Intro to Abstract Algebra
Spring 2020 – February 11
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Answer Key
Quiz 3 – Section 6
Total: 10 / 10

1. Write each of the following as a single cycle or a product of disjoint cycles: 2 / 2

   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 5 & 3 & 2 \end{pmatrix}$

   Solution: $(2\ 6)(3\ 4\ 5)$

   (b) $(1\ 2\ 3)^{-1}\,(2\ 3)\,(1\ 2\ 3)$

   Solution: $(1\ 2\ 3)^{-1}\,(2\ 3)\,(1\ 2\ 3) = (1\ 3\ 2)\,(2\ 3)\,(1\ 2\ 3) = (1\ 2)$

2. (a) Write $(a_1 a_2 \cdots a_k)^{-1}$ in cycle notation (without the symbol for inverse). 2 / 2
   Solution: $(a_1 a_2 \cdots a_k)^{-1} = (a_k a_{k-1} a_{k-2} \cdots a_3 a_2 a_1)$.

   (b) For which values of k will every k-cycle be its own inverse?
   Solution: For $k = 1$ and $k = 2$, every k cycle will be its own inverse.

   Although you didn't need to provide an explanation on the quiz, here is why:

   Let $\alpha = (a_1 a_2 \cdots a_k)$ be a k-cycle in S_n, where evidently $k \leq n$. Then

   $$\alpha \circ \alpha = (a_1 a_2 \cdots a_k)(a_1 a_2 \cdots a_k) = \begin{cases} (a_1 a_3 \cdots a_{k-2} a_k a_2 a_4 \cdots a_{k-3} a_{k-1}) & \text{if } k \text{ is odd} \\ (a_1 a_3 \cdots a_{k-3} a_{k-1})(a_2 a_4 \cdots a_{k-2} a_k) & \text{if } k \text{ is even} \end{cases}$$

   Note that these are the ways we write $\alpha \circ \alpha$ *uniquely* as a product of disjoint cycles.

   Now if α is its own inverse, then $\alpha \circ \alpha = (1)$. So if k is odd, then

   $$(1) = \alpha \circ \alpha = (a_1 a_3 \cdots a_{k-2} a_k a_2 a_4 \cdots a_{k-3} a_{k-1}),$$

   which, by uniqueness of cycle decomposition, is only possible if $k = 1$. Remember that we can write the identity (1) in n different ways:

   $$(1) = (1)(2) = (1)(2)(3) = \cdots = (1)(2)(3)\cdots(n).$$

   If k is even, then

   $$(1)(2) = (1) = \alpha \circ \alpha = (a_1 a_3 \cdots a_{k-3} a_{k-1})(a_2 a_4 \cdots a_{k-2} a_k).$$

   which, by uniqueness of cycle decomposition, is only possible if $k = 2$.

3. Let $\sigma \in S_n$ be a 3-cycle. How many conjugates does σ have? (You may use the fact that any two k-cycles in S_n are conjugate to each other.) 2 / 2

Proof. By the hint, we know that any conjugate of a 3-cycle is also a 3-cycle. Hence, we need to count the number of 3-cycles in S_n. Consider an arbitrary 3-cycle $(a_1a_2a_3) \in S_n$. Then, we have n choices for a_1, $n-1$ choices for a_2, and $n-2$ choices for a_3. So far, we've counted $n(n-1)(n-2)$ 3-cycles in S_n. However, we have counted some cycles too many times: We know that

$$(a_1a_2a_3) = (a_2a_3a_1) = (a_3a_1a_2),$$

and hence there are 3 ways to write the same 3-cycle. Therefore, there are $\dfrac{n(n-1)(n-2)}{3}$ 3-cycles in S_n. □

4. Assume that α and β are dijoint cycles representing elements of S_n, say $\alpha = (a_1a_2\cdots a_s)$ and $\beta = (b_1b_2\cdots b_t)$ with $a_i \neq b_j$ for all i and j. 4 / 4

   (a) Compute $(\alpha\circ\beta)(a_k)$ and $(\beta\circ\alpha)(a_k)$ for $1 \le k \le s$. [Here $(\alpha\circ\beta)(a_k)$ denotes the image of a_k under the mapping $\alpha\circ\beta$; that is, (a_k) is not 1-cycle.]
   Solution:

$$\begin{aligned}
(\alpha\circ\beta)(a_k) &= \alpha(\beta(a_k)) \\
&= \alpha(a_k). && \text{(since } \beta \text{ fixes } a_k.\text{)} \\
(\beta\circ\alpha)(a_k) &= \beta(\alpha(a_k)) \\
&= \alpha(a_k). && \text{(since } \beta \text{ fixes } \alpha(a_k).\text{)}
\end{aligned}$$

   (b) Compute $(\alpha\circ\beta)(b_k)$ and $(\beta\circ\alpha)(b_k)$ for $1 \le k \le t$.
   Solution:

$$\begin{aligned}
(\alpha\circ\beta)(b_k) &= \alpha(\beta(b_k)) \\
&= \beta(b_k). && \text{(since } \alpha \text{ fixes } \beta(b_k).\text{)} \\
(\beta\circ\alpha)(b_k) &= \beta(\alpha(b_k)) \\
&= \beta(b_k). && \text{(since } \alpha \text{ fixes } b_k.\text{)}
\end{aligned}$$

   (c) Compute $(\alpha\circ\beta)(m)$ and $(\beta\circ\alpha)(m)$ for $1 \le m \le n$ with $m \neq a_i$ and $m \neq b_j$ for all i and j.

Solution:

$$\begin{aligned}
(\alpha \circ \beta)(m) &= \alpha(\beta(m)) \\
&= \alpha(m) && \text{(since } \beta \text{ fixes } m.) \\
&= m. && \text{(since } \alpha \text{ fixes } m.) \\
(\beta \circ \alpha)(m) &= \beta(\alpha(m)) \\
&= \beta(m) && \text{(since } \alpha \text{ fixes } m.) \\
&= m. && \text{(since } \beta \text{ fixes } m.)
\end{aligned}$$

(d) What do parts (a), (b), (c), taken together, prove about the relationship between $\alpha \circ \beta$ and $\beta \circ \alpha$?

Answer: We showed for *any* number ℓ, that $(\alpha \circ \beta)(\ell) = (\beta \circ \alpha)(\ell)$. In other words, $\alpha \circ \beta$ and $\beta \circ \alpha$ are really the *same* map: $\alpha \circ \beta = \beta \circ \alpha$. Therefore, we proved that when two cycles are disjoint, they commute with each other.