Intro to Abstract Algebra
Spring 2020 – February 4
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Answer Key
Quiz 2 – Section 5
Total: 26 / 26

*Show **all** of your work in the space provided.*

1. Determine if the following set of numbers forms a group with the given operation: 2 / 2

$$(\{-1, 0, 1\}, +)$$

Proof. $(\{-1, 0, 1\}, +)$ is not a group, because the set is not closed under the "+" operation:

$$1 + 1 = 2 \notin \{-1, 0, 1\}.$$

□

2. Verify that $\{2^m : m \in \mathbb{Z}\}$ is a group with respect to multiplicaiton. Identify12 / 12 clearly the properties of $\mathbb{Z}$ (and/or $\mathbb{R}$) that you use.

Proof. $\{2^m : m \in \mathbb{Z}\}$ is a group. For associativity,

$$(2^\ell \cdot 2^m) \cdot 2^n = 2^\ell \cdot (2^m \cdot 2^n) \ \forall \ell, m, n \in \mathbb{Z}$$

because

$$\begin{aligned} \text{LHS} &= 2^{\ell+m} \cdot 2^n \\ &= 2^{(\ell+m)+n} \\ &= 2^{\ell+(m+n)} && \text{(by associativity of } (\mathbb{Z},+)) \\ &= 2^\ell \cdot (2^m \cdot 2^n) \\ &= \text{RHS}. \end{aligned}$$

The identity is $2^0 \in \{2^m : m \in \mathbb{Z}\}$, because for all $m \in \mathbb{Z}$,

$$\begin{aligned} 2^0 \cdot 2^m &= 2^{0+m} \\ &= 2^m && \big(\text{since 0 is the identity of } (\mathbb{Z},+), \text{ we have } 0+m=m=m+0\big) \\ &= 2^{m+0} \\ &= 2^m \cdot 2^0. \end{aligned}$$

The element 2^{-m} is in the set, and serves as the inverse of 2^m, because

$$\begin{aligned} 2^m \cdot 2^{-m} &= 2^{m+(-m)} \\ &= 2^0 && \big(-m \text{ is the inverse of } m \text{ in } (\mathbb{Z},+), \text{ so } m+(-m)=0=(-m)+m\big) \\ &= 2^{(-m)+m} \\ &= 2^{-m} \cdot 2^m. \end{aligned}$$

□

3. Let S be a nonempty set, let G be a group, and let G^S denote the set of all 12 / 12 mappings from S to G. Find an operation on G^S that will yeild a group.

Proof. Let $f, g \in G^S$. Define an operation $*$ on G^S by

$$(f * g)(s) = f(s) *_G g(s) \text{ for all } s \in S,$$

where $*_G$ denotes the operation for the group G. Then $f * g \in G^S$. We need to check that $(G^S, *)$ yeilds a group.

For associativity, $(f * g) * h = f * (g * h)$ holds for all $f, h, g \in G^S$ because by the associativity of the group $(G, *_G)$,

$$(f(s) *_G g(s)) *_G h(s) = f(s) *_G (g(s) *_G h(s))$$

for all $s \in S$.

Let $\mathrm{id} : S \to G$ be given by $s \mapsto 1_G$ for all $s \in S$, where 1_G is the identity of G. Then $\mathrm{id} \in G^S$, and $\mathrm{id} * f = f = f * \mathrm{id}$ for all $f \in G^S$ because for all $s \in S$,

$$\begin{aligned}(\mathrm{id} * f)(s) &= \mathrm{id}(s) *_G f(s) \\ &= 1_G *_G f(s) \\ &= f(s) \\ &= f(s) *_G 1_G \\ &= f(s) *_G \mathrm{id}(s) \\ &= (f * \mathrm{id})(s)\end{aligned}$$

Finally, if $f \in G^S$, let $f^{-1} : S \to G$ be given by $s \mapsto (f(s))^{-1}$ for all $s \in S$. Then $f^{-1} \in G^S$, and $f * f^{-1} = \mathrm{id} = f^{-1} * f$ because for all $s \in S$,

$$\begin{aligned}(f^{-1} * f)(s) &= f^{-1}(s) *_G f(s) \\ &= (f(s))^{-1} *_G f(s) \\ &= 1_G \\ &= f(s) *_G (f(s))^{-1} \\ &= f(s) *_G f^{-1}(s) \\ &= (f * f^{-1})(s).\end{aligned}$$

□