$\boxed{\text{I}}$ Find the length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \leq x \leq 2$$

$\boxed{\text{Solution:}}$

$$\left(\frac{dy}{dx}\right)^2 = \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 + \frac{1}{16x^4} - 2(x^2)\left(\frac{1}{4x^2}\right)$$

$$= x^4 + \frac{1}{16x^4} - \frac{1}{2}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = x^4 + \frac{1}{16x^4} + \frac{1}{2}$$

$$= \left(x^2 + \frac{1}{4x^2}\right)^2$$

$$\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 x^2 + \frac{1}{4x^2}\, dx$$

$$= \frac{x^3}{3} - \frac{1}{4x}\Big|_{x=1}^{x=2} = \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{64}{24} - \frac{3}{24} - \frac{8}{24} + \frac{6}{24} = \frac{59}{24}$$

$\boxed{2}$ The following curve is rotated about the y-axis. Find the area of the resulting surface:

$$x = \sqrt{a^2 - y^2}, \quad 0 \le y \le a/2$$

Solution:

$$\left(\frac{dx}{dy}\right)^2 + 1 = \frac{y^2}{a^2-y^2} + \frac{a^2-y^2}{a^2-y^2} = \frac{a^2}{a^2-y^2}$$

$$\int_0^{a/2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^{a/2} \sqrt{a^2-y^2} \sqrt{\frac{a^2}{a^2-y^2}}\, dy$$

$$= 2\pi \int_0^{a/2} \sqrt{a^2}\, dy$$

$$= 2\pi \int_0^{a/2} |a|\, dy$$

$$= 2\pi a \left(y\right)\Big|_0^{a/2}$$

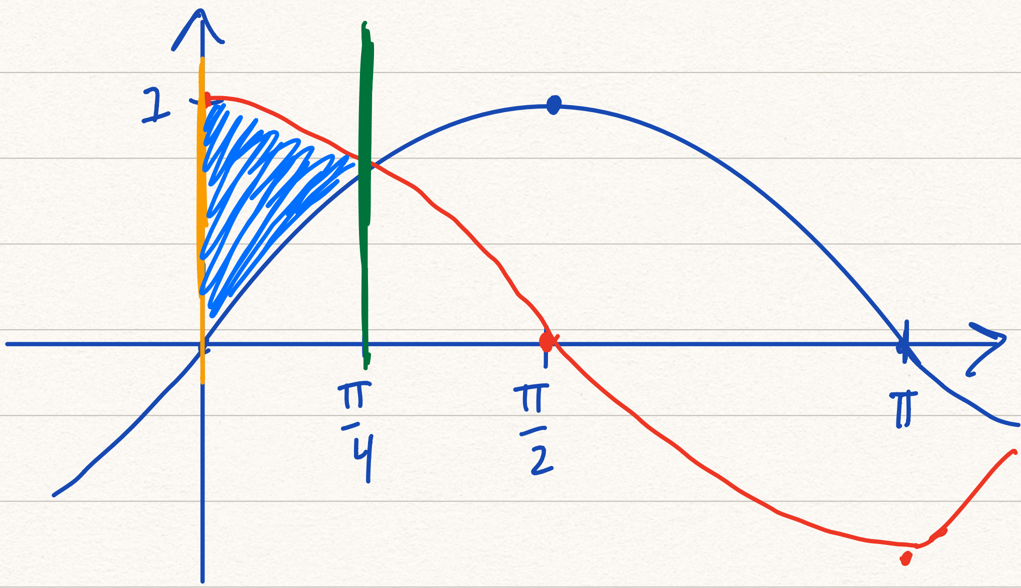
$$= 2\pi a (a/2) = \pi a^2$$

$\boxed{3.}$ Find the centroid of the region bounded by the given curves:

$$y = \sin x \;, \quad y = \cos x, \quad x = 0, \quad x = \frac{\pi}{4}$$

$\boxed{\text{Solution:}}$

$x = 0$, $x = \pi/4$



• $A = \int_0^{\pi/4} \cos x - \sin x \, dx = \sin x + \cos x \Big|_0^{\pi/4}$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1)$$

$$= \frac{2\sqrt{2}}{2} - 1$$

$$= \sqrt{2} - 1$$

• $\bar{x} = \dfrac{1}{A} \displaystyle\int_0^{\pi/4} x(\cos x - \sin x)\, dx$

$$= \frac{1}{A}\int_0^{\pi/4} x\cos x\,dx + \frac{1}{A}\int_0^{\pi/4} x(-\sin x)\,dx$$

$u = x \quad dv = \cos x\,dx$

$du = dx \quad v = \sin x$

$u = x \quad dv = -\sin x\,dx$

$du = dx \quad v = \cos x$

$$= \frac{1}{A}\left[x\sin x\Big|_0^{\pi/4} - \int_0^{\pi/4}\sin x\,dx\right] + \frac{1}{A}\left[x\cos x\Big|_0^{\pi/4} - \int_0^{\pi/4}\cos x\,dx\right]$$

$$= \frac{1}{\sqrt{2}-1}\Big[x\sin x + \cos x + x\cos x - \sin x\Big]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}-1}\left[\left(\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\right) - \left(0 + 1 + 0 - 0\right)\right]$$

$$= \frac{1}{\sqrt{2}-1}\left[\frac{\pi\sqrt{2} + 4\sqrt{2} + \pi\sqrt{2} - 4\sqrt{2}}{8} - 1\right]$$

$$= \frac{1}{\sqrt{2}-1}\left[\frac{2\pi\sqrt{2}}{8} - 1\right]$$

$$= \frac{1}{\sqrt{2}-1}\left[\frac{\pi}{4}\cdot\sqrt{2} - \frac{4}{4}\right]$$

$$= \frac{\pi\sqrt{2} - 4}{4\sqrt{2} - 4}$$

$$\cdot\ \bar{y} = \frac{1}{A}\int_0^{\pi/4} \frac{1}{2}(\cos^2 x - \sin^2 x)\, dx$$

$$= \frac{1}{2A}\int_0^{\pi/4} \cos^2 x\, dx - \frac{1}{2A}\int_0^{\pi/4} \sin^2 x\, dx$$

$$= \frac{1}{2A}\int_0^{\pi/4} \frac{1}{2}(1+\cos 2x)\, dx - \frac{1}{2A}\int_0^{\pi/4} \frac{1}{2}(1-\cos 2x)dx$$

$$= \frac{1}{4A}\left[x + \frac{1}{2}\sin 2x - x + \frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \frac{1}{4\sqrt{2}-4}\left[\left(\frac{\pi}{4} + \frac{1}{2}(1) - \frac{\pi}{4} + \frac{1}{2}(1)\right) - (0+0-0-0)\right]$$

$$= \frac{1}{4\sqrt{2}-4}\left[1 - 0\right]$$

$$= \frac{1}{4\sqrt{2}-4}$$