Intro to Abstract Algebra
Spring 2020 – January 28
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Answer Key
Quiz 1 – Section 5
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*Show **all** of your work in the space provided.*

1. Let $a, b \in \mathbb{R}$ with $a \neq 0$. Let A denote the set of mappings $\alpha_{a,b} : \mathbb{R} \to \mathbb{R}$ defined by $\alpha_{a,b}(x) = ax + b$ for each $x \in \mathbb{R}$. Show that $(A, \circ)$ where $\circ$ is the composition operation, is a group. 9 / 9

Proof. First, note that for all $a, a', b, b' \in \mathbb{R}$ with $a, a' \neq 0$, and for all $x \in \mathbb{R}$, we have

$$(\alpha_{a',b'} \circ \alpha_{a,b})(x) = a'(ax + b) + b' = \alpha_{a'a,a'b+b'}(x).$$

In other words $\alpha_{a',b'} \circ \alpha_{a,b} = \alpha_{a'a,a'b+b'}$.

To show that the set A is a group with the operation $\circ$, we must show that this operation is *associative*, that there exists an element in A that serves as an *identity*, and that each element in A has an *inverse*.

For associativity, we have

$$\begin{aligned}\alpha_{a'',b''} \circ (\alpha_{a',b'} \circ \alpha_{a,b}) &= \alpha_{a'',b''} \circ \alpha_{a'a,a'b+b'} \\ &= \alpha_{a''a'a,a''(a'b+b')+b''},\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}(\alpha_{a'',b''} \circ \alpha_{a',b'}) \circ \alpha_{a,b} &= \alpha_{a''a',a''b'+b''} \circ \alpha_{a,b} \\ &= \alpha_{a''a'a,a''(a'b+b')+b''},\end{aligned}$$

showing associativity.

The element $\alpha_{1,0} \in A$ serves as an identity under the operation $\circ$, since

$$\begin{aligned}\alpha_{1,0} \circ \alpha_{a,b} &= \alpha_{1\cdot a,1\cdot b+0} = \alpha_{a,b} \\ \alpha_{a,b} \circ \alpha_{1,0} &= \alpha_{a\cdot 1,a\cdot 0+b} = \alpha_{a,b}.\end{aligned}$$

Finally, if $\alpha_{a,b} \in A$, the element $\alpha_{a^{-1},-ba^{-1}}$ is such that

$$\begin{aligned}\alpha_{a,b} \circ \alpha_{a^{-1},-ba^{-1}} &= \alpha_{aa^{-1},-aba^{-1}+b} = \alpha_{1,0} \\ \alpha_{a^{-1},-ba^{-1}} \circ \alpha_{a,b}, &= \alpha_{a^{-1}a,a^{-1}b-ba^{-1}} = \alpha_{1,0}.\end{aligned}$$

which means $(\alpha_{a,b})^{-1} = \alpha_{a^{-1},-ba^{-1}}$, and hence each element in A has an inverse.

Therefore, the set A, together with the operation $\circ$, defines a group. □

2. Let $GL(2,\mathbb{R})$ be the set of 2×2 real matrices with non-zero determinant. Show that $GL(2,\mathbb{R})$ with matrix multiplication operation is a group. (No need to show associativity.) 6 / 6

Proof. We only need to show that $GL(2,\mathbb{R})$ (together with the operation of matrix multiplication) has an identity element, and that each element in $GL(2,\mathbb{R})$ has an inverse.

First, the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ serves as the identity element for $GL(2,\mathbb{R})$, since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{R})$.

Next, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{R})$, then the element

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is such that

$$\begin{aligned} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and similarly,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right] &= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

However, we need to check that the element $\frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ is in fact an element of $GL(2,\mathbb{R})$. We need to check that this element has nonzero determinant. So,

$$\det \left[\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right] = \frac{1}{ad-bc} \det \left[\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right] = \frac{1}{ad-bc}(ad-bc) = 1.$$

Hence, $\frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \in GL(2,\mathbb{R})$, which means $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, completing the proof that $GL(2,\mathbb{R})$ is a group. ∎