \wedge \cdots \wedge \alpha_e^n(X_e^1, \dots, X_e^n) \\
&= \det[\alpha_e^i(X_e^i)] \\
&= 1.
\end{aligned}$$

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Exercise 21.8. Orientability of a parallelizable manifold

Show that a parallelizable manifold is orientable. (In particular, this shows again that every Lie group is orientable.)

*** I obtained solutions from the back of the book for all those problems in Section 22 which had them. Sorry Jesse, I did not manage my time well on this assignment.***

Exercise 22.4. Smooth outward-pointing vector field along the boundary

Show that the vector field $X = \sum \rho_\alpha X_\alpha$ defined in the proof of Proposition 22.10 is a smooth outward-pointing vector field along ∂M.

Proof. Let $p \in \partial M$ and let $(U, x^1, \ldots, x^n)$ be a coordinate neighborhood of p. Write

$$X_{\alpha,p} = \sum_{i=1}^{n} a^i(X_{\alpha,p}) \frac{\partial}{\partial x^i}\bigg|_p .$$

Then

$$X_p = \sum_\alpha \rho_\alpha(p) X_{\alpha,p} = \sum_{i=1}^{n} \sum_\alpha \rho_\alpha(p) a^i(X_{\alpha,p}) \frac{\partial}{\partial x^i}\bigg|_p .$$

Since $X_{\alpha,p}$ is outward pointing, the coefficient $a^n(X_{\alpha,p})$ is negative by Exercise 22.3. Because $\rho_\alpha(p) \geq 0$ for all α with $\rho_\alpha(p)$ positive for at lest on α, the coefficient $\sum_\alpha \rho_\alpha(p) a^i(X_{\alpha,p})$ of $\partial/\partial x^n|_p$ in X_p is negative. Again by Exercise 22.3, this proves that X_p is outward pointing.

The smoothness of the vector field X follows from the smoothness of the partition of unity ρ_α and of the coefficient functions $a^i(X_{\alpha,p})$ as functions of p. ☕

Exercise 22.5. Boundary orientation

Let M be an oriented manifold with boundary, ω an orientation form for M, and X a C^∞ outward-pointing vector field along ∂M.

(a) If τ is another orientation form on M, then $\tau = f\omega$ for a C^∞ everywhere-positive function f on M. Show that $\iota_X \tau = f \iota_X \omega$ and therefore, $\iota_X \tau \sim \iota_X \omega$ on ∂M. (Here "$\sim$" is the equivalence relation defined in Subsection 21.4.)
Solution: Since ι_X is $C^\infty(M)$-linear, we have $\iota_X \tau = \iota_X(f\omega) = f \iota_X \omega$.

(b) Prove that if Y is another C^∞ outward-pointing vector field along ∂M, then $\iota_X \omega \sim \iota_Y \omega$ on ∂M.

Proof. By Proposition 22.11, both $\iota_X \omega$ and $\iota_Y \omega$ are smooth nowhere vanishing $(n-1)$-forms on ∂M, i.e., $\iota_X \omega, \iota_Y \omega \in \Lambda^{n-1}(T^*\partial M)$. Since ∂M is an $(n-1)$-dimensional manifold, both $\iota_X \omega$ and $\iota_Y \omega$ are top dimensional forms on ∂M, and hence $\iota_X \omega = f \iota_Y \omega$ for some nowhere-vanishing $f \in C^\infty(M)$. ☕

Exercise 22.6. Induced atlas on the boundary

Assume $n \geq 2$ and let (U, ϕ) and (V, ψ) be two charts in an oriented atlas of an orientable n- manifold M with boundary. Prove that if $U \cap V \cap \partial M \neq \varnothing$, then the restriction of the transition function $\psi \circ \phi^{-1}$ to the boundary $B := \phi(U \cap V) \cap \partial\mathcal{H}^n$,

$$(\psi \circ \phi^{-1})|_B : \phi(U \cap V) \cap \partial\mathcal{H}^n \to \psi(U \cap V) \cap \partial\mathcal{H}^n,$$

has positive Jacobian determinant. (*Hint:* Let $\phi = (x_1, \ldots, x_n)$ and $\psi = (y_1, \ldots, y_n)$. Show that the Jacobian matrix of $\psi \circ \phi^{-1}$ in local coordinates is block triangular with $J(\psi \circ \phi^{-1})|_B$ and $\dfrac{\partial y^n}{\partial x^n}$ as the diagonal blocks, and that $\dfrac{\partial y^n}{\partial x^n} > 0$.) Thus, if $\{(U_\alpha, \phi_\alpha)\}$ is an oriented atlas for a manifold M with boundary, then the induced atlas $\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M})\}$ for ∂M is oriented.

Proof. Let $r^1, \ldots, r^n$ be the standard coordinates on the upper half-space $\mathcal{H}^n$. As a shorthand, we write $a = (a^1, \ldots, a^n 1)$ for the first $n1$ coordinates of a point in $\mathcal{H}^n$. Since the transition function

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subset \mathcal{H}^n$$

takes boundary points to boundary points and interior points to interior points, (i) $(r^n \circ \psi \circ \phi^{-1})(a, 0) = 0$ and (ii) $(r^n \circ \psi \circ \phi^{-1})(a, t) > 0$ for $t > 0$, where $(a, 0)$ and (a, t) are points in $\phi(U \cap V) \subset \mathcal{H}^n$.

Let $x^j = r^j \circ \phi$ and $y^i = r^i \circ \phi$ be the local coordinates on the charts (U, ϕ) and (V, ψ) respectively. In particular, $y^n \circ \phi^{-1} = r^n \circ \psi \circ \phi^{-1}$. Differentiating (i) with respect to r^j gives

$$\left.\frac{\partial y^n}{\partial x^j}\right|_{\phi^{-1}(a,0)} = \left.\frac{\partial (y^n \circ \phi^{-1})}{\partial r^j}\right|_{(a,0)} = \left.\frac{\partial (r^n \circ \psi \circ \phi^{-1})}{\partial r^j}\right|_{(a,0)} = 0 \quad \text{for } j = 1, \ldots, n-1.$$

From (i) and (ii),

$$\begin{aligned}\left.\frac{\partial y^n}{\partial x^n}\right|_{\phi^{-1}(a,0)} = \left.\frac{\partial (y^n \circ \phi^{-1})}{\partial r^n}\right|_{(a,0)} &= \lim_{t \to 0^+} \frac{(y^n \circ \phi^{-1})(a,t) - (y^n \circ \phi^{-1})(a,0)}{t} \\ &= \lim_{t \to 0^+} \frac{(y^n \circ \phi^{-1})(a,t)}{t} \geq 0\end{aligned}$$

since both t and $(y^n \circ \phi^{-1})(a, t)$ are positive.

The Jacobian matrix of $J = [\partial y^i / \partial x^j]$ of the overlapping charts U and V at a point $p = \phi^{-1}(a, 0)$ in $U \cap V \cap \partial M$ therefore has the form

$$J = \begin{pmatrix} \dfrac{\partial y^1}{\partial x^1} & \cdots & \dfrac{\partial y^1}{\partial x^{n-1}} & \dfrac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial y^{n-1}}{\partial x^1} & \cdots & \dfrac{\partial y^{n-1}}{\partial x^{n-1}} & \dfrac{\partial y^{n-1}}{\partial x^n} \\ 0 & \cdots & 0 & \dfrac{\partial y^n}{\partial x^n} \end{pmatrix} = \begin{pmatrix} A & * \\ 0 & \dfrac{\partial y^n}{\partial x^n} \end{pmatrix}$$

where the upper left $(n-1) \times (n-1)$ block $A = [\partial y^i / \partial x^j]_{1 \leq i,j \leq n-1}$ is the Jacobian matrix od the induces charts $U \cap \partial M$ and $V \cap \partial M$ on the boundary. Since $\det J(p) > 0$ and $\partial y^n / \partial x^n(p) > 0$, we have $\det A(p) > 0$. ☕

Exercise 22.7. Boundary orientation of the left half-space

Let M be the left half-space

$$\{(x^1,\dots,x^n)\in\mathbb{R}^n \ \mid x_1 \le 0\},$$

with orientation form $dx^1\wedge\cdots\wedge dx^n$. Show that an orientation form for the boundary orientation on $\partial M = \{(0,x^2,\dots,x^n)\in R^n\}$ is $dx^2\wedge\wedge dx^n$.

Unlike the upper half-space $\mathcal{H}^n$, whose boundary orientation takes on a sign (Example 22.13), this exercise shows that the boundary orientation for the left half-space has no sign. For this reason some authors use the left half-space as the model of a manifold with boundary.

Proof. Because a smooth outward-pointing vector field along ∂M is $\partial/\partial x^1$, by definition an orientation form of the boundary orientation on ∂M is the contraction

$$\iota_{\partial/\partial x^1}(dx^1\wedge dx^2\wedge\cdots\wedge dx^n) = dx^2\wedge\cdots\wedge dx^n.$$

Exercise 22.8. Boundary orientation on a cylinder

Let M be the cylinder $S^1\times[0,1]$ with the counterclockwise orientation when viewed from the exterior. Describe the boundary orientation on $C_0 = S^1\times\{0\}$ and $C_1 = S^1\times\{1\}$.
Solution:
Define $f:\mathbb{R}^3\to\mathbb{R}$ by $f(x,y,z) = x^2+y^2$. Then M is a compact subset of the regular submanifold $f^{-1}(1)$. (Since $f_* = (2x\ 2y\ 0)$ fails to be surjective if and only if $x=y=0$, then 1 is a regular value of f.)

Now, the tangent space T_pM at a point $p\in M$ can be identified with $\operatorname{Ker} df$. We have $df = 2xdx+2ydy$, and if $X_p = a\partial/\partial x + b\partial/\partial + c\partial/\partial z\in T_pM$, then $0 = df(X_p) = 2xa+2yb$, which is satisfied by $a=-y$ and $b=-x$. Since $df(\partial/\partial z) = 0$, we have an ordered basis $\{-y\partial/\partial x + x\partial/\partial, \partial/\partial z\}$ which gives the counterclockwise orientation of M.

Now, $\partial/\partial z$ is an outward pointing vector on C_1. We orient C_1 by $-x\partial/\partial y + y\partial/\partial x$. To check that this coincides with the orientation on M, we check by using the outward vector first rule:

$$\left(\frac{\partial}{\partial z}, -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right) \sim \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right),$$

and the latter is the orientation on M.

Similarly, $-\partial/\partial z$ is an outward pointing vector on C_0, and we orient C_0 by $x\partial/\partial y - y\partial/\partial x$ and see that this coincides with the orientation on M since

$$\left(-\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \sim \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right).$$

Exercise 22.9. Boundary orientation on a sphere

Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Show that on orientation form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1},$$

where the caret $\widehat{}$ over dx^i indicates that dx^i is omitted. (*Hint:* An outward-pointing vector field on S^n is the radial vector field $X = \sum x^i \partial/\partial x^i$.)

Proof. We have the standard orientation $dx^1 \wedge \cdots \wedge dx^{n+1}$ on $\mathbb{R}^{n+1}$ and since the closed unit ball is a subset of $\mathbb{R}^{n+1}$, this form can be used to orient its boundary S^n. Using the hint, an outward pointing vector field on S^{n+1} is $X = \sum x^i \partial/\partial x^i$, and so an orientation form on S^n is the contraction

$$\begin{aligned}\omega = \iota_X(dx^1 \wedge \cdots \wedge dx^{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i-1} dx^i(X) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1} \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}.\end{aligned}$$

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Exercise 22.10. Orientation on the upper hemisphere of a sphere

Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Let U be the upper hemisphere $U = \{x \in S^n \mid x^{n+1} > 0\}$. It is a coordinate chart on the sphere with coordinates $x^1, \dots, x^n$

(a) Find an orientation form on U in terms of $dx^1, \dots, dx^n$.
Solution:
As in Exercise 22.9, we have $dx^1 \wedge \cdots \wedge dx^{n+1}$ as an orientation form on the closed unit ball. An outward pointing vector field on U is $\partial/\partial x^{n+1}$, and so an orientation form on U is

$$\begin{aligned}\iota_{\partial/\partial x^{n+1}}(dx^1 \wedge \cdots \wedge dx^{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i-1} dx^i(\partial/\partial x^{n+1}) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1} \\ &= (-1)^{(n+1)-1} dx^{n+1}(\partial/\partial x^{n+1}) dx^1 \wedge \cdots \wedge dx^n \\ &= (-1)^n dx^1 \wedge \cdots \wedge dx^n,\end{aligned}$$

where the second equality follows from the fact that $dx^i(\partial/\partial x^{n+1}) = \delta_i^{n+1}$.

(b) Show that the projection map $\pi : U \to \mathbb{R}^n$,

$$\pi(x^1, \dots, x^n, x^{n+1}) = (x^1, \dots, x^n),$$

is orientation-preserving if and only if n is even.

Proof. Let $\omega = (-1)^n dx^1 \wedge \cdots \wedge dx^n$ be the orientation form on U obtained in part (a). Let $\tau = dx^1 \wedge \cdots \wedge dx^n$ be the standard orientation form on $\mathbb{R}^n$. Note that $\pi(U) = \mathring{D}^n$, where $\mathring{D}^n$ is the unit disk in $\mathbb{R}^n$.

We want to check that the diffeomorphism $\pi : (U, [\omega]) \to (\mathring{D}^n, \tau)$ is orientation-preserving when n is even. To that end, let $p \in U$ and let $e_1, \dots, e_n$ be a basis for T_pU. Since is a linear map, $\pi_* = \pi$, and so $\pi_*(e_i) = e_i$ for all $1 \le i \le n$, (since π is the identity on the first n coordinates). Then

$$\omega_p(e_1, \dots, e_n) = (-1)^n dx^1 \wedge \cdots \wedge dx^n(e_1, \dots, e_n)$$

and

$$\begin{aligned}(\pi^*\tau)_p(e_1, \dots, e_n) &= \tau_{\pi(p)}(\pi_{*,p}e_1, \dots, \pi_{*,p}e_n) \\ &= \tau_p(e_1, \dots, e_n) \\ &= dx^1 \wedge \cdots \wedge dx^n(e_1, \dots, e_n).\end{aligned}$$

Hence $[\omega] = [\pi^*\tau]$ if and only if n is even. ☕

Exercise 22.11. Antipodal map on a sphere and the orientability of $\mathbb{R}P^n$

(a) The antipodal map $a : S^n \to S^n$ on the n-sphere is defined by

$$a(x^1, \ldots, x^{n+1}) = (-x^1, \ldots, -x^{n+1}).$$

Show that the antipodal map is orientation-preserving if and only if n is odd.

Proof. Using the orientation form $\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$ from Exercise 22.9, we want to show that $a : (S^n, [\omega]) \to (S^n, [\omega])$ is orientation preserving, i.e., that $[a^*\omega] = [\omega]$. Let $p \in S^n$ and $(e_1, \ldots, e_{n+1})$ be a basis for T_pS^n. Note that $a_* = a$ and so $a_*(e_i) = -e_i$ for all $1 \leq i \leq n+1$. Then

$$\begin{aligned} (a^*\omega)_p(e_1, \ldots, e_{n+1}) &= \omega_{a(p)}(a_*e_1, \ldots, a_*e_{n+1}) \\ &= \omega_{-p}(-e_1, \ldots, -e_{n+1}) \\ &= (-1)^{n+1}\omega(e_1, \ldots, e_{n+1}), \end{aligned}$$

and so $[a^*\omega] = [\omega]$ if and only if n is odd. ☕

(b) Use part (a) and Problem 21.6 to prove that an odd-dimensional real projective space $\mathbb{R}P^n$ is orientable.

Intro to Manifolds, Tu – *End of Section Exercises*

Exercise 23.4. Stokess theorem

Prove Stokes's theorem for $\mathbb{R}^n$ and for $\mathcal{H}^n$.

Proof. (*Stokes's theorem for* $\mathbb{R}^n$)
Let ω be an $(n-1)$-form on $\mathbb{R}^n$ with compact support. Then ω will have the form

$$\omega = \sum_i f_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \widehat{dx^i} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

Since ω has compact support in $\mathbb{R}^n$, there exists $a \in \mathbb{R}$ such that supp $f_i \subsetneq [-a,a]^n$ for all i. Then

$$\begin{aligned}
d\omega &= \sum_i \sum_j \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_i \frac{\partial f_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n && \text{(wedge product is 0 unless } j = i\text{)} \\
&= \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

Since $\partial\mathbb{R}^n = \varnothing$, then $\int_{\partial\mathbb{R}^n} \omega = 0$. Notice that for each i, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x^i} dx^i &= \int_{-a}^{a} \frac{\partial f_i}{\partial x^i} dx^i \\
&= f_i(\dots, x^{i-1}, a, x^{i+1}, \dots) - f_i(\dots, x^{i-1}, -a, x^{i+1}, \dots) \qquad (🌍) \\
&= 0
\end{aligned}$$

since supp $f_i \subsetneq [-a,a]^n$. So

$$\begin{aligned}
\int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^n} \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\
&= \sum_i (-1)^{i-1} \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n \\
&= \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \left(\frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} (0) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= 0
\end{aligned}$$

which gives Stoke's Theorem in $\mathbb{R}^n$.

(*Stokes's theorem for $\mathcal{H}^n$*)

We use the same $(n-1)$-form ω from the first part and consider it as a form on $\mathcal{H}^n$. Notice that $dx^n \equiv 0$ on $\partial\mathcal{H}^n$ since $\partial\mathcal{H}^n$ is defined by the equation $x^n = 0$. So for $i < n$, we have $f_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \widehat{dx^i} \wedge dx^{i+1} \wedge \cdots \wedge dx^n \equiv 0$ on $\partial\mathcal{H}^n$. So

$$\begin{aligned}
\int_{\partial\mathcal{H}^n} \omega &= \int_{\partial\mathcal{H}^n} \sum_i f_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \widehat{dx^i} \wedge dx^{i+1} \wedge \cdots \wedge dx^n \\
&= \sum_i \int_{\partial\mathcal{H}^n} f_i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \int_{\partial\mathcal{H}^n} f_n dx^1 \cdots dx^{n-1}.
\end{aligned}$$

On the other hand, first notice that we have

$$\begin{aligned}
\int_0^\infty \frac{\partial f_n}{\partial x^n} dx^n = \int_0^a \frac{\partial f_n}{\partial x^n} dx^n &= f_n(x^1, \ldots, x^{n-1}, a) - f_n(x^1, \ldots, x^{n-1}, 0) \\
&= -f_n(x^1, \ldots, x^{n-1}, 0).
\end{aligned}$$

So

$$\begin{aligned}
(-1)^{n-1} \int_{\mathcal{H}^n} \frac{\partial f_n}{\partial x^n} dx^1 \cdots dx^n &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial f_n}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1} \\
&= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \left(\int_0^a \frac{\partial f_n}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1} \\
&= (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathcal{H}^n} d\omega &= \int_{\mathcal{H}^n} \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\
&= \sum_i (-1)^{i-1} \int_{\mathcal{H}^n} \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n \\
&= \sum_i^{n-1} (-1)^{i-1} \int_{\mathcal{H}^{n-1}} \left(\int_{-\infty}^\infty \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n + (-1)^{n-1} \int_{\mathcal{H}^n} \frac{\partial f_n}{\partial x^n} dx^1 \cdots dx^{n-1} \\
&= 0 + (-1)^{n-1} \int_{\mathcal{H}^n} \frac{\partial f_n}{\partial x^n} dx^1 \cdots dx^{n-1} \qquad \text{(by (🌍) applied to all } i < n) \\
&= (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1} \\
&= \int_{\partial\mathcal{H}^n} f_n dx^1 \cdots dx^{n-1}
\end{aligned}$$

where the last equality follows from the fact that $(-1)^n\mathbb{R}^{n-1}$ is precisely $\partial\mathcal{H}^n$ with its boundary orientation. ☕

Exercise 23.5. Area form on the sphere S^2

Prove that the area form ω on S^2 in Example 23.11 is equal to the orientation form

$$xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$

of S^2 in Problem 22.9.

Proof. The area form in Example 23.11 is

$$\omega = \begin{cases} \dfrac{dy \wedge dz}{x} & \text{for } x \neq 0 \\[2ex] \dfrac{dz \wedge dx}{y} & \text{for } y \neq 0 \\[2ex] \dfrac{dx \wedge dy}{z} & \text{for } z \neq 0 \end{cases}$$

We can describe S^2 as all the points in $\mathbb{R}^3$ which satisfy the equation $x^2 + y^2 + z^2 = 1$. Taking the exterior derivative of this equation and dividing by 2 we obtain $xdx + ydy + zdz = 0$. So, $dx = (-ydy - zdz)/x$, which gives

$$dx \wedge dy = \frac{z}{x} dy \wedge dz \quad \text{and} \quad dx \wedge dz = \frac{-y}{x} dy \wedge dz.$$

So

$$\begin{aligned} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy &= xdy \wedge dz + \frac{y^2}{x} dy \wedge dz + \frac{z^2}{x} dx \wedge dy \\ &= x + \frac{y^2}{x} + \frac{z^2}{x} dy \wedge dz \\ &= \frac{x^2}{x} + \frac{y^2}{x} + \frac{z^2}{x} dy \wedge dz \\ &= \frac{dy \wedge dz}{x} \end{aligned}$$

when $x \neq 0$. Similarly we obtain the other equations describing ω when $y \neq 0$ and $z \neq 0$. ☕

Exercise 24.1. Nowhere-vanishing 1-forms

Prove that a nowhere-vanishing 1-form on a compact manifold cannot be exact.

Proof. We show the contrapositive statement. Let M be a compact manifold and suppose ω is an exact 1-form on M. Then there exists a smooth function $f \in C^\infty(M)$ such that $df = \omega$. Since M is compact, f attains maximum (or minimum) value at on M by the Extreme Value Theorem. Suppose f attains a maximum at $p \in M$. Then $\omega = df_p = 0$, and so ω is not nowhere vanishing. ☕

Exercise 24.2. Cohomology in degree zero

Suppose a manifold M has infinitely many connected components. Compute its de Rham cohomology vector space $H^0(M)$ in degree 0. (*Hint:* By second countability, the number of connected components of a manifold is countable.)

Proof. By the hint, the number connected components of M is countable. Since there are no nonzero exact 0- forms on M, we have $H^0(M) = Z^0(M) = \{\text{closed 0-forms}\}$. Suppose f is a closed 0-form on M and let $(U, x^1, \ldots, x^n)$ be a chart on M. Then

$$0 = df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

This means that the partial derivatives of f are all zero on U, i.e., f is constant on U. Since f must be constant on each connected component of M, then f can be represented by real-valued sequence: $f = (a_1, a_2, a_3, \ldots)$. Thus $H^0(M) = \mathbb{R}^{\mathbb{N}}$. ☕

Vector Calculus, Colley – *Exercises*

Exercise 8.2.4. A robot arm is constructed in $\mathbb{R}^3$ by anchoring a rod of length 2 to the origin (using a ball joint so that the rod may swivel freely) and attaching to the free end of the rod another rod of length 1 (which may also swivel freely). Show that the set of states of this robot arm may be described by a smooth parametrized 4-manifold in $\mathbb{R}^6$.

Solution:
A point (x_1, y_1, z_1) in a state of the rod of length 2 can be described in spherical coordinates by

$$(x_1, y_1, z_1) = (2\sin\varphi_1\cos\theta_1, 2\sin\varphi_1\sin\theta_1, 2\cos\varphi_1).$$

Similarly, a point (x_2, y_2, z_2) in a state of the rod of length 1 can be described by

$$\begin{aligned}(x_2, y_2, z_2) &= (x_1 + 2\sin\varphi_2\cos\theta_2, y_1 + 2\sin\varphi_2\sin\theta_2, z_1 + 2\cos\varphi_2)\\ &= (2\sin\varphi_1\cos\theta_1 + 2\sin\varphi_2\cos\theta_2, 2\sin\varphi_1\sin\theta_1 + 2\sin\varphi_2\sin\theta_2, 2\cos\varphi_1 + 2\cos\varphi_2).\end{aligned}$$

Let $D = [0,\pi] \times [0, 2\pi) \times [0,\pi] \times [0,2\pi)$, and define a map $\boldsymbol{X} : D \to \mathbb{R}^6$ by

$$X(\varphi_1, \theta_1, \varphi_2, \theta_2) = (x_1, y_1, z_1, x_2, y_2, z_2),$$

where x_1, y_1, z_1, x_2, y_2, and z_2 are as above. Then $\boldsymbol{X}$ is smooth since each of its component functions are smooth.

We now show that $\boldsymbol{X}$ in injective, except possibly on the boundary. Suppose $\boldsymbol{X}(\varphi_1, \theta_1, \varphi_2, \theta_2) = \boldsymbol{X}(\tilde{\varphi_1}, \tilde{\theta_1}, \tilde{\varphi_2}, \tilde{\theta_2})$. This would imply that $\cos\varphi_1 = \cos\tilde{\varphi_1}$, and since $\varphi_1, \tilde{\varphi_1} \in [0,\pi]$, we must have $\varphi_1 = \tilde{\varphi_1}$. This then yields $2\cos\varphi_2 = 2\cos\tilde{\varphi_2}$ (from the last component), which gives $\varphi_2 = \tilde{\varphi_2}$. Using these equations in the first and second components, we see that $\theta_1 = \tilde{\theta_1}$ since we can restrict ourselves to values away from the boundary, i.e., on $(0,\pi) \times (0, 2\pi) \times (0,\pi) \times (0, 2\pi)$. Finally, the fourth and fifth components are deduced to

$$\cos\theta_2 = \cos\tilde{\theta_2}, \quad \text{and} \quad \sin\theta_2 = \sin\tilde{\theta_2},$$

respectively, which gives $\theta_2 = \tilde{\theta_2}$. Then

$$\begin{aligned}\boldsymbol{T}_{\varphi_1} &= \frac{\partial \boldsymbol{X}}{\partial\varphi_1} = (2\cos\varphi_1\cos\theta_1, 2\cos\varphi_1\sin\theta_1, -2\sin\varphi_1, 2\cos\varphi_1\cos\theta_1, 2\cos\varphi_1\sin\theta_1, -2\sin\varphi_1)\\ \boldsymbol{T}_{\theta_1} &= \frac{\partial \boldsymbol{X}}{\partial\theta_1} = (-2\sin\varphi_1, 2\sin\varphi_1\cos\theta_1, 0, -2\sin\varphi_1, 2\sin\varphi_1\cos\theta_1, 0)\\ \boldsymbol{T}_{\varphi_2} &= \frac{\partial \boldsymbol{X}}{\partial\varphi_2} = (0,0,0,\cos\varphi_2\cos\theta_2, \cos\varphi_2\sin\theta_2, -\sin\varphi_2)\\ \boldsymbol{T}_{\theta_2} &= \frac{\partial \boldsymbol{X}}{\partial\theta_2} = (0,0,0,-\sin\varphi_2\sin\theta_2, \sin\varphi_2\cos\theta_2, 0).\end{aligned}$$

Now, consider the equation $c_1\boldsymbol{T}_{\varphi_1} + c_2\boldsymbol{T}_{\theta_1} + c_3\boldsymbol{T}_{\varphi_2} + c_2\boldsymbol{T}_{\theta_2} = (0,0,0,0,0,0)$. Because we are concerned about linear independence of $\boldsymbol{T}_{\varphi_1}, \boldsymbol{T}_{\theta_1}, \boldsymbol{T}_{\varphi_2}, \boldsymbol{T}_{\theta_2}$ on an open neighborhood of a point in $\boldsymbol{X}(D)$, we can again restrict ourselves to points away from the boundary. First, notice that our equation gives $-2c_1\sin\varphi_1 = 0$, which means $c_1 = 0$ since $\sin\varphi_1 \neq 0$ for $\varphi_1 \in (0,\pi)$. We then have $-2c_3\sin\varphi_2 = 0$ and so $c_3 = 0$. Then $-2c_2\sin\varphi_1 = 0$ so that $c_2 = 0$, and then $c_4 = 0$. Hence $\boldsymbol{T}_{\varphi_1}, \boldsymbol{T}_{\theta_1}, \boldsymbol{T}_{\varphi_2}, \boldsymbol{T}_{\theta_2}$ are linearly independent, which completes the problem.

Exercise 8.2.6. Let a, b, and c be positive constants and $\boldsymbol{x} : [0, \pi] \to \mathbb{R}^3$ the smooth path given by $\boldsymbol{x}(t) = (a \cos t, b \sin t, ct)$. If $\omega = bdx - ady + xydz$, calculate $\int_{\boldsymbol{x}} \omega$.

Solution:
First, we have

$$\omega_{\boldsymbol{x}(t)} = bdx - ady + ab \cos t \sin t dz \quad \text{and} \quad \boldsymbol{T}_t = \frac{\partial T}{\partial t} = (-a \sin t, b \cos t, c),$$

and so

$$\omega_{\boldsymbol{x}(t)}(\boldsymbol{T}_t) = -ab \sin t - ab \sin t + abc \cos t \sin t.$$

Then

$$\int_{\boldsymbol{x}} \omega = \int_0^\pi \omega_{\boldsymbol{x}(t)}(T_t) dt = \int_0^\pi (-ab \sin t - ab \sin t + abc \cos t \sin t) dt = -2ab.$$

Exercise 8.2.10. Consider the helicoid parametrized as

$$\boldsymbol{X}(u_1, u_2) = (u_1 \cos 3u_2, u_1 \sin 3u_2, 5u_2), \quad 0 \le u_1 \le 5, 0 \le u_2 \le 2\pi.$$

Let S denote the underlying surface of the helicoid and let Ω be the orientation 2-form defined in terms of $\boldsymbol{X}$ as

$$\Omega = \iota_N (dx \wedge dy \wedge dz).$$

where $N = (-5 \sin 3u_2, 5 \cos 3u_2, -3u_1)$.

(a) Explain why the parametrization $\boldsymbol{X}$ is incompatible with Ω.
Solution:
We have

$$\boldsymbol{T}_{u_1} = (\cos 3u_2, \sin 3u_2, 0) \quad \text{and} \quad \boldsymbol{T}_{u_2} = (-3u_1 \sin 3u_2, 3u_1 \cos 3u_2, 5),$$

and so

$$\begin{aligned}
\Omega_{\boldsymbol{X}(u_1,u_2)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}) &= (\iota_N(dx \wedge dy \wedge dz))_{\boldsymbol{X}(u_1,u_2)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}) \\
&= \det \begin{bmatrix} -5 \sin 3u_2 & \cos 3u_2 & -3u_1 \sin 3u_2 \\ 5 \cos 3u_2 & \sin 3u_2 & 3u_1 \cos 3u_2 \\ -3u_1 & 0 & 5 \end{bmatrix} \\
&= (-5 \sin 3u_2)(5 \sin 3u_2) - (5 \cos 3u_2)(5 \cos 3u_2) \\
&\qquad - 3u_1(3u_1 \cos^2 3u_2 + 3u_1 \sin^2 3u_2) \\
&= -25 - 9u_1^2 < 0
\end{aligned}$$

and so the parametrization $\boldsymbol{X}$ is incompatible with Ω since $\Omega_{\boldsymbol{X}(u_1,u_2)}(T_{u_1}, T_{u_2}) < 0$.

(b) Modify the parametrization $\boldsymbol{X}$ to one having the same underlying surface S but that is compatible with Ω.
Solution:
Define a parametrization $\tilde{\boldsymbol{X}}(u_1, u_2) := \boldsymbol{X}(u_2, u_1)$. This corresponds to interchanging columns 2 and 3 in the determinant computed in (a), and so $\Omega_{\tilde{\boldsymbol{X}}(u_1,u_2)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}) > 0$.

(c) Alternatively, modify the orientation 2-form Ω to Ω' so that the original parametrization $\boldsymbol{X}$ is compatible with Ω'.
Solution:
Define $\Omega' := \iota_N(dy \wedge dx \wedge dz)$. This corresponds to interchanging the rows 1 and 2 in the determinant computed in (a), and so $\Omega'_{\boldsymbol{X}(u_1,u_2)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}) > 0$.

(d) Calculate $\int_S \omega$, where $\omega = zdx \wedge dy - (x^2+y^2)dy \wedge dz$ and S is oriented using Ω.
Solution:
We have

$$\begin{aligned}\omega_{\boldsymbol{X}(u_1,u_2)} &= 5u_2 dx \wedge dy - (u_1^2 \cos^2 3u_2 + u_1^2 \sin^2 3u_2) dy \wedge dz,\\ \boldsymbol{T}_{u_1} &= \frac{\partial \boldsymbol{X}}{\partial u_1} = (\cos 3u_2, \sin 3u_2, 0)\\ \boldsymbol{T}_{u_2} &= \frac{\partial \boldsymbol{X}}{\partial u_2} = (-3u_1 \sin 3u_2, 3u_1 \cos 3u_2, 5).\end{aligned}$$

Then

$$\begin{aligned}\omega_{\boldsymbol{X}(u_1,u_2)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}) &= 5u_2 \det \begin{bmatrix} \cos 3u_2 & -3u_1 \sin 3u_2 \\ \sin 3u_2 & 3u_1 \cos 3u_2 \end{bmatrix}\\ &\qquad - (u_1^2 \cos^2 3u_2 + u_1^2 \sin^2 3u_2) \det \begin{bmatrix} \sin 3u_2 & 3u_1 \cos 3u_2 \\ 0 & 5 \end{bmatrix}\\ &= 15u_1u_2 - 5u_1^2(\sin 3u_2 \cos 3u_2 + \sin^3 2u_2).\end{aligned}$$

Since the parametrization $\boldsymbol{X}$ is orientation-reversing (by part (a)), we have

$$\int_S \omega = -\int_0^{2\pi} \int_0^5 \omega_{\boldsymbol{X}(u_1,u_2)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}) du_1 du_2.$$

So,

$$\begin{aligned}\int_S \omega &= -\int_0^{2\pi} \int_0^5 15u_1u_2 - 5u_1^2(\sin 3u_2 \cos 3u_2 + \sin^3 3u_2) du_1 du_2\\ &= -\int_0^{2\pi} \left(\frac{15}{2}u_1^2 u_2\right)\Bigg|_{u_1=0}^{u_1=5} du_2 + \int_0^{2\pi} \int_0^5 (5u_1^2(\sin 3u_2 \cos 3u_2 + \sin^3 3u_2) du_1 du_2\\ &= -\frac{375}{2} \left(\frac{u_2^2}{2}\right)\Bigg|_{u_2=0}^{u_2=2\pi} + 0\\ &= -375\pi^2.\end{aligned}$$

Exercise 8.2.11. Let M be the subset of $\mathbb{R}^3$ given by $\{(x, y, z) \ \mid x^2 + y^2 - 6 \leq z \leq 4 - x^2 - y^2\}$. Then M may be parametrized as a 3-manifold via

$$\boldsymbol{X} : D \to \mathbb{R}^3; \boldsymbol{X}(u_1, u_2, u_3) = (u_1 \cos u_2, u_1 \sin u_2, u_3),$$

where

$$D = \{(u_1, u_2, u_3) \in \mathbb{R}^3 \ \mid 0 \leq u_1 \leq \sqrt{5}, 0 \leq u_2 < 2\pi, u_1^2 - 6 \leq u_3 \leq 4 - u_1^2\}.$$

(a) Orient M by using the 3-form $\Omega = dx \wedge dy \wedge dz$. Show that the parametrization, when smooth, is compatible with this orientation.
Solution:
We have

$$\begin{aligned} \boldsymbol{T}_{u_1} &= \frac{\partial \boldsymbol{X}}{\partial u_1} = (\cos u_2, \sin u_2, 0) \\ \boldsymbol{T}_{u_2} &= \frac{\partial \boldsymbol{X}}{\partial u_2} = (-u_1 \sin u_2, u_1 \cos u_2, 0) \\ \boldsymbol{T}_{u_3} &= \frac{\partial \boldsymbol{X}}{\partial u_3} = (0, 0, 1) \end{aligned}$$

and

$$\begin{aligned} \Omega_{\boldsymbol{X}(u_1,u_2,u_3)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}, \boldsymbol{T}_{u_3}) &= \det \begin{bmatrix} \cos u_2 & -u_1 \sin u_2 & 0 \\ \sin u_2 & u_1 \cos u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= u_1 \cos^2 u_2 + u_1 \sin^2 u_2 \\ &= u_1. \end{aligned}$$

So when $u_1 > 0$, $\boldsymbol{X}$ is compatible with the orientation form Ω.

(b) Identify ∂M and parametrize it as a union of to 2-manifolds (i.e., as a piecewise smooth surface).
Solution:
There are two pieces to ∂M: One which corresponds to when $z = x^2 + y^2 - 6$ and the other when $z = 4 - x^2 - y^2$. These intersect when $x^2 + y^2 = 5$, i.e., when $z = -1$. So ∂M can be written

$$\partial M = \{(x, y, z) \ \mid z = x^2 + y^2 - 6, z \leq -1\} \cup \{(x, y, z) \ \mid z = 4 - x^2 - y^2, z \geq -1\}.$$

Then we have parametrizations for each piece:

$$\boldsymbol{Y} : [0, \sqrt{5}] \times [0, 2\pi) \to \mathbb{R}^3; \boldsymbol{Y}(s_1, s_2) = (s_1 \cos s_2, s_1 \sin s_2, s_1^2 - 6)$$

and

$$\boldsymbol{Z} : [0, \sqrt{5}] \times [0, 2\pi) \to \mathbb{R}^3; \boldsymbol{Y}(s_1, s_2) = (s_1 \cos s_2, s_1 \sin s_2, 1 - s_1^2).$$

(c) Describe the outward-pointing unit vector $\boldsymbol{V}$, varying continuously along each smooth piece of ∂M, that is normal to ∂M. Give formulas for it in terms of the parametrizations used in part (b).
Solution:
Let U and W be the two portions of ∂M, where U corresponds to $\boldsymbol{Y}$ and W corresponds to $\boldsymbol{Z}$ from part (b).

Notice that U is a portion of the 0 level set of the function $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 + y^2 - z - 6$. Hence an outward pointing vector to U is $\nabla F = (2x, 2y, -1)$. Written in terms of $\boldsymbol{Y}$, $(2s_1 \cos s_2, 2s_1 \sin s_2, -1)$.

Similarly, W is a portion of the 0 level set of the function $G : \mathbb{R}^2 \to \mathbb{R}$, $G(x, y) = x^2 + y^2 + z - 4$. Hence an outward pointing vector to W is $\nabla G = (2x, 2y, 1)$. Written in terms of $\boldsymbol{Z}$, $(2s_1 \cos s_2, 2s_1 \sin s_2, 1)$.

Exercise 8.2.13. Calculate $\int_S \omega$ where S is the portion of the cylinder $x^2 + z^2 = 4$ with $-1 \le y \le 3$, oriented by the outward normal vector $(x, 0, z)$, and $\omega = z dx \wedge dy + e^{y^2} dz \wedge dx + x dy \wedge dz$.

Solution:
S can be parametrized by $\boldsymbol{X} : D \to \mathbb{R}^3$, $\boldsymbol{X}(r, \theta) = (2\sin\theta, r, 2\cos\theta)$ where $D = [-1, 3] \times [0, 2\pi)$. Let $N = (x, 0, z)$ and orient S by the 2-form $\Omega = \iota_N(dx \wedge dy \wedge dz)$. Then

$$\boldsymbol{T}_r = \frac{\partial \boldsymbol{X}}{\partial r} = (0, 1, 0)$$
$$\boldsymbol{T}_\theta = \frac{\partial \boldsymbol{X}}{\partial \theta} = (2\cos\theta, 0, -2\sin\theta)$$

and

$$\Omega_{\boldsymbol{X}(r,\theta)}(\boldsymbol{T}_r, \boldsymbol{T}_\theta) = \det \begin{bmatrix} 2\sin\theta & 0 & 2\cos\theta \\ 0 & 1 & 0 \\ 2\cos\theta & 0 & -2\sin\theta \end{bmatrix} = -4.$$

Hence $\boldsymbol{X}$ is orientation-reversing. Also,

$$\begin{aligned} \omega_{\boldsymbol{X}(r,\theta)}(\boldsymbol{T}_r, \boldsymbol{T}_\theta) &= 2\cos\theta \det \begin{bmatrix} 0 & 2\cos\theta \\ 1 & 0 \end{bmatrix} + e^{r^2} \det \begin{bmatrix} 0 & -\sin\theta \\ 0 & 2\cos\theta \end{bmatrix} + 2\sin\theta \det \begin{bmatrix} 1 & 0 \\ 0 & -2\sin\theta \end{bmatrix} \\ &= -4. \end{aligned}$$

Finally,

$$\int_S \omega = -\int\int_D \omega_{\boldsymbol{X}(r,\theta)}(\boldsymbol{T}_r, \boldsymbol{T}_\theta) = -\int_{-1}^{3}\int_0^{2\pi} -4 = 32\pi.$$

Exercise 8.3.11. Verify the generalized Stokes's theorem for the 3-manifold

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = 8 - 2y^2 - 2z^2 - 2w^2, x \geq 0\}$$

and the 2-form $\omega = xydz \wedge dw$. (*Hint:* First compute $\int_{\partial M} \omega$).

Solution:
Using the hint, we first compute $\int_{\partial M} \omega$. We have

$$\partial M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = 0, 8 = 2y^2 + 2z^2 + 2w^2\}.$$

So for $(x, y, z, w) \in \partial M$, $\omega_{(x,y,z,w)} = 0$. Hence $\int_{\partial M} \omega = 0$.

We can parametrize M by the map $\boldsymbol{X} : B \to \mathbb{R}^4$, $\boldsymbol{X}(u_1, u_2, u_3) = (8 - 2u_1^2 - 2u_2^2 - 2u_3^2, u_1, u_2, u_3)$, where $B = \{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 \leq 4\}$. Now

$$d\omega = ydx \wedge dz \wedge dw + xdy \wedge dz \wedge dw,$$

and

$$\begin{aligned}
\boldsymbol{T}_{u_1} &= \frac{\partial \boldsymbol{X}}{\partial u_1} = (-4u_1, 1, 0, 0) \\
\boldsymbol{T}_{u_2} &= \frac{\partial \boldsymbol{X}}{\partial u_2} = (-4u_2, 0, 1, 0) \\
\boldsymbol{T}_{u_3} &= \frac{\partial \boldsymbol{X}}{\partial u_3} = (-4u_3, 0, 0, 1)
\end{aligned}$$

So

$$\begin{aligned}
(d\omega)_{\boldsymbol{X}(u_1,u_2,u_3)}(\boldsymbol{T}_{u_1}, \boldsymbol{T}_{u_2}, \boldsymbol{T}_{u_3}) = u_1 \det \begin{bmatrix} -4u_1 & -4u_2 & -4u_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
+ (8 - 2u_1^2 - 2u_2^2 - 2u_3^2) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= -4u_1^2 + 8 - 2(u_1^2 - u_2^2 - u_3^2).
\end{aligned}$$

Switching to spherical coordinates, we get

$$\begin{aligned}
\int_M d\omega &= \int \int \int_B d\omega (8 - 2(u_1^2 - u_2^2 - u_3^2) - 4u_1^2) du_1 du_2 du_3 \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} (8 - 2\rho^2 - 4\rho^2 \sin^2\varphi \cos^2\theta)\rho^2 \sin\varphi \ d\rho d\varphi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} (8\rho^2 \sin\varphi - 2\rho^4 \sin\varphi - 4\rho^4 \sin^3\varphi \cos^2\theta)\rho^2 \sin\varphi \ d\rho d\varphi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} [8/3(8)\sin\varphi - 2/5(32)\sin\varphi - 4/5(32) sin^3\varphi \cos\theta] \ d\varphi d\theta \\
&= 8 \int_0^{2\pi} [-8/3\cos\varphi + 8/5\cos\varphi]_0^{\pi} \ d\theta - \frac{4}{5}(32) \int_0^{2\pi} \int_0^{\pi} sin^3\varphi \cos\theta \ d\varphi d\theta \\
&= 8(2\pi)\frac{32}{15} - \frac{4}{5}(32)\left(\frac{4}{3}\pi\right) \\
&= 0.
\end{aligned}$$

Chasing Chains with *Chain Chasing Charlie*

Suppose that the following diagram commutes and both rows are exact. Assume that the first, second, fourth, and fifth vertical maps are isomorphisms and prove that the middle vertical map is an isomorphism.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "i"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "i'"] & E'
\end{tikzcd}$$

Proof. Injectivity: Suppose $\gamma(c) = 0$. Then $0 = h'(\gamma(c)) = \delta(h(c))$ which means $h(c) \in \operatorname{Ker}\delta = 0$. Since $\operatorname{Ker} h = \operatorname{Im} g$, there exists $b \in B$ such that $g(b) = c$. Then $g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0$, and so $\beta(b) \in \operatorname{Ker} g' = \operatorname{Im} f'$. So there exists $a' \in A'$ such that $f'(a') = \beta(b)$. Since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Then $\beta(f(a)) = f'(\alpha(a)) = f'(a') = \beta(b)$. Since β is injective , $f(a) = b$. Since $\operatorname{Im} f = \operatorname{Ker} g$, $g(b) = 0$, and so $c = g(b) = 0$. Hence γ is injective.

Surjectivity: Let $c' \in C'$. Since δ is surjective, there exists $d \in D$ such that $\delta(d) = h'(c')$. Then $\epsilon(i(d)) = i'(\delta(d)) = i'(h'(c')) = 0$ since $h'(c') \in \operatorname{Im} h' = \operatorname{Ker} i'$. So $i(d) \in \operatorname{Ker}\epsilon = 0$, which means $i(d) = 0$ and so $d \in \operatorname{Ker} i = \operatorname{Im} h$. So there exists $c \in C$ such that $h(c) = d$. Then $h'(\gamma(c')) = \delta(h(c)) = \delta(d) = h'(c')$. So $(\gamma(c) - c') \in \operatorname{Ker} h' = Imagg'$, which means there exists $b' \in B'$ such that $g'(b') = \gamma(c) - c'$. Since β is surjective , there exists $b \in B$ such that $\beta(b) = b'$. Then $\gamma(c) - c' = g'(b) = g'(\beta(b)) = \gamma(g(b))$, which implies $\gamma(c - g(b)) = c'$, and hence γ is surjective.