Intro to Abstract Algebra | **Answer Key**
Spring 2020 – March 9 | Quiz 6 – Sections 9 & 10
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1. Let G be a permutation group on S, and define a relation $\sim$ on S by 3 / 3

$$a \sim b \quad \text{iff} \quad \alpha(a) = b \quad \text{for some } \alpha \in G.$$

Then $\sim$ is an equivalence relation (you do **not** need to prove this). Find the equivalence classes and a complete set of equivalence class representatives in each of the following special cases:

2. For polynomials $f(x)$ and $g(x)$ with real coefficients, let $f(x) \sim g(x)$ mean that $f'(x) = g'(x)$ (where the primes denote derivatives). Give a complete set of equivalence class representatives. (A polynomial with real coefficients is an expression of the form $a_0 + a_1x + \cdots + a_nx^n$ were $a_0, a_1, \ldots, a_n \in \mathbb{R}$.) 2 / 2

Proof. Two polynmoials have the same derivative if and only if they are of the same degree and have the same non-constant coefficients. Consider the set

$$S = \{1 + a_1x + a_2x^2 + \cdots + a_nx^n : n \in \mathbb{Z}_{\geq 1}, a_i \in \mathbb{R}\}$$

If $f(x) = b_0 + b_1x + \cdots + b_nx^n$ is any polynomial with real coefficients, then the polynomial

$$g(x) = 1 + b_1x + b_2x^2 + \cdots + b_nx^n$$

is an element in the set S such that $f'(x) = g'(x)$. Now we need to show that $g(x)$ is a *unique* element of S. So, suppose that

$$h(x) = 1 + c_1x + c_2x^2 + \cdots + c_mx^m$$

is another element of S such that $f(x) \sim h(x)$, or in other words $f'(x) = h'(x)$. In order for this to be true, it must be that $m = n$, i.e. f and h must have the same degree. Then, we obtain

$$b_1+2b_2x+3b_3x^2+\cdots+nb_nx^{n-1} = f'(x) = h'(x) = c_1+2c_2x+3c_3x^2+\cdots+nc_nx^{n-1}.$$

But this means that

$$(b_1 - c_1) + 2(b_2 - c_2)x + 3(b_3 - c_3)x^2 + \cdots + n(b_n - c_n)x^{n-1} = 0,$$

which implies $(b_i - c_i) = 0$ for all $1 \leq i \leq n$, or in other words $b_i = c_i$ for all $1 \leq i \leq n$. Therefore $g(x) = h(x)$. So, the set S is a complete set of equivalence class representatives. □

3. There are ten integers x such that $-25 < x < 25$ and $x \equiv 3 \mod 5$. Find them all. 1 / 1

Solution: Said another way, these are all the integers between -25 and 25 which are 3 more than multiple of 5. One way to do this problem is to simply pick one number which is 3 more than a multiple of 5, say 13, and then repeatedy add/subtract 5.

$$-22, -17, -12, -7, -2, 3, 8, 13, 18, 23.$$

4. For each pair a, b, find the unique integers q and r such that $a = bq + r$ with $0 \leq r < b$. 3 / 3

   (a) $a = 19, b = 5$.
   Solution:
   $$q = 3, r = 4 \implies 19 = (5)(3) + 4.$$

   (b) $a = -7, b = 5$.
   Solution:
   $$q = -2, r = 3 \implies -7 = (5)(-2) + 3.$$

   (c) $a = 11, b = 17$.
   Solution:
   $$q = 0, r = 11 \implies 11 = (17)(0) + 11.$$

5. Consider the following statement: 1 / 1

$$\text{If } a \equiv b \mod n, \text{ then } a^2 \equiv b^2 \mod n^2.$$

If the statement is true, give a proof. If it is false, give a counterexample.

Solution: The statement is false. Notice that $1 \equiv 4 \mod 3$, but $1^2 = 1 \equiv 1 \mod 9$ and $4^2 = 16 \equiv 7 \mod 9$. There are many other counterexamples you could have chosen.