Intro to Abstract Algebra
Spring 2020 – March 3
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Answer Key
Quiz 5 – Section 9
Total: 12 / 12

1. If $\mathcal{P} = \{\{1,4,5\},\{2,3\}\}$, then $\mathcal{P}$ is a partition of $\{1,2,3,4,5\}$. For the corresponding equivalence relation $\sim$, which of the following are true? (Circle True or False). 4 / 4

   (a) $4 \sim 5$

   (True) False

   (b) $3 \sim 3$

   (True) False

   (c) $1 \sim 2$

   True (False)

   (d) $5 \sim 1$

   (True) False

2. Define a relation $\sim$ on the set $\mathbb{N}$ of natural numbers by 2 / 2

$$a \sim b \text{ iff } a = b \cdot 10^k \text{ for some } k \in \mathbb{Z}.$$

   Give a complete set of equivalence class representatives.

   Solution:

   As was mentioned during the quiz, you only needed to give an answer for this one, since it was proven in class. The answer is:

$$\{a \in \mathbb{N} : 10 \not| a\} = \{1, 2, \ldots, 9, 11, 12 \ldots, 19, 21, 22, \ldots\}$$

3. For $x, y \in \mathbb{R}$, let $x \sim y$ mean that $xy > 0$. Which properties of an equivalence relation does $\sim$ satisfy? 2 / 2

Solution:

The relation $\sim$ is not reflexive. We must have $x \sim x$ *for all* $x \in \mathbb{R}$. This is true for any $x \in \mathbb{R}$ that is not zero. But, $0 \not\sim 0$ since $0 \cdot 0 \not> 0$.

Suppose $x, y \in \mathbb{R}$ and $x \sim y$, i.e. $xy > 0$. Then $yx = xy > 0$, which means $y \sim x$, and hence $\sim$ is symmetric.

Suppose $x, y, z \in \mathbb{R}$, $x \sim y$, and $y \sim z$. Then $xy > 0$ and $yz > 0$. This means that either x, y, z are all negative or all positive. In either case, $xz > 0$, meaning $x \sim z$, and hence $\sim$ is transitive.

4. How many different equivalence relations are there on a four-element set? 2 / 2

Solution:

There are 15. Since equivalence relations are in bijection with partitions, here are all 15:

$$\{\{a,b,c,d\}\}, \quad \{\{a\},\{b,c,d\}\}, \quad \{\{b\},\{a,c,d\}\}, \quad \{\{c\},\{a,b,d\}\},$$

$$\{\{d\},\{a,b,c\}\}, \quad \{\{a,b\},\{c,d\}\}, \quad \{\{a,d\},\{b,c\}\}, \quad \{\{a,c\},\{b,d\}\},$$

$$\{\{a\},\{b\},\{c\},\{d\}\} \quad \{\{a\},\{b\},\{c,d\}\}, \quad \{\{a\},\{c\},\{b,d\}\}, \quad \{\{a\},\{d\},\{b,c\}\},$$

$$\{\{b\},\{c\},\{a,d\}\}, \quad \{\{b\},\{d\},\{a,c\}\}, \quad \{\{c\},\{d\},\{a,b\}\}.$$

5. For points (x_1, y_1) and (x_2, y_2) in a plane with rectangular coordinate system, let $(x_1, y_1) \sim (x_2, y_2)$ mean that $x_1 - x_2$ is an integer. 2 / 2

   (a) Prove that $\sim$ is an equivalence relation.

   Proof. $(x_1, y_1) \sim (x_1, y_1)$ since $x_1 - x_1 = 0 \in \mathbb{Z}$, so $\sim$ is reflexive.
   Suppose $(x_1, y_1) \sim (x_2, y_2)$, i.e. $x_1 - x_2 \in \mathbb{Z}$. Then

$$x - 2 - x_1 = -(x_1 - x_2) \in \mathbb{Z}$$

   which means $(x_2, y_2) \sim (x_1, y_1)$, so $\sim$ is symmetric.

Suppose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, i.e. $x_1 - x_2 \in \mathbb{Z}$ and $x_2 - x_3 \in \mathbb{Z}$. Then

$$x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3) \in \mathbb{Z}$$

since $\mathbb{Z}$ is closed under addition. Hence $(x_1, y_1) \sim (x_3, y_3)$, so $\sim$ is transitive. Since $\sim$ is reflexive, symmetric, and transitive, then it is an equivalence relation. □

(b) Give a complete set of equivalence class representatives.

Solution:

Consider the set

$$S = \{(x, 0) \in \mathbb{R}^2 : x \in [0, 1)\}.$$

For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, (for example, $\lfloor -3.14 \rfloor = -4$ and $\lfloor 10 \rfloor = 10$). Then $x - \lfloor x \rfloor \in [0, 1)$.

Now, suppose $(x, y) \in \mathbb{R}^2$. Consider the element $(x - \lfloor x \rfloor, 0) \in S$. Notice that

$$x - (x - \lfloor x \rfloor) = \lfloor x \rfloor \in \mathbb{Z},$$

which means $(x, y) \sim (x - \lfloor x \rfloor, 0)$. Suppose $(a, 0)$ is another element of S such that $x - a \in \mathbb{Z}$. Since $0 \leq a < 1$, we have

$$x - 1 < x - a \leq x.$$

Since $x - a$ is an integer, then $x - a$ must be equal to $\lfloor x \rfloor$, which means $a = x - \lfloor x \rfloor$. Hence S is a complete set of representatives.