Topology Qual Prep Review 2017

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Contents

1 General Topology/Algebraic Topology

1.1 "Basics"

1. **Iowa Qual, Fall 2005**

   Let $\mathbb{R}P(n)$ be the quotient space obtained from $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation, that two points are equivalent if they are scalar multiples of one another. Prove that $\mathbb{R}P(n)$ is second countable, Hausdorff, and compact.

   Proof. Write $\mathbb{R}^{n+1}_*$ in place of $\mathbb{R}^{n+1} \setminus \{0\}$. Let $q : \mathbb{R}^{n+1}_* \to \mathbb{R}P(n)$ be the quotient map, and denote equivalence classes in $\mathbb{R}P(n)$ by brackets: $[x]$.

   We first show that the quotient map is open. Consider the maps $f_\lambda : \mathbb{R}^{n+1}_* \to \mathbb{R}^{n+1}_*, f_\lambda(x) = \lambda x$ where $\lambda \in \mathbb{R}^{n+1}_*$. Notice that f_λ is a homeomorphism since it has inverse $f_{1/\lambda}$, and both $f_\lambda$and $f_{1/\lambda}$ have component functions which are linear, and hence are continuous. In particular, f_λ is open. Now, let $U \subseteq \mathbb{R}^{n+1}_*$ be open. We want to show that $q(U)$ is open. By definition of the quotient topology, $q(U)$ is open if $q^{-1}(q(U)$ is open. Notice that

   $$q^{-1}(q(U)) = q^{-1}\left(\bigcup_{x\in U}[x]\right) = \bigcup_{x\in U} q^{-1}([x]) = \bigcup_{\substack{x\in U\\ \lambda\in\mathbb{R}^{n+1}_*}} \lambda x = \bigcup_{\lambda\in\mathbb{R}^{n+1}_*} f_\lambda(U),$$

   and so, being the union of open sets, $q^{-1}(q(U))$ is open.

   Now recall that $\mathbb{R}^{n+1}$ is second countable, with countable basis consisting of all products of intervals with rational endpoints, call it $\{U_i\}$. Then $\{q(U_i)\}$ is a countable basis for $\mathbb{R}P(n)$.

   Since the quotient map is open, the quotient space is Hausdorff if and only if the defining relation is closed. So, define

   $$R = \{(x, y) \in \mathbb{R}^{n+1}_* \times \mathbb{R}^{n+1}_* \mid \exists \lambda \in \mathbb{R}^{n+1}_* \text{ such that } \lambda x = y\}.$$

   Now $(x_0, \ldots, x_n, y_0, \ldots, y_n) = (x, y) \in R$ precisely when $\det \begin{bmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{bmatrix} = 0$, since the first row is a multiple of the second. For $i < j$, where $i, j \in \{0, \ldots, n\}$, define maps

   $$D_{i,j}(x, y) = \det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix}.$$

   Since the determinant is continuous (since it is a polynomial in the global coordinates of $\mathbb{R}^{n+1}$), we get a continuous map

   $$f(x, y) = \sum_{0\le i<j\le n} D_{i,j}(x, y).$$

   Since $R = f^{-1}(\{0\})$, then R is closed, being the preimage of a closed set under a continuous map.

   Finally, notice that for $[x] \in \mathbb{R}P(n)$, we can find $x' \in S^n$ such that $[x] = [x']$ since we can take $\lambda = 1/||x||$, and $x' = \lambda x$. So, the restriction $q|_{S^n}$ is a surjection onto $\mathbb{R}P(n)$, and since S^n is compact and q is continuous, then $\mathbb{R}P(n) = q(S^n)$ is compact. □

2. **Iowa Qual, Fall 2006**

Define the uniform and box topologies on a product of topological spaces. Let $X = \mathbb{R}^J$ be the product of a countable number of copies of the real numbers. Prove that the product, uniform, and box topologies yield three distinct, non-homeomorphic topologies on X.

Proof. Given $\{X_\alpha\}$, a collection of topological spaces, the *box topology* on $X := \prod_\alpha X_\alpha$ consists of basis elements of the form $\prod_\alpha U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open. In the *product topology*, we take as basis elements $\prod_\alpha U_\alpha$ where again $U_\alpha \subseteq X_\alpha$ is open, with the restriction that for all but finitely many α, $U_\alpha = X_\alpha$.The *uniform topology* on X is that topology induced by the metric

$$d((x_n), (y_n)) = \sup_{n \in \mathbb{N}}\{\min\{|x_n - y_n|, 1\}\}.$$

Now, letting $X = \mathbb{R}^{\mathbb{N}}$, we will exhibit topological properties possessed by X in one topology, but not in the others.

<u>X is connected in the Product, but not in the Uniform or Box:</u>

Consider X in the product topology and define $\tilde{\mathbb{R}}^n$ to be the set of all sequences in X which are zero after the nth coordinate. Being homeomorphic to $\mathbb{R}^n$, $\tilde{\mathbb{R}}^n$ is connected. Then taking $\mathbb{R}^\infty := \bigcup_{n \in \mathbb{N}} \tilde{\mathbb{R}}^n$, we get that $\mathbb{R}^\infty$ is connected, since it is the union of connected spaces which share the point $(0, 0, \dots)$. We now show that in fact $X = \overline{\mathbb{R}^\infty}$, which will show that X is connected, since it is the closure of a connected space. So, let $(x_n) \in X$, and $U = \prod_{n \in \mathbb{N}} U_n$ be a neighborhood of (x_n). Since we are in the product topology, there exists $N \in \mathbb{N}$ such that $U_k = \mathbb{R}$ for all $k > N$. Then,

$$(x_1, x_2, \dots, x_N, 0, 0 \dots) \in U \cap \mathbb{R}^\infty,$$

and so $X = \overline{\mathbb{R}^\infty}$.

Now, define A to be the set of bounded sequences in X, and B to be the set of unbounded sequences in X. Evidently these sets are disjoint, nonempty, and their union is all of X. We show that A and B are open in both the box and uniform topologies, and therefore form a separation of X in each topology. Let $(x_n) \in X$. Then the set

$$U = \prod_{n \in \mathbb{N}} (x_n - \epsilon, x_n + \epsilon) \qquad (0 < \epsilon < 1)$$

is open in the box topology and contains (x_n). If $(x_n) \in A$ with bound M, then U is contained in A, since any sequence $(y_n) \in U$ will be bounded (by $M + \epsilon$). If, on the other hand, $(x_n) \in B$, the set U is contained in B since any sequence in U will be unbounded.

In the uniform topology, the δ-ball $B((x_n), \delta)$, $0 < \delta < 1$, is an open set containing (x_n). By similar reasoning as above, $(x_n) \in B((x_n), \delta) \subset A$ if $(x_n) \in A$, and $(x_n) \in B((x_n), \delta) \subset B$ if $(x_n) \in B$. Hence A and B are indeed open in the box topology.

X is metrizable in the Uniform, but not in the Box:

Evidently X is metrizable in the uniform topology. Now, since metrizable spaces are first countable, we show that X is not first countable, and hence not metrizable, in the box topology. Suppose it were. Then for a point $(x_n) \in X$, we find a countable basis at (x_n), $\{U_i\}_{i=1}^{\infty}$. Write $U_i = \prod_{j=1}^{\infty} U_{ij}$ for each i. Then, we can choose $V_i \subset U_{ii}$ open so that $(x_n) \in V_i$ and $\overline{V_i} \subset U_{ii}$. But then $V = \prod_{i=1}^{\infty} V_i$ is a neighborhood of (x_n) which does not contain any U_i, a contradiction. □

3. **Iowa Qual, Fall 2006**

Let X be $S^2 \setminus \{(0,0,\pm 1)\}$, that is X is the result of removing the north and south poles from the unit sphere. Define two points of X to be equivalent if and only if they lie on the same great circle through the north and south poles. Identify the quotient space of these equivalence classes, giving an explicit homeomorphism.

Proof. □

4. **Iowa Qual, Fall 2007**

Suppose that $p : X \to Z$ is a quotient map and $f : X \to Y$ is continuous. Prove that there exists a continuous map $\overline{f} : Z \to Y$ with $\overline{f} \circ p = f$ if and only if for every x_1, x_2 with $p(x_1) = p(x_2)$ it is the case that $f(x_1) = f(x_2)$.

Proof. ($\Rightarrow$) If $p(x_1) = p(x_2)$. Then indeed $f(x_1) = \overline{f}(p(x_1)) = \overline{f}(p(x_2)) = f(x_2)$.

($\Leftarrow$) Notice that for any $[x] \in Z$, the map f is constant on $p^{-1}([x])$ since for any $x_1, x_2 \in p^{-1}([x])$, we have $p(x_1) = p(x_2) \implies f(x_1) = f(x_2)$. Therefore, the map $\overline{f} : Z \to Y$ given by $[x] \mapsto f(p^{-1}([x])) = f(x)$ is well-defined. Moreover,

$$\overline{f}(p(x)) = f(p^{-1}(p(x))) = f(p^{-1}([x])) = f(x),$$

Finally, to see that $\overline{f}$ is continuous, let U be open in Y. Since f is continuous, $f^{-1}(U)$ is open in X. So $p^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U)$ is open, and by definition of the quotient topology, $\overline{f}^{-1}(U)$ is open in Z. Hence $\overline{f}$ is continuous. □

5. **Iowa Qual, Spring 2008**

Let X be a topological space and let

$$\Delta = \{(x,x) \in X \times X\}.$$

Prove that X is Hausdorff if and only if $\Delta \subset X \times X$ is closed.

Proof. ($\Rightarrow$) Suppose X is Hausdorff and pick $(x,y) \notin \Delta$. Then since X is Hausdorff, there exists disjoint neighborhoods U and V of x and y, respectively. Then $U \times V$ is a neighborhood of (x,y) not intersecting Δ, since otherwise if $(w,z) \in U \times V \cap \Delta$, then $w = z \in U \cap V$, a contradiction. Hence Δ is closed.

($\Leftarrow$) Now suppose Δ is closed and pick $x \neq y$ in X. Since Δ^c is open and $(x,y) \notin \Delta$, then by definition of the product topology, there exists open sets U, V of X containing x, y, respectively, so that $(x,y) \in U \times V \subset \Delta^c$. Now if $w \in U \cap V$ then $(w,w) \in \Delta \cap U \times V$, a contradiction, since $U \times V$ is contained in Δ^c. Hence U, V are disjoint neighborhoods of the distinct points x, y, and so X is Hausdorff. □

6. **Iowa Qual, Spring 2008**

Prove that if X is compact, Y is Hausdorff, and $f : X \to Y$ is continuous, one-to-one and onto then f is a homeomorphism.

Proof. We need only to show that f^{-1} is continuous, or equivalently, that f is a closed map. So, take $C \subseteq X$ closed. Since X is compact, and C is closed, then C is compact:

If $\mathcal{O}$ is an open cover of C then $\mathcal{O} \cup X - C$ is an open cover of X, and so has a finite subcover. If $X - C$ is among the finite subcover, get rid of it. If not, leave the finite subcover alone. What remains is a finite subcover of C.

Since f is continuous, $f(C)$ is compact:

If $\mathcal{O}$ is an open cover of $f(C)$, then $f^{-1}(\mathcal{O})$ is an open cover of C, and therefore has a finite subcover since C is compact, say $\mathcal{U}$. Then $f(\mathcal{U})$ is a finite subcover of $f(C)$.

Since $f(C)$ is a compact subspace of a Hausdorff space, then $f(C)$ is closed:

If $x \in Y \setminus f(C)$, then for each $c \in f(C)$, pick disjoint neighborhoods U_c, V_c of x, c, respectively. Then $\{V_c\}$ covers the compact set $f(C)$ and so there exists $c_1, \ldots, c_n$ so that $\{V_{c_i}\}_{i=1}^n$ covers $f(C)$. Then $U := \cap_{i=1}^n U_{c_i}$ is a neighborhood of x disjoint from $f(C)$, since otherwise, if $z \in U \cap f(C)$, then $z \in V_{c_j}$ for some j and $z \in U_{c_j}$, a contradiction. So $f(C)$ is closed. □

7. **Iowa Qual, Fall 2013**

Let $\mathcal{O}$ be the collection of intervals $I_a = (a, \infty)$ in $\mathbb{R}$ including $I_\infty = \varnothing$ and $I_{-\infty} = \mathbb{R}$. Show that this is a topology on $\mathbb{R}$ and describe the closure of a set $A \subset \mathbb{R}$.

Proof. Since $I_\infty = \varnothing$ and $I_{-\infty} = \mathbb{R}$, we need only to check that arbitrary unions and finite intersections of elements of $\mathcal{O}$ lie in $\mathcal{O}$. So let $J := \bigcup_\alpha I_\alpha$ be an arbitrary union of elements in $\mathcal{O}$. Let $\beta = \inf\{\alpha\}$, so that $J = I_\beta$. Now let $K = \bigcap_{i=1}^n I_{a_i}$. Let $b = \max_i\{a_i\}$. Then $K = I_b$. Hence $\mathcal{O}$ is indeed a topology on $\mathbb{R}$.

By definition, the closure of A is the intersection of all closed sets containing A; that is

$$\overline{A} = \bigcap_{A \subseteq (-\infty, a]} (-\infty, a].$$

Let $\beta = \sup\{a \in A\}$. Then any closed set containing A will contain J_β, and so $\overline{A} = J_\beta$.

□

8. **Iowa Qual, Fall 2015**

Let $A \subset \mathbb{R}^\omega$ be defined by $A = \{(x_i) \in \mathbb{R}^\omega | x_i = 0 \text{ for all but finitely many } i\}$.

- **a.** Is A dense in $\mathbb{R}^\omega$ with the product topology? Prove your answer.
- **b.** is A dense in $\mathbb{R}^\omega$ with the box topology? Prove your answer.

Proof. The set A is indeed dense in $\mathbb{R}^\omega$ in the product topology:

Let $(x_n) \in X$, and $U = \prod_{n\in\mathbb{N}} U_n$ be a neighborhood of (x_n). Since we are in the product topology, there exists $N \in \mathbb{N}$ such that $U_k = \mathbb{R}$ for all $k > N$. Then,

$$(x_1, x_2, \dots, x_N, 0, 0 \dots) \in U \cap A,$$

and so $\mathbb{R}^\omega = \overline{A}$.

But of course, A is not dense in $\mathbb{R}^\omega$ in the box topology. Let $(x_n) = (1, 1, 1, \dots,) \in \mathbb{R}^\omega$. Then the set

$$U = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right) \times \dots$$

is a neighborhood of (x_n) not intersecting A, since no point in U can have *any* 0 component. □

9. **Iowa Qual, Fall 2016**

Let S^1 be the circle $S^1 = \{e^{2\pi i t} : t \in \mathbb{R}\}$. Define an equivalence relation on S^1 where two points are identified if $t_1 - t_2$ is an integer multiple of $\sqrt{2}$.

- **a.** Prove that the quotient space is not Hausdorff.
- **b.** Describe all continuous functions on the quotient space.

Proof.

- **a.** Let $p : S^1 \to S^1_*$ be the quotient map, and denote equivalence classes in S^1_* by brackets: $\left[e^{2\pi i t}\right]$. Notice that

$$p\left(e^{2\pi i s}\right) = \left[e^{2\pi i t}\right] \iff s - t = n\sqrt{2} \text{ for some } n \in \mathbb{Z} \iff s = n\sqrt{2} + t \text{ for some } n \in \mathbb{Z},$$

and so

$$p^{-1}\left(\left[e^{2\pi i t}\right]\right) = \left\{e^{2\pi i s} : \exists n \in \mathbb{Z} \text{ such that } e^{2\pi i s} = e^{2\pi i (n\sqrt{2}+t)}\right\} = \bigcup_{n\in\mathbb{Z}} e^{2\pi i (n\sqrt{2}+t)}$$

For all $n \in \mathbb{Z}$, define maps [1] $f_n : S^1 \to S^1$ by $e^{2\pi i t} \mapsto e^{2\pi i (n\sqrt{2}+t)}$. Then f_n is a homeomorphism since f_n has inverse f_{-n}, and moreover, both f_n and f_{-n} are restrictions of the continuous multiplication map $\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. In particular, f_n is the restriction of μ to $\{e^{2\pi i t (n\sqrt{2})}\} \times S^1$, and similarly for f_{-n}. Then

$$p^{-1}\left(\left[e^{2\pi i t}\right]\right) = \bigcup_{n\in\mathbb{Z}} e^{2\pi i (n\sqrt{2}+t)} = \bigcup_{n\in\mathbb{Z}} f_n\left(e^{2\pi i t}\right).$$

Now, the quotient space S^1_* is Hausdorff if and only if p is open and the defining relation is closed. So, suppose U is open in S^1, $U = \{e^{2\pi i t} : x < t < y\}$ for some $x, y \in \mathbb{R}$. Then by definition of the quotient topology, $p(U)$ is open in S^1_* if $p^{-1}(p(U))$ is open in S^1. So

$$p^{-1}(p(U)) = p^{-1}\left(\bigcup_{x<t<y} \left[e^{2\pi i t}\right]\right) = \bigcup_{x<t<y} p^{-1}\left(\left[e^{2\pi i t}\right]\right) = \bigcup_{x<t<y} \bigcup_{n\in\mathbb{Z}} f_n\left(e^{2\pi i t}\right)$$
$$= \bigcup_{n\in\mathbb{Z}} f_n(U),$$

[1] f_n is (left) multiplication by $e^{2\pi i (n\sqrt{2})}$.

and since the f_n's are homeomorphisms, they are in particular open maps, so that $f_n(U)$ is open for all $n \in \mathbb{Z}$, and hence $p^{-1}(p(U))$ is open.

Now, define

$$R = \left\{ \left(e^{2\pi is}, e^{2\pi it}\right) \in S^1 \times S^1 : \exists n \in \mathbb{Z} \text{ such that } e^{2\pi is} = e^{2\pi i(n\sqrt{2}+t)} \right\}.$$

Consider the map $F : S^1 \times S^1 \to S^1$ given by

$$\left(e^{2\pi is}, e^{2\pi it}\right) \mapsto \frac{e^{2\pi is}}{e^{2\pi it}}.$$

Notice that F is the restriction of a composition of the continuous multiplication and inversion maps in $\mathbb{C}$, and hence F is continuous. Consider the set $A = \left\{e^{2\pi in\sqrt{2}}\right\}_{n\in\mathbb{Z}}$. This set is dense in S^1. So if $x \in S^1 \setminus A$, then any neighborhood of x in S^1 intersects a point in A, i.e., the complement of A in S^1 is not open and so A is open. Then $R = F^{-1}(A)$, and hence R is open. Therefore, S^1_* is not Hausdorff.

- **b.** Suppose $f : S^1 \to X$ is a continuous map into some topological space X. If

  $$f\left(e^{2\pi it}\right) = f\left(e^{2\pi is}\right)$$

  whenever $t - s = n\sqrt{2}$ for some $n \in \mathbb{Z}$, then f descends to a continuous map $\overline{f} : S^1_* \to X$ by defining $\overline{f}\left(\left[e^{2\pi it}\right]\right) = f\left(p^{-1}\left(\left[e^{2\pi it}\right]\right)\right)$. Then $\overline{f}$ is well-defined since f is constant on the fibers above points in S^1_*, and $\overline{f}$ is continuous since if U is open in X, then $f^{-1}(U)$ is open in S^1, and hence $p^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U)$ is open in S^1, which gives that $\overline{f}^{-1}(U)$ is open in S^1_* by definition of the quotient topology.

  Hence the continuous maps on S^1_* are precisely those which come from continuous maps on S^1 which are constant on the fibers above points in S^1_*.

□

10. **UGA Qual, Fall 2016**

Let $\mathcal{S}, \mathcal{T}$ be topologies on a set X. Show that $\mathcal{S} \cap \mathcal{T}$ is a topology on X. Give an example to show that $\mathcal{S} \cup \mathcal{T}$ need not be a topology.

Proof. Since $\varnothing$ and X are in $\mathcal{S}$ and $\mathcal{T}$, then $\varnothing, X \in \mathcal{S} \cap \mathcal{T}$. We show that arbitrary unions and finite intersections of elements of $\mathcal{S} \cap \mathcal{T}$ lie in $\mathcal{S} \cap \mathcal{T}$.

Let $U = \bigcap_\alpha U_\alpha$ be a union of elements of $\mathcal{S} \cap \mathcal{T}$. Since $U_\alpha \in \mathcal{S}$ for all α, then $U \in \mathcal{S}$ since $\mathcal{S}$ is a topology, and similarly, $U \in \mathcal{T}$. Hence $U \in \mathcal{S} \cap \mathcal{T}$.

Let $V = \bigcup_{i=1}^n V_i$ be an intersection of elements of $\mathcal{S} \cap \mathcal{T}$. Since then $V \in \mathcal{S}$ since $\mathcal{S}$ is a topology and each $V_i \in \mathcal{S}$, and similarly, $V \in \mathcal{T}$. Hence $V \in \mathcal{S} \cap \mathcal{T}$.

Let $X = \{a, b, c, d\}$, $\mathcal{S} = \{\varnothing, X, \{a\}\}$, and $\mathcal{T} = \{\varnothing, X, \{b, c\}\}$. Then $\mathcal{S} \cup \mathcal{T} = \{\varnothing, X, \{a\}, \{b, c\}\}$, however $\mathcal{S} \cup \mathcal{T}$ is not a topology since $\{a\} \cup \{b, c\} = \{a, b, c\} \notin \mathcal{S} \cup \mathcal{T}$.

□

1.2 Connectedness

1. **Iowa Qual, Fall 2005**

   Give an example of a space that is connected but not path connected. Justify your answer.

   Solution: Consider the unit square $I^2 = [0,1]^2 = \{x \times y \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ in the order topology. Being a linear continuum, I^2 is connected. Suppose I^2 was path connected. Then there exists a continuous map $f : [a,b] \to I^2$ with $f(a) = 0 \times 0, f(b) = 1 \times 1$, i.e., a path from 0×0 to 1×1. By the Intermediate Value Theorem, $f([a,b]) = I^2$, i.e., f is surjective.

   For each $x \in [0,1]$, the set $\{x\} \times (0,1)$ is open. Hence $U_x = f^{-1}(\{x\} \times (0,1))$ is open in $[a,b]$. Moreover, the collection $\{U_x\}_{x \in [0,1]}$ is pairwise disjoint, for if $z \in U_{x_1} \cap U_{x_2}$, $x_1 \neq x_2$, then $f(z) \in (\{x_1\} \times (0,1)) \cap (\{x_2\} \times (0,1)) = \varnothing$, a contradiction. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each $x \in [0,1]$, we can pick $q_x \in U_x \cap \mathbb{Q}$. Since the U_x's are disjoint, this gives an injection $x \mapsto q_x$ of $[0,1]$ into $\mathbb{Q}$, and so the cardinality of $[0,1]$ is at most countable, a contradiction. Hence I^2 is not path connected.

2. **Iowa Qual, Fall 2006**

   Suppose $A = \cup A_\alpha$, where each A_α is connected, and so that there is a point x common to all A_α. Prove that A is connected.

   Proof. We first prove the following: *If X is a topological space with nonempty connected subspace B, and C, D is a separation of X, then B lies completely in C or D.*

   By definition of the subspace topology, $B \cap C$ and $B \cap D$ are open in B and moreover, $B \cap C$ and $B \cap D$ are disjoint and $B = (B \cap C) \cup (B \cap D)$. If however, $B \cap C$ and $B \cap D$ were both nonempty, we would obtain a separation of B, contradicting the connectedness of B. Hence one of $B \cap C$ and $B \cap D$ is empty, giving that B lies completely in C or D.

   Now let $x \in \bigcap_\alpha A_\alpha$, and suppose C, D is a separation of A, and without loss of generality, suppose $x \in C$. By what was shown above, A_α must lie completely in C or D for all α. Since $x \in A_\alpha \cap C$ for all α, then we must have $A_\alpha \subseteq C$ for all α, i.e., $A \subseteq C$, and so D is empty, a contradiction. □

3. **Iowa Qual, Fall 2007**

   Suppose that X and Y are connected, nonempty topological spaces. Prove that $X \times Y$ is connected.

   Proof. Fix $(a,b) \in X \times Y$. For each $x \in X$, the set $\{x\} \times Y$ is connected, being homeomorphic to Y, and similarly, $(X \times \{b\})$ is connected. Then $T_x = (\{x\} \times Y) \cup (X \times \{b\})$ is connected, being the union of connected spaces sharing the point (x,b) in common. But then

   $$X \times Y = \bigcup_{x \in X} T_x,$$

   since $\bigcup_{x \in X} T_x \subseteq X \times Y$, and if $(x,y) \in X \times Y$, then $(x,y) \in \{x\} \times Y \subset T_x$. Hence $X \times Y$ is connected, being the union of connected spaces which share the point (a,b) in common. □

4. **Iowa Qual, Fall 2013**

   Show that the subspace $X \subset \mathbb{R}^2$ consisting of all points (x, y) where at least one of the coordinates is rational is connected.

   Solution:
   We show that X is path connected and hence connected. Let $(a, b), (c, d) \in X$. We find a path from (a, b) to (c, d). There are four cases to consider:

$$\text{(i) } a, c \in \mathbb{Q}, \quad \text{(ii) } a, d \in \mathbb{Q}, \quad \text{(iii) } b, c \in \mathbb{Q}, \quad \text{and} \quad \text{(iv) } b, d \in \mathbb{Q}.$$

   Case (i): Define continuous maps

$$\begin{aligned} f_0 &: [0,1] \to X,\ t \mapsto (a, (1-t)b), \\ f_1 &: [1,2] \to X,\ t \mapsto ((2-t)a + (t-1)c, 0), \\ f_2 &: [2,3] \to X,\ t \mapsto (c, (t-2)d). \end{aligned}$$

   Notice that these maps do indeed map into X, since each map fixes a rational coordinate. Now, we also have

$$f_0(1) = (a, 0) = f_1(1) \text{ and } f_1(2) = (c, 0) = f_2(2).$$

   So by the Pasting Lemma, the map

$$f : [0,3] \to X,\ t \mapsto \begin{cases} f_0(t) & \text{if } t \in [0,1] \\ f_1(t) & \text{if } t \in [1,2] \\ f_2(t) & \text{if } t \in [2,3] \end{cases}$$

   is continuous since the components of f agree on their overlap. Moreover, $f(0) = (a, b)$ and $f(3) = (c, d)$, and so we have a path from (a, b) to (c, d), as desired.

   Case (ii):

   Again define continuous maps

$$\begin{aligned} g_0 &: [0,1] \to X,\ t \mapsto (a, (1-t)b + td), \\ g_1 &: [1,2] \to X,\ t \mapsto ((2-t)a + (t-1)c, d), \end{aligned}$$

   Then $g_0(1) = (a, d) = g_1(1)$ and so by the Pasting Lemma, the map

$$g : [0,2] \to X,\ t \mapsto \begin{cases} g_0(t) & \text{if } t \in [0,1] \\ g_1(t) & \text{if } t \in [1,2] \end{cases}$$

   is continuous and $g(0) = (a, b)$, $g(2) = (c, d)$.

   Cases (iii) and (iv) are similar.

1.3 Compactness

1. **Iowa Qual, Fall 2005 (Poudel)**

   Prove the *Lebesgue number lemma.* That is if X is a compact metric space and $\mathcal{U}$ is an open cover of X, there exists $\epsilon > 0$ so that if D is any subset having diameter less that or equal to ϵ then there exists $U \in \mathcal{U}$ with $D \subset U$.

Proof. Since (X,d) is compact, we find $U_1,\dots,U_n \in \mathcal{U}$ such that their union covers X. Define $V_i = X \setminus U_i$ for all $1 \le i \le n$. If $U_i = X$ for any i, then any open set with any diameter in X is contained in an element of $\mathcal{U}$. So we suppose $U_i \subsetneq X$ for all i so that each V_i is nonempty.

The map $x \mapsto d(x,V_i) = \inf_{z\in V_i} d(x,z)$ is continuous: If $x,y \in X$, then for all $z \in V_i$, we have

$$d(x,V_i) \le d(x,z) \le d(x,y) + d(y,z),$$

which gives $d(x,V_i) - d(x,y) \le d(y,z)$ and so $d(x,V_i) - d(x,y) \le d(y,V_i)$, i.e., $d(x,V_i) - d(y,V_i) \le d(x,y)$. Switching roles of x and y, we get $d(y,V_i) - d(x,V_i) \le d(x,y)$, and so

$$|d(x,V_i) - d(y,V_i)| \le d(x,y).$$

Hence our map is continuous. So the averaging function

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x,V_i)$$

is continuous. Notice that f is strictly positive, since for each $x \in X$, there is a $V_i \not\ni x$. Hence $d(x,V_i) = \delta > 0$, and so $f(x) \ge \delta/n > 0$.

Since f is continuous and X is compact, f attains a minimum value by the Extreme Value Theorem, say $\epsilon > 0$. We claim this is the desired ϵ. Pick $A \subset X$ with diameter less than ϵ. Then if $x_0 \in A$, $A \subseteq B(x_0,\epsilon)$. Pick $j \in \{1,\dots,n\}$ for which $d(x_0,V_j)$ is maximal. Then

$$\epsilon \le f(x_0) \le d(x_0,V_j) =: \gamma.$$

Then $A \subseteq B(x_0,\epsilon) \subseteq B(x_0,\gamma) \subseteq X \setminus V_j = A_j$. □

2. **Iowa Qual, Fall 2015**

Let X be a compact metric space and suppose that $f : X \to X$ is an isometry, i.e. $d(f(x),f(y)) = d(x,y)$ for all $x,y \in X$. Prove that f is a homeomorphism.

Proof. First, its is easy to see that f is injective:

$$f(x) = f(y) \implies 0 = d(f(x),f(y)) = d(x,y) \implies x = y.$$

Now, f is continuous: If $\epsilon > 0$ and $d(x,y) < \epsilon$, then

$$d(f(x),f(y)) = d(x,y) < \epsilon.$$

Suppose f is not surjective, and pick $x \in X \setminus f(X)$. Since f is continuous and X is compact, $f(X)$ is compact, and since X is Hausdorff, $f(X)$ is closed. Hence we can find $\epsilon > 0$ such that $B(x,\epsilon) \cap f(X) = \varnothing$. Then for natural numbers $n < m$,

$$\begin{aligned} d(f^n(x), f^m(x)) = d(f^{n-1}(x), f^{m-1}(x)) = \cdots &= d(f^{n-(n-2)})(x), f^{m-(n-2)}) \\ &= d(f^{n-(n-1)})(x), f^{m-(n-1)}) \\ &= d(x, f^{m-n}(x)) \ge \epsilon. \end{aligned}$$

So $\{f^n(x)\}$ is a sequence in X with no convergent subsequence, a contradiction since X is a compact metric space. Finally, f^{-1} is continuous: If $\epsilon > 0$ and $d(x,y) < \epsilon$, then

$$d(f^{-1}(x), f^{-1}(y)) = d(f(f^{-1}(x), f(f^{-1}(y))) = d(x,y) < \epsilon.$$

□

3. **Iowa Qual, Spring 2008**

Tube Lemma Suppose that X is compact and $X \times \{y\} \subset U$ where $U \subset X \times Y$ is open. Prove that there exists $W \subset Y$ open so that $X \times \{y\} \subset X \times W \subset U$.

Proof. For all $x \in X$, since U is open, then by definition of the product topology, there exists $U_x \subseteq X$ open and $V_x \subseteq Y$ open so that

$$x \times y \in U_x \times V_x \subseteq U.$$

Since X is compact and $\{U_x\}_{x \in X}$ is an open cover of X, there exists $x_1, \ldots x_n \in X$ so that $X \subseteq \bigcup_{i=1}^n U_{x_i}$. Define $W = \bigcap_{i=1}^n V_{x_i}$. Then $X \times \{y\} \subseteq X \times W$ and

$$\bigcup_{i=1}^n (U_{x_i} \times V_{x_i}) \subseteq U.$$

It remains to show that $X \times W \subseteq U$. To that end, let $x \times w \in X \times W$. Let U_{x_j} be the open set in $\{U_{x_i}\}_{i=1}^n$ containing x. Since $w \in V_{x_j}$, then

$$x \times w \in U_{x_j} \times V_{x_j} \subseteq \bigcup_{i=1}^n (U_{x_i} \times V_{x_i}) \subseteq U.$$

□

4. **Iowa Qual, January 2011**

- **a.** PROVE: If a space X is compact, Hausdorff, and connected, and X has at least two points, then X is uncountable.
- **b.** PROVE: If a space X is compact, Hausdorff, and perfect (i.e., each point of X is a limit point of X) then X is uncountable.

Proof.

- **a.** We first show that for any $x \in X$ and any $U \subseteq X$ open and nonempty, there exists $V \subseteq U$ open and nonempty so that $\overline{V} \not\ni x$.

  First,

  $$\text{choose } y \in U \text{ different from } x. \qquad (*)$$

  We argue why $(*)$ is possible: If $x \notin U$, this choice is possible simply because U is nonempty. If $x \in U$ and U contains no other points, then $U = \{x\}$ is closed, since singleton sets are closed in Hausdorff spaces. Hence U is both open and closed in X, and is neither empty nor all of X, which contradicts the connectedness of X.

Hence $(*)$ is possible, and since X is Hausdorff, we find a neighborhood W of y not containing x. Then $V = U \cap W$ is an open set in U for which $x \notin \overline{V}$.

We show that any map $f : \mathbb{N} \to X$ is not surjective, giving that $|\mathbb{N}| < |X|$, i.e., X is uncountable. Let $f(n) = x_n$. Let $x_1 \in X$, and using $U = X$ as above, there exists $V_1 \subset X$ open and nonempty such that $x \notin \overline{V_1}$. For $n > 1$, given V_{n-1} open an nonempty such that $x_{n-1} \notin \overline{V_{n-1}}$, there exists a nonempty open set $V_n \subseteq V_{n-1}$ such that $x_n \notin \overline{V_n}$. This gives an descending chain of closed sets

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \dots,$$

and hence the collection $\{V_n\}_{n\in\mathbb{N}}$ has the finite intersection property, i.e., any finite intersection of sets from this collection have nonempty intersection. Since X is compact, this means that there exists $x \in \bigcap_{n\in\mathbb{N}} \overline{V_n}$. But then $f(m) = x_m \neq x$ for any m, since $x \in \overline{V_n}$ for all $n \in \mathbb{N}$, but $x_m \notin \overline{V_m}$.

- **b.** The proof here is the same, except we need to argue differently why $(*)$ is possible:

  If $x \notin U$, this choice is possible simply because U is nonempty. Since x is a limit point in X, every neighborhood of x intersects X at a point other than x. So if $x \in U$, then U contains a point y different from x.

□

5. **Iowa Qual, Fall 2013**

   Is the intersection of two compact sets always compact? Prove or give a counterexample.

   Solution: Counterexample:

   Let $x, y \in \mathbb{R} \setminus \mathbb{N}$ and let $X = \mathbb{N} \cup \{x, y\}$. Give X the following topology:

$$\mathcal{T} = \{\varnothing, X, \mathbb{N}, \mathbb{N} \cup \{x\}, \mathbb{N} \cup \{y\}, \{1\}, \{2\}, \dots\}.$$

Then, any open cover for $\mathbb{N} \cup \{x\}$ must contain $\mathbb{N} \cup \{x\}$ itself, or X. Hence any open cover of $\mathbb{N} \cup \{x\}$ has a finite subcover, and so $\mathbb{N} \cup \{x\}$ is compact. Similarly, $\mathbb{N} \cup \{y\}$ is compact. However

$$(\mathbb{N} \cup \{x\}) \cap (\mathbb{N} \cup \{y\}) = \mathbb{N}$$

is not compact since $\{\{n\}\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{N}$ with no finite subcover.

6. **Iowa Qual, Fall 2013**

   Is the closure of a compact set always compact? Prove or give a counterexample.

   Solution: Counterexample:

   Let $X = \mathbb{N} \cup \{x\}$, for a point $x \in \mathbb{R} \setminus \mathbb{N}$. Define all nonempty open sets in X to be those which contain x. Then $\{x\}$ is compact. We claim $\overline{\{x\}} = X$. Indeed, if $y \in X$, and U is an open set containing y, then by construction $x \in U$, and so every neighborhood of every point of X intersects $\{x\}$. But $\bigcup_{n\in\mathbb{N}} \{x, n\}$ is an open cover of X with no finite subcover.

7. **Iowa Qual, Winter 2016**

- **a.** Show that compactness implies limit point compactness.
- **b.** Give an example of a space that is limit point compact but not compact. Explain why your example works.

Solution:

- **a.** We show the contrapositive: If X is compact and $A \subseteq X$ has no limit points, then A is finite. The assumption gives $\overline{A} = A \cup A' = A$, and so A is closed. Since closed subspaces of compact spaces are compact, A is compact. Let $a \in A$. If every neighborhood of a intersects A at a point other than a, then $a \in A' = \varnothing$, a contradiction. Hence for each $a \in A$, there exists a neighborhood U_a of a intersecting the point a alone. Then $\{U_a\}_{a \in A}$ is an open cover of A, and since A is compact, the open cover must admit a finite subcover of A. Since each U_a intersects A at a single point, and finitely many such sets cover A, then A is finite.
- **b.** Let $X = \mathbb{N} \times \{a, b\}$ in the product topology, where $\mathbb{N}$ is given the discrete topology and $\{a, b\}$ is given the indiscrete topology. We show that X is limit point compact but not compact. First, X is not compact since $\{\{n\} \times \{a, b\}\}_{n \in \mathbb{N}}$ is an open cover of X with no finite subcover.

  Now, let $A \subseteq X$ be an infinite set and suppose $(n, a) \in A$. We show that (n, b) is a limit point of A, which will show that X is limit point compact. Let Y be a neighborhood of (n, b). Then by definition of the product topology, there exists basic open sets $U \subseteq \mathbb{N}$ and $V \subseteq \{a, b\}$ such that $(n, b) \in U \times V \subseteq Y$. But what can V be? Since $\{a, b\}$ was given the indiscrete topology and $b \in V$, then we must have $V = \{a, b\}$. Then $(n, a) \in U \times V \subseteq Y$, giving that any neighborhood of (n, b) intersects A, i.e., (n, b) is a limit point of A.

1.4 Separation/Countability Axioms

1. **Iowa Qual, Fall 2005**

   Prove that every compact Hausdorff space is normal. That is prove that if A and B are disjoint closed subsets of the compact Hausdorff space X, then there are open sets U and V so that $U \cap V = \varnothing$, with $A \subset U, B \subset V$.

Proof. We first show that A (and B) are compact in X: If $\mathcal{U}$ is an open cover of A, then $\mathcal{U} \cup (X \setminus A)$ is an open cover of X, and since X is compact, there exists a finite subcover of X from this collection. If $X \setminus A$ is in this finite collection, remove it; if not, leave it alone. What remains is a finite collection of sets from $\mathcal{U}$ which cover A.

Fix $a \in A$. Since X is Hausdorff, for every $b \in B$, we can choose disjoint open sets U_b, V_b of a, b respectively. Then $\{V_b\}$ is an open cover of B, and since B is compact, there exists $b_1, \ldots, b_n \in B$ such that $\{V_{b_i}\}_{i=1}^n$ covers B. Then

$$a \in \bigcap_{i=1}^{n} U_{b_i} =: U_a \quad \text{and} \quad B \subseteq \bigcup_{i=1}^{n} V_{b_i} =: V_a$$

and $U_a \cap V_a = \varnothing$, since otherwise $x \in U_a \cap V_a$ means that $x \in V_{b_i}$ for some i and also $x \in U_{b_i}$, a contradiction.

Repeating this for each $a \in A$, we obtain an open cover $\{U_a\}_{a\in A}$ of A, and a collection of covers $\{V_a\}_{a\in A}$ for B. Since A is compact, there exists $a_1, \dots a_m \in A$ such that $A \subseteq \bigcup_{j=1}^m U_{a_j}$. Define

$$U = \bigcup_{j=1}^{m} U_{a_j} \quad \text{and} \quad V = \bigcap_{j=1}^{m} V_{a_j}.$$

Then U, V are open and contain A, B respectively. If $x \in U \cap V$, then $x \in U_{a_j}$ for some j and $x \in V_{a_j}$; but since $U_{a_j} \cap V_{a_j} = \varnothing$, we get a contradiction. Hence $U \cap V = \varnothing$. $\square$

2. **Iowa Qual, Fall 2006**

Prove or give a counterexample: The product of two regular spaces is regular.

Proof. We first prove the following: X is regular if and only if for every $x \in X$ and for every neighborhood U of x, there exists a neighborhood V of x such that $\overline{V} \subset U$.

First suppose X is regular, and x, U are as above. Then $X \setminus U$ is closed, and since X is regular, there exists disjoint open sets V, W containing x and $X \setminus U$, respectively. Then $\overline{V} \cap (X \setminus U) = \varnothing$ since for any $z \in X \setminus U$, W is a neighborhood of z not intersecting V. Hence $\overline{V} \subset U$.

Conversely, let $x \in X$, and A a closed set in X not containing x. Then $X \setminus A$ is open and contains x, and so by hypothesis, there exists a neighborhood V of x such that $\overline{V} \subset X \setminus A$. Then V and $X \setminus \overline{V}$ are disjoint open sets containing x and A, respectively.

Now, the proof essentially follows from the fact that the closure of a product of spaces is the product of the closures of those spaces (and indeed this will work for arbitrary products: in the product topology, of course).

Let X and Y be regular spaces. Suppose $x \times y \in X \times Y$ and N is a neighborhood of $x \times y$. By definition of the product topology, there exists basic open sets $U \subset X, V \subset Y$ containing x and y, respectively, such that $x \times y \in U \times V \subseteq N$. Since both X and Y are regular, there exists open sets $W \subset X, Z \subset Y$ containing x and y, respectively, such that $\overline{W} \subset U$ and $\overline{Z} \subset V$. Then

$$x \times y \in \overline{W \times Z} = \overline{W} \times \overline{Z} \subset U \times V \subseteq N,$$

and hence $X \times Y$ is regular. $\square$

3. **Iowa Qual, Fall 2007**

A space is *separable* if it has a countable dense subset. Prove that a topological space is separable if it is second countable, and that a metric space is second countable if it is separable. *There are two things to prove here.*

Proof. Suppose X is second countable with countable basis $\{U_n\}_{n\in\mathbb{N}}$. For each n, pick $x_n \in U_n$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a countable dense subset of X; evidently it is countable, and it is dense in X since if $x \in X$ and U is a neighborhood of x, then there exists a basic open set U_n such that $x \in U_n \subset U$, and $x_n \in U$.

Now suppose (X, d) is a separable metric space, with countable dense subset S. We claim that $\mathcal{B} = \{B(s, 1/n) : s \in S, n \in \mathbb{N}\}$ is a countable basis for X. First, $\mathcal{B}$ is countable since it is a collection of countably many balls at countably many points. We show that $\mathcal{B}$ is a basis.

Let $x \in X$ and U^2 a neighborhood of x. Then there exists $\epsilon > 0$ such that $B(x, \epsilon)$ is contained in U. Now choose $n \in \mathbb{N}$ such that $1/n < \epsilon/2$. Since S is dense in X, there exists $s \in S$ such that $s \in B(x, 1/n)$. Now if $y \in B(s, 1/n)$, then

$$d(y, x) \leq d(y, s) + d(s, x) < 1/n + 1/n < \epsilon/2 + \epsilon/2 = \epsilon,$$

and so $B(s, 1/n) \subseteq B(x, \epsilon)$. Therefore $x \in B(s, 1/n) \subseteq B(x, \epsilon) \subseteq U$, and hence $\mathcal{B}$ is a countable basis for X. □

4. **UGA Qual, Fall 2016**

Prove that a metric space X is normal, i.e., if $A, B \subset X$ are closed and disjoint then there exist open sets $A \subset U \subset X, B \subset V \subset X$ such that $U \cap V = \varnothing$.

Proof. Fix $a \in A$ and let $\delta_a = \inf_{b \in B} d(a, b)$. Suppose $\delta_a = 0$. Then for every $n \in \mathbb{N}$, there exists $b_n \in B$ such that $d(a, b_n) < 1/n$. But then $\{b_n\} \to a$, and since B is closed, it contains its limits points, and so $a \in B$, a contradiction since A and B are disjoint. Hence $\delta_a > 0$. Similarly, define $0 < \delta_b = \inf_{a \in A} d(b, a)$ for all $b \in B$. We claim

$$U = \bigcup_{a \in A} B(a, \delta_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \delta_b/2)$$

are the desired open sets. Evidently $A \subseteq U$ and $B \subseteq V$. Suppose $x \in U \cap V$. Then there exists $a \in A$ and there exists $b \in B$ such that $x \in B(a, \delta_a/2) \cap B(b, \delta_b/2)$. Assume without loss of generality that $\delta_a \leq \delta_b$. Then

$$\delta_b \leq d(a, b) \leq d(a, x) + d(x, b) < \delta_a/2 + \delta_b/2 \leq 2\delta_b/2 = \delta_b,$$

a contradiction. □

1.5 Metrization

1. **Iowa Qual, Winter 2016**

Prove the Urysohn metrization theorem (Every regular space X with a countable basis is metrizable.). You may assume that there exists a countable collection of functions $f_n : X \to [0, 1]$ such that for any point $x_0 \in X$ and any neighborhood U of x_0, there is some n so that f_n is positive at x_0 and 0 outside of U.

Proof. To show that X is metrizable, we will embed X into the metrizable space $\mathbb{R}^{\mathbb{N}}$ (in the product topology). To that end, consider the map

$$F : X \to \mathbb{R}^{\mathbb{N}}, x \mapsto (f_1(x), f_2(x), \dots).$$

[2] The neighborhood U is coming from the topology induced by the metric on X; we are essentially showing that $\mathcal{B}$ is finer than the topology induced by the metric, which will suffice to prove that $\mathcal{B}$ is a basis. See Munkre's Lemmas 13.2 and 13.3.

Since each f_n is continuous and X has the product topology, then F is continuous. To see that F is injective, suppose $x \neq y$, and let U be a neighborhood of x not containing y (such a neighborhood exists since X is Hausdorff). Then there exists n so that $f_n(x) > 0$ and $f_n(y) = 0$, and so $F(x) \neq F(y)$.

Now, we show that F is a homeomorphism onto its image $Z := F(X)$. Since F is continuous, we need only to show that $F^{-1} : Z \to X$ is continuous, or equivalently, that F is an open map. So, let $U \subseteq X$ be open and pick $z \in F(U)$. We find a neighborhood W of z contained in $F(U)$.

Let $x \in U$ be the unique point such that $F(x) = z$. Then there exists n such that $f_n(x) > 0$ and $f_n(y) = 0$ for all $y \in X \setminus U$. Let $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be projection onto the n-th coordinate. Let

$$V = \pi_n^{-1}(0, \infty).$$

We claim $W = V \cap Z$ is the desired neighborhood of z contained in $F(U)$. First, $z \in W$ since $z \in F(U) \subseteq Z$ and

$$\pi_n(z) = \pi_n(F(x)) = f_n(x) > 0 \implies z \in V.$$

Now, if $w \in V \cap Z$, then $\pi_n(w) > 0$ and there exists $a \in X$ with $F(a) = w$. So

$$f_n(a) = \pi_n(F(a)) = \pi_n(w) > 0 \implies a \in U$$

Hence $w = F(a) \in F(U)$. So $z \in W \subseteq F(U)$. □

2. **Munkres, §33, Problem 4**

Let X be normal. Prove that there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$ if and only if A is a closed G_δ set in X. (Recall that a G_δ set is one which can be expressed as the countable intersection of open sets.)

Proof. ($\Rightarrow$) Suppose such a function $f : X \to [0,1]$ exists. Since singleton sets are closed in $[0,1]$, f is continuous, and $A = f^{-1}(\{0\})$, then A is closed. □

1.6 Homotopy, Fundamental Group, Seifert-Van Kampen

1. **Iowa Qual, Fall 2013**

Consider the space obtained by removing from $\mathbb{R}^3$ the unit circle in the xy-plane and the z-axis. Determine the fundamental group of this space.

Solution:

Let Z be the z-axis and U be the unit circle in the xy-plane, and $X = \mathbb{R}^3 \setminus (Z \cup U)$. For every $\alpha \in [0, 1/2)$, let $T \times \{\alpha\}$ be the object obtained by rotating around the z-axis the circle centered at $(0,1,0)$ of radius $1/2 - \alpha$ in the yz-plane. In particular, $T \times \{\alpha\}$ is a torus for each α. Then $T \times [0, 1/2)$ is a solid torus without the inner circle.

We can then deformation retract X onto $T \times [0, 1/2)$. Then for each $\alpha \in (0, 1/2)$, we can do a straight-line homotopy from $T \times \{\alpha\}$ to $T \times \{0\}$, or, perform a deformation retract of $T \times [0, 1/2)$ onto $T \times \{0\}$, leaving just a torus, which has fundamental group $\mathbb{Z} \times \mathbb{Z}$. (The base point $(0, 3/2, 0)$ is remained fixed by each homotopy, and so we can consider $\pi_1(X, (0, 3/2, 0)) \cong \pi_1(T \times \{0\}, (0, 3/2, 0)) \cong \mathbb{Z} \times \mathbb{Z}$.)

2. **Iowa Qual, Fall 2014**

What is the fundamental group of

$$X = \{(x, y, z) \in \mathbb{R}^3 | x \neq 0 \text{ or } y \neq 0\},$$

where we use $(1, 0, 0)$ as the base point. *X is the complement of the z-axis.*

Solution: Since the projection map of $\mathbb{R}^3$ onto the xy-plane is a retract, we can define a homotopy

$$H : X \times [0, 1] \to \mathbb{R}^3, H(x, y, z, t) = (x, y, z(1 - t)),$$

which is a deformation retract of X onto the xy-plane without the origin, call it A. Using the retraction

$$r : A \to S^1, (x, y, z) \mapsto \frac{(x, y, z)}{\sqrt{x^2 + y^2}},$$

we can define a homotopy

$$G : A \times [0, 1] \to \mathbb{R}^3, G(x, y, z, t) = (1 - t)\frac{(x, y, z)}{\sqrt{x^2 + y^2}}$$

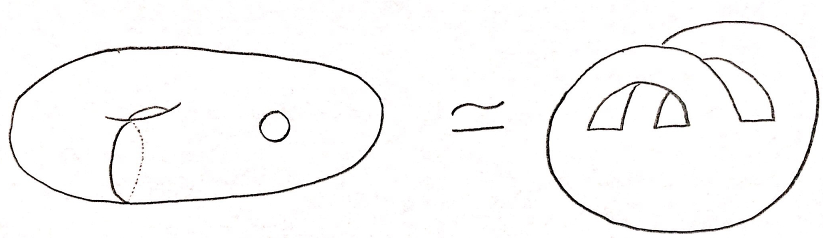
which is a deformation retract of A onto the unit circle S^1 in the xy-plane. The point $(1, 0, 0)$ is fixed during each homotopy, and since the fundamental group does not change under homotopy, we have $\pi_1(X, (1, 0, 0)) \cong \pi_1(S^1, (1, 0, 0)) \cong \mathbb{Z}$.

3. **Iowa Qual, Fall 2015**

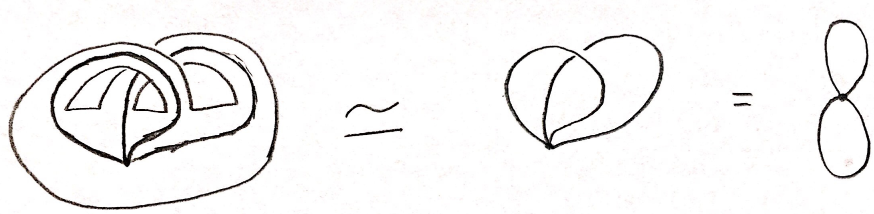
Find the fundamental group of $T^2 = S^1 \times S^1$ with k points removed.

Solution:

We proceed by induction on k to show that $\pi_1(T^2 \setminus \{k \text{ pts}\})$ is isomorphic to the free product on $k + 1$ generators. If $k = 1$, then we have the T^2 with one boundary component,

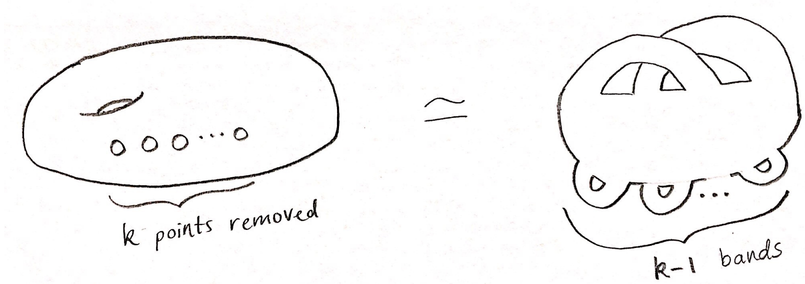


which is homotopy equivalent to the rose with $k + 1 = 2$ pedals:

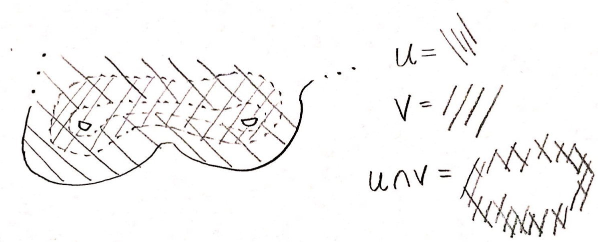


And hence $\pi_1(T^2 \setminus \{\text{pt}\}) \cong \mathbb{Z} * \mathbb{Z}$, the free product on two generators. Now let $k > 1$ and suppose for induction that $\pi_1(T^2 \setminus \{k-1 \text{ pts}\}) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \text{ factors}}$.

Notice that $X = T^2 \setminus \{k \text{ pts}\}$ is a torus with k boundary components:



Let V be a neighborhood containing two of the boundary components of X, and let U be the complement of a neighborhood contained in V which also contains two boundary components:



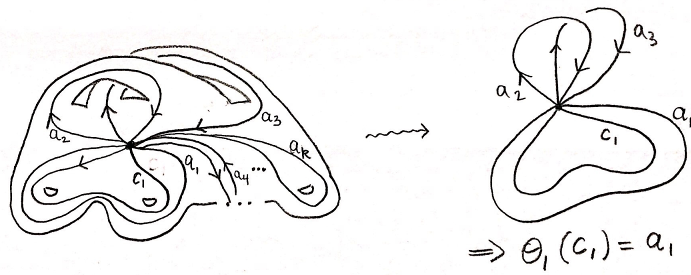
Then $U \cup V = X$, where $U = T^2 \setminus \{(k-1) \text{ pts}\}$, V is a disk with two boundary components, and $U \cap V$ is an annulus. Then V is homotopic to a rose with two pedals, and $U \cap V$ is homotopic to S^1, which gives $\pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$ and $\pi_1(U \cap V) \cong \mathbb{Z}$. By induction, we have $\pi_1(U) \cong \mathbb{Z} * \cdots * \mathbb{Z}$, the free product on k generators. Write

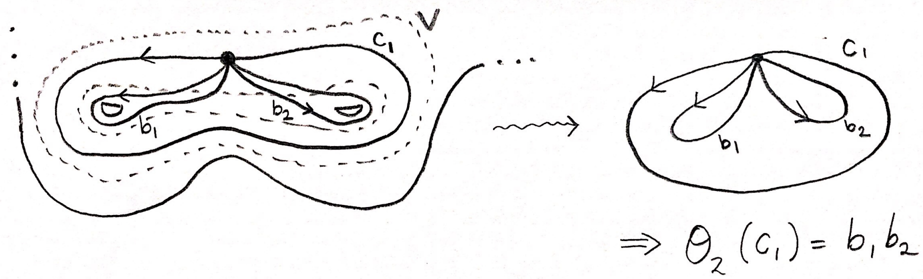
$$\begin{aligned} \pi_1(U) &= \langle a_1, \ldots, a_k | \ \rangle \\ \pi_1(V) &= \langle b_1, b_2 | \ \rangle \\ \pi_1(U \cap V) &= \langle c_1 | \ \rangle \end{aligned}$$

Let $\theta_1 : \pi_1(U \cap V) \to \pi_1(U) \to \pi_1(U \cup V)$ be the composition of the induced maps of inclusions, and similarly, define $\theta_2 : \pi_1(U \cap V) \to \pi_1(V) \to \pi_1(U \cup V)$. Then by the Seifert Van-Kampen Theorem, we have

$$\pi_1(X) = \langle a_1, \ldots, a_k, b_1, b_2 \ | \ \theta_1(c_1) = \theta_2(c_1) \rangle,$$

So, we find $\theta_1(c_1)$ and $\theta_2(c_1)$:





Hence

$$\pi_1(X) = \langle a_1, \ldots, a_k, b_1, b_2 \mid a_1 = b_1 b_2 \rangle = \langle a_2, \ldots, a_k, b_1, b_2 | \, \rangle \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k+1 \text{ factors}}.$$

4. **Iowa Qual, Fall 2015**

Let X be the so-called Hawaiian earring, which is defined by $X = \cup_{i=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$. So X is the union of the circles with center $(1/n, 0)$ and radius $1/n$ for $n = 1, 2, 3, \ldots$. Let Y be the quotient space formed by starting with $\mathbb{R}^1$ and defining $x \sim y$ if either $x = y$ or $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic.

Proof. We show that X is compact and that Y is not compact, showing that these spaces cannot be homeomorphic.

First, let $\{x_k\}$ be a Cauchy in X converging to $x \in \mathbb{R}^2$. We show that $x \in X$, proving that X is closed. If there exists C_i containing infinitely many x_k's, then there is a subsequence $\{x_{k_\ell}\} \subset C_i$, and since C_i is closed, $\{x_{k_\ell}\} \to x \in C_i$, and so X is closed. If all C_i contain at most finitely many x_k's, we claim $x = (0, 0)$. Indeed, let $\epsilon > 0$; then there exists N large enough so that $C_i \subset B(0, \epsilon)$ for all $i \geq N$. Since $\cup_{i=1}^{N-1} C_i$ contains finitely many points of $\{x_k\}$, then $B(0, \epsilon)$ contains infinitely many x_k's, and hence $\{x_k\} \to (0, 0) \in X$, and so X is closed. Moreover, X is bounded since $|x| < 5$ for all $x \in X$. Therefore, as a closed and bounded subset of $\mathbb{R}^2$, X is compact.

We exhibit an open cover of Y which contains no finite subcover of Y. Let brackets denote equivalent classes in Y, and let $q : \mathbb{R} \to Y$ be the quotient map. Consider the open interval $I = (-1/2, 1/2)$. Then $[0] \in q(I) =: U$. Notice that

$$q^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \left(n - \frac{1}{2}, n + \frac{1}{2} \right),$$

which is open in $\mathbb{R}$. Hence U is a neighborhood of $[0]$ in Y. Now if $W_n = (n, n+1)$, then

$$V_n := q(W_n) = \bigcup_{x \in (n, n+1)} [x],$$

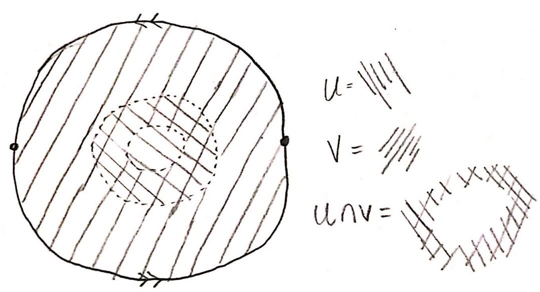
and $q^{-1}(V_n) = (n, n+1)$ is open in $\mathbb{R}$, and hence V_n is open in Y. Then $U \cup \bigcup_{n \in \mathbb{Z}} V_n$ is an open cover of Y which has no finite subcover. Indeed, if we remove U from the cover then we do not cover $[0]$; if we remove any V_n from the cover, then we do not cover the point $[n + 1/2]$. Hence Y is not compact. $\square$

5. **Iowa Qual, Winter 2016**

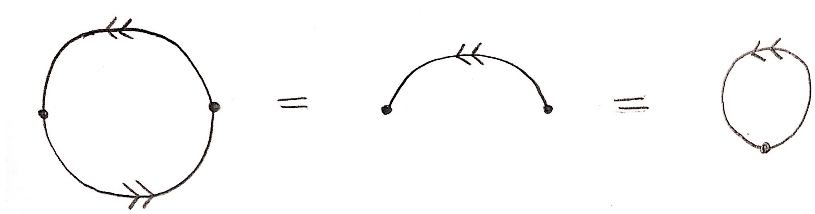
Find the fundamental group of the object obtained by identifying each point of the boundary of a disk with its antipodal point.

Solution:

This space is $\mathbb{R}P(2)$. We use the Seifert Van-Kampen Theorem to find its fundamental group. First, define U to be a disk within the disk, and let V be the complement of a disk which lies inside U:



Then U is homotopy equivalent to a point, and $U \cap V$ is homotopy equivalent to S^1; so $\pi_1(U) = 1$ and $\pi_1(U \cap V) = \mathbb{Z}$. Now, V retracts to the boundary of the disk, and so we need to find the fundamental group of S^1 with antipodal points identified:



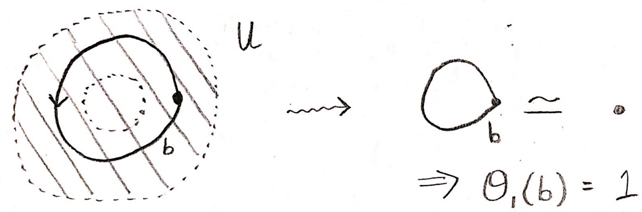
and so V is homotopy equivalent to S^1, giving $\pi_1(V) = \mathbb{Z}$. Write

$$\pi_1(U) = \langle | \; \rangle$$
$$\pi_1(V) = \langle a | \; \rangle$$
$$\pi_1(U \cap V) = \langle b | \; \rangle$$

Let $\theta_1 : \pi_1(U \cap V) \to \pi_1(U) \to \pi_1(U \cup V)$ be the composition of the induced maps of inclusions, and similarly, define $\theta_2 : \pi_1(U \cap V) \to \pi_1(V) \to \pi_1(U \cup V)$. Then by the Seifert Van-Kampen Theorem, we have

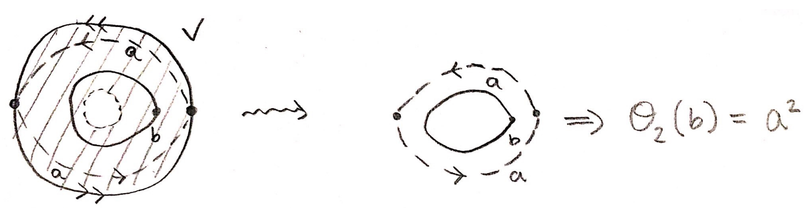
$$\pi_1(\mathbb{R}P(2)) = \langle a \mid \theta_1(b) = \theta_2(b) \rangle ,$$

and so we need to find $\theta_1(b)$ and $\theta_2(b)$. For $\theta_1(b)$,



Now, we consider the generator a of $\pi_1(V)$:

Now, when we include b into the V, we see that b traverses the upper and lower semicircles of the boundary of the disk, but since these are identified, we get that b "overlaps" a twice, which gives $\theta_1(b) = a^2$:



Hence

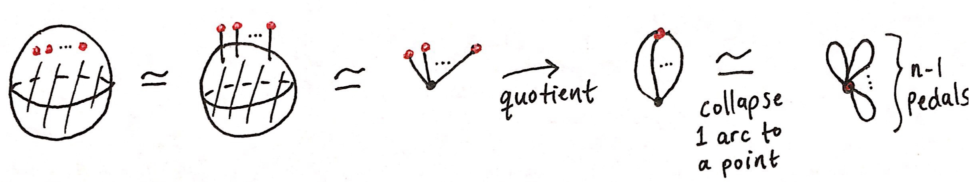
$$\pi_1(\mathbb{R}P(2)) = \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

6. **Iowa Qual, Winter 2016**

Let $D^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. Let $A = \{a_1, a_2, \ldots, a_n\} \subset D^3$ be a subset of distinct points in the 3-ball. Compute the fundamental group of the quotient space $\pi_1(D^3/A, b)$ where $b \in D^3 \setminus A$.

Solution:

We have



and so $\pi_1(D^3/A, b) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n-1 \text{ factors}}$.

7. **Iowa Qual, Fall 2016**

- **a.** State the Van-Kampen theorem. In particular, explicitly define all maps.
- **b.** Suppose that X, Y, Z are the yz, xz, xy-planes in $\mathbb{R}^3$ and S^2 is the unit 2-sphere in $\mathbb{R}^3$.

$$S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}, \qquad X = \{(x,y,z) : x = 0\}$$

$$Y = \{(x,y,z) : y = 0\}, \qquad Z = \{(x,y,z,) : z = 0\}.$$

If B is the space obtained as the union of these four sets, endowed with the subspace topology, then compute the fundamental group of B.

Solution:

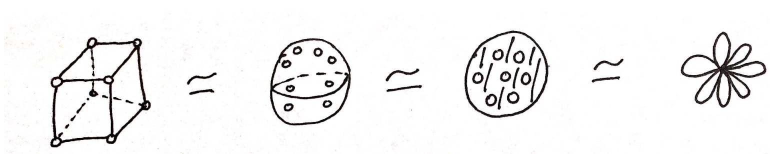
- **a.**
- **b.**

8. **Iowa Qual, Winter 2017**

Find the fundamental group of the space obtained by taking the surface of a cube and removing all corner points.

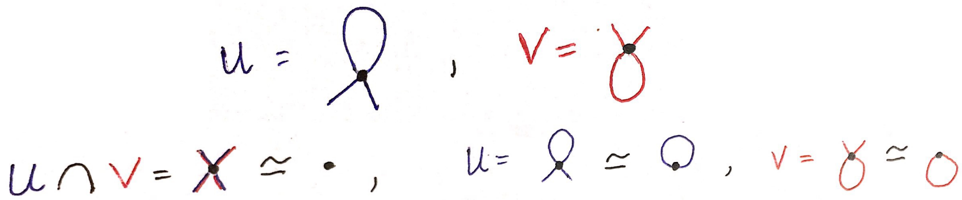
Solution:

This space X is homotopy equivalent to the rose with 7 pedals:



and so $\pi_1(X) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{7 \text{ factors}}$.

We also show here that the rose with n pedals, which we will call R_n, does indeed have fundamental group the free product on n generators. By induction: For $n = 1$, $R_1 = S^1$, which has fundamental group $\mathbb{Z}$. For $n = 2$, let U and V be as shown:



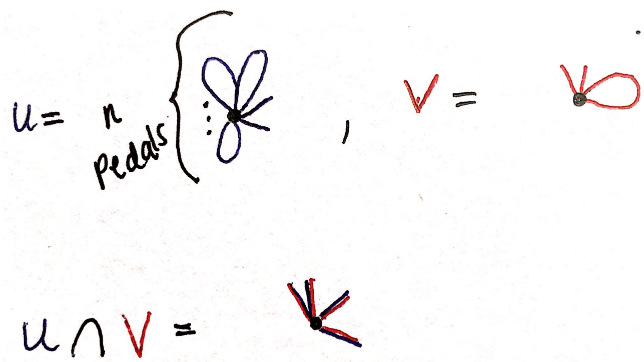
Write

$$\pi_1(U) = \langle a \mid \rangle$$
$$\pi_1(V) = \langle b \mid \rangle$$
$$\pi_1(U \cap V) = \langle \mid \rangle$$

Let $\theta_1 : \pi_1(U \cap V) \to \pi_1(U) \to \pi_1(U \cup V)$ be the composition of the induced maps of inclusions, and similarly, define $\theta_2 : \pi_1(U \cap V) \to \pi_1(V) \to \pi_1(U \cup V)$. Then by the Seifert Van-Kampen Theorem, we have

$$\pi_1(R_2) = \langle a, b \mid \theta_1(1) = \theta_2(1)\rangle = \langle a, b \mid \rangle = \mathbb{Z} * \mathbb{Z}.$$

Now suppose for induction that $\pi_1(R_n) = \langle a_1, \ldots, a_n \mid \rangle$. Let U and V be in R_{n+1} as shown:



Then U is homotopy equivalent to R_n, V is homotopy equivalent so S^1, and $U \cap V$ is homotopy equivalent to a point. Write

$$\pi_1(U) = \langle a_1, \ldots, a_n \mid \rangle$$
$$\pi_1(V) = \langle a_{n+1} \mid \rangle$$
$$\pi_1(U \cap V) = \langle \mid \rangle$$

Then

$$\pi_1(R_{n+1}) = \langle a_1, \ldots, a_{n+1} \mid \theta_1(1) = \theta_2(1)\rangle = \langle a_1, \ldots, a_{n+1} \mid \rangle = \mathbb{Z} * \cdots * \mathbb{Z}.$$

9. **Iowa Qual, Winter 2017**

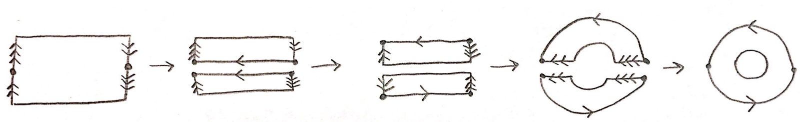
Suppose that $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is the unit 2-sphere in $\mathbb{R}^3$. If $Y \subset S^2$ is a subset of n distinct points then compute the fundamental group of the quotient space: $\pi_1(S^2/Y, [Y])$. Prove that your answer is correct.

10. **UGA Qual, Fall 2016 (Camacho)**

Let S_k be the space obtained by removing k disjoint open discs from the sphere S^2, to leave a surface whose boundary is k circles. Form X_k by gluing k Möbius bands onto S_k, one for each circle boundary component of S_k (by identifying the boundary circle of a Möbius band homeomorphically with a given boundary component circle). Use Van Kampen's theorem to calculate $\pi_1(X_k)$ for each $k > 0$ and identify X_k in terms of the classification of surfaces.

Solution:

Let the following sequence convince you that a Möbius band is $\mathbb{R}P(2) \setminus B^2$:



Moreover, $\mathbb{R}P(2) \setminus B^2 \simeq S^1/\sim$, where $\sim$ is the antipodal relation:

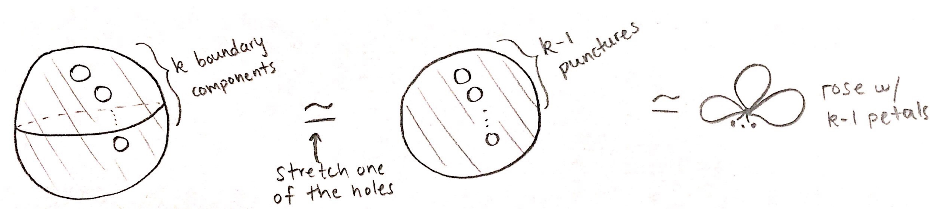
Hence, the effect of gluing in k Möbius bands onto S_k is that of identifying antipodal points of each of the boundary components of S_k. We claim that $\pi_1(X_k) = \langle \alpha_1, \ldots, \alpha_k \mid \alpha_1^2 \cdots \alpha_k^2 = 1 \rangle$. To do this, we first show that the fundamental group of a non-orientable surface of genus i with $n - i$ boundary components (for $i \leq k$, of course), call it $N_{i,k-i}$, has fundamental group

$$\pi_1(N_{i,k-i}) = \langle \epsilon, \epsilon_1, \ldots, \epsilon_{i-1}, \alpha_i, \ldots, \alpha_{k-1} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_{i-1}^2 \alpha_i \cdots \alpha_{k-1} \rangle .$$

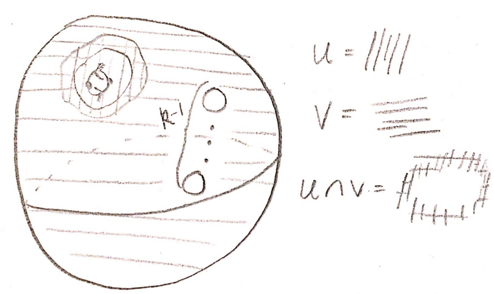
If we can show this, then setting $i = k$, we get

$$\begin{aligned}\pi_1(X_k) = \pi_1(N_{k,0}) &= \langle \epsilon, \epsilon_1, \ldots, \epsilon_{k-1} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_{k-1}^2 \rangle \\ &= \langle \epsilon_1, \ldots, \epsilon_k \mid \epsilon_1^2 \cdots \epsilon_k^2 = 1 \rangle . \qquad \text{(letting } \epsilon = \epsilon_k^{-1}\text{)}\end{aligned}$$

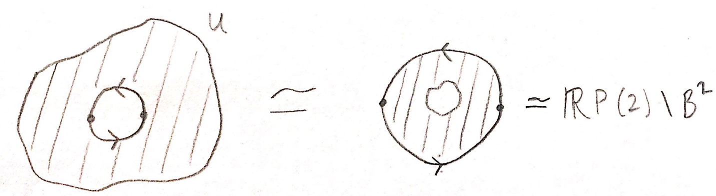
Note that $N_{0,k}$ is simply S^2 with k boundary components, which is homotopy equivalent to a rose with $k - 1$ petals, R_{k-1}:



We proceed by induction; start with $i = 1$. Let U be a neighborhood of the boundary component which has antipodal points identified, and let V be the complement of a neighborhood which is inside U:



Then U is a Möbius band:



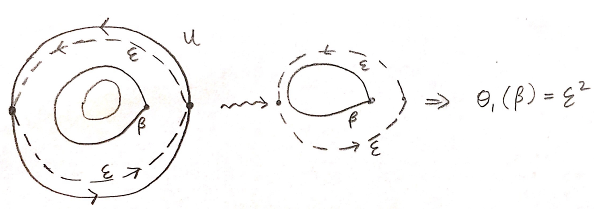
Also, $V \simeq N_{0,k} \simeq R_{k-1}$, and $U \cap V \simeq S^1$. Write

$$\begin{aligned} \pi_1(U) &= \langle \epsilon | \ \rangle \\ \pi_1(V) &= \langle \alpha_1, \ldots, \alpha_{k-1} | \ \rangle \\ \pi_1(U \cap V) &= \langle \beta | \ \rangle \end{aligned}$$

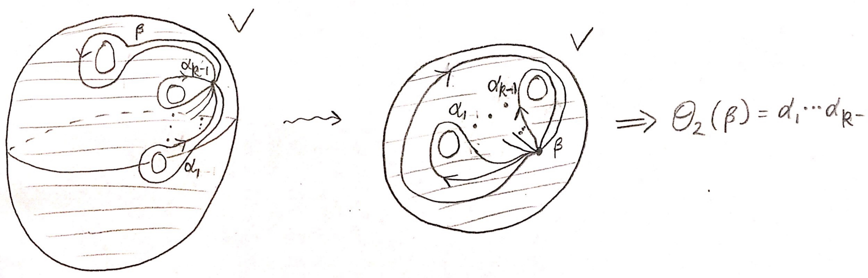
Let $\theta_1 : \pi_1(U \cap V) \to \pi_1(U) \to \pi_1(U \cup V)$ be the composition of the induced maps of inclusions, and similarly, define $\theta_2 : \pi_1(U \cap V) \to \pi_1(V) \to \pi_1(U \cup V)$. Then by the Seifert Van-Kampen Theorem, we have

$$\pi_1(N_{1,k-1}) = \langle \epsilon, \alpha_1, \ldots, \alpha_{k-1} \ | \ \theta_1(\beta) = \theta_2(\beta) \rangle \,.$$

So, we have



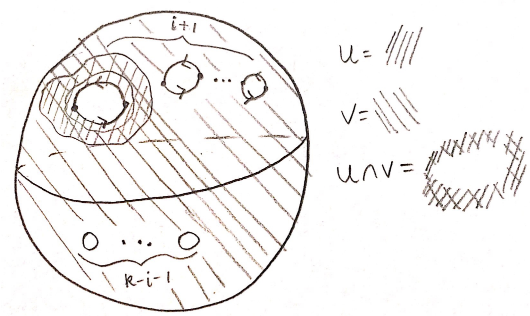
and



Therefore $\pi_1(N_{1,k-1}) = \langle \epsilon, \alpha_1, \ldots, \alpha_{k-1} \mid \epsilon^2 = \alpha_1 \ldots, \alpha_{k-1} \rangle$, as desired. Now for the inductive step. Let $1 \leq i < k$, and suppose for induction that

$$\pi_1(N_{i,k-i}) = \langle \epsilon, \epsilon_1, \ldots, \epsilon_{i-1}, \alpha_i, \ldots, \alpha_{k-1} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_{i-1}^2 \alpha_i \cdots \alpha_{k-1} \rangle .$$

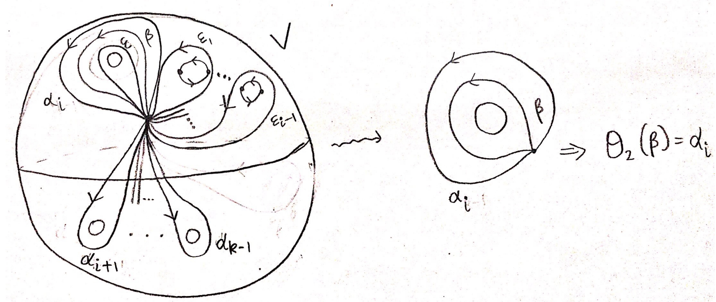
Now consider $N_{i+1,k-i-1}$, and let U be a neighborhood of one of the boundary components of $N_{i+1,k-i-1}$ which has its antipodal points identified, and let V be the complement of a neighborhood which is inside U:



Then U is a Möbius band, $V \simeq N_{i,k-i}$, and $U \cap V \simeq S^1$. Write

$$\begin{aligned} \pi_1(U) &= \langle \epsilon_i | \ \rangle \\ \pi_1(V) &= \langle \epsilon, \epsilon_1, \ldots, \epsilon_{i-1}, \alpha_i, \ldots, \alpha_{k-1} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_{i-1}^2 \alpha_i \cdots \alpha_{k-1} \rangle \\ \pi_1(U \cap V) &= \langle \beta | \ \rangle . \end{aligned}$$

Just as before, we get $\theta_1(\beta) = \epsilon_i^2$. Now for $\theta_2(\beta)$, we have:



And so by the Seifert Van-Kampen Theorem, $\pi_1(N_{i+1,k-i-1})$ is

$$\begin{aligned}
&\left\langle \epsilon_i, \epsilon, \epsilon_1, \ldots, \epsilon_{i-1}, \alpha_i, \ldots, \alpha_{k-1} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_{i-1}^2 \alpha_i \cdots \alpha_{k-1}, \theta_1(\beta) = \theta_2(\beta) \right\rangle \\
&= \left\langle \epsilon, \epsilon_1, \ldots, \epsilon_i, \alpha_i, \ldots, \alpha_{k-1} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_{i-1}^2 \alpha_i \cdots \alpha_{k-1},\ \epsilon_i^2 = \alpha_i \right\rangle \\
&= \left\langle \epsilon, \epsilon_1, \ldots, \epsilon_i, \alpha_{i+1}, \ldots, \alpha_{k-2} \mid \epsilon^2 = \epsilon_1^2 \cdots \epsilon_i^2 \alpha_{i+1} \cdots \alpha_{k-2} \right\rangle,
\end{aligned}$$

which is exactly what we were looking for!

1.7 Covering Spaces

1. **Iowa Qual, Fall 2014 (Nevalainen)**

   Suppose B is connected, and locally path connected and Hausdorff. Suppose that $p : E \to B$ is a covering map. Prove that for any two $b_1, b_2 \in B$, the cardinality of the fibers $p^{-1}(b_1)$ and $p^{-1}(b_2)$ are the same.

2. **Iowa Qual, Fall 2014 (Aceves)**

   Prove that if $p : E \to B$ is a covering map with B and E connected, locally path connected and Hausdorff, and $\gamma : [0,1] \to B$ is continuous with $\gamma(0) = b$, and $e \in p^{-1}(b)$ that there exists a unique continuous map $\tilde{\gamma} : [0,1] \to E$ with $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = e$. *That is, prove the existence and uniqueness of liftings of paths for covering maps of sufficiently nice topological spaces.*

3. **Iowa Qual, Fall 2015 (Oswald)**

   Let $X = \mathbb{R}^3 \setminus \ell$ where $\ell = \{(0,0,z) : z \in \mathbb{R}\}$ is the z-axis. Find a nontrivial connected covering space $\pi : Y \to X$. Prove that your example satisfies the required properties.

4. **Iowa Qual, Fall 2016 (Aceves)**

   Show how to construct a n-sheeted cover of a genus g surface for any positive integer n.

5. **Iowa Qual, Winter 2017 (Sanadhya)**

   Suppose that $P = \mathbb{R}^2 \setminus \{(0,0)\}$ is the plane minus the origin.

   - (a) Compute the fundamental group $\pi_1(P, (0,1))$.
   - (b) Construct a space X which is homeomorphic to the universal cover of P. (Do not prove that X is the universal cover of P.)
   - (c) Describe the action of $\pi_1(P, (0,1))$ on X.

6. **UGA Qual, Fall 2015 (Malachi)**

   Explicitly give a collection of deck transformations on $\{(x,y) | -1 \leq x \leq 1, -\infty < y < \infty\}$ such that the quotient is a Möbius band.

1.8 Simplicial Homology

1. **General Topology, Fall 2016, Exam 1 (Oswald)**

   Recall that the *Klein bottle* K is obtained from a rectangle by identifying two of its opposite sides with matching orientations, and the other two of its sides with reversed orientations.

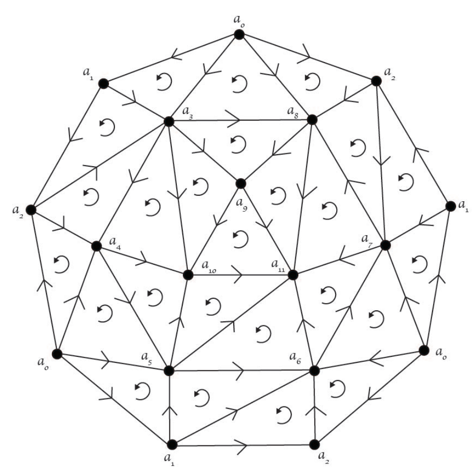
   - (a) Give a triangulation of the Klein bottle.
   - (b) Recall that in an n-complex, we call an $(n-1)$ simplex σ_i *interior* if it is a face of precisely two n-simplices $\overline{\sigma_i}$ and $\overline{\overline{\sigma_i}}$. Recall further that an orientation of an n-complex is *coherent* provided

     $$[\overline{\sigma_i}, \sigma_i] = -[\overline{\overline{\sigma_i}}, \sigma_i]$$

     for every interior $(n-1)$-simplex σ_i. Prove that your triangulation of the Klein bottle from part (a) can admit no coherent orientation.

2. **General Topology, Fall 2016, Exam 2 (Wickrama)**

   Let K denote the 2-complex shown below, with the indicated orientations. Compute the homology groups of K. Since K has more than 9 vertices, be sure to include the a's in your notation (e.g. write $\langle a_1 a_2\rangle$ instead of $\langle 12\rangle$)



3. **UGA Qual, January 2016 (Wood)**

   Give a list without repetitions of all compact surfaces (orientable or non-orientable and with or without boundary) that have Euler characteristic negative one. Explain why there are no repetitions on your list.

4. **UGA Qual, January 2016 (Wickrama)**

   Give an example, with explanation, of a closed curve in a surface which is not null homotopic but is null homologous.

2 Smooth Manifolds

2.1 "Basics"

1. **Iowa Qual, Fall 2005**

   - **a.** Define T_pM where M is a smooth manifold and $p \in M$.
   - **b.** If $F : M \to N$ is a smooth map of smooth manifolds define $F_* : T_pM \to T_{F(p)}N$.
   - **c.** Prove the *chain rule*, that is, if $F : M \to N$ and $G : N \to P$ are smooth maps of smooth manifolds then $(G \circ F)_* = G_* \circ F_*$.

   Solution:

   - **a.**
   $$T_pM = \left\{ D : C^\infty(M) \to \mathbb{R} : \begin{array}{l} D \text{ is } \mathbb{R}\text{-linear and } D \text{ satisfies the Leibniz rule:} \\ (D(fg))(p) = (Df)g(p) + f(p)(Dg) \end{array} \right\}$$
   - **b.** For all $X_p \in T_pM$, and for all $f \in C^\infty(N)$, the map $F_{*,p}(X_p) \in T_pN$ is given by
   $$(F_{*,p}(X_p))(f) = X_p(F^*f) = X_p(f \circ F).$$
   - **c.**

     Proof. Let $p \in M$, $X_p \in T_pM$, and $f \in C^\infty(P)$. Then
     $$((G \circ F)_{*,p}(X_p))(f) = X_p((G \circ F)^* f) = X_p(f \circ G \circ F),$$
     and on the other hand
     $$\begin{aligned} ((G_{*,F(p)} \circ F_{*,p})(X_p))(f) &= G_{*,F(p)}(F_{*,p}(X_p))(f) \\ &= F_{*,p}(X_p)(f \circ G) \\ &= X_p(f \circ G \circ F). \end{aligned}$$
     □

2. **Iowa Qual, Fall 2005**

   State and prove the *local immersion theorem.*

   Theorem (Local Immersion Theorem)**.** *Let $F : M^m \to N^n$ be a smooth map of smooth manifolds. If F is an immersion at $p \in M$, then there exists charts $(U, \varphi), (V, \psi)$ about p and $F(p)$, respectively so that $\psi \circ F \circ \varphi^{-1} = i : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$, where i is inclusion.*

   Proof. Let $(U, \varphi) = (U, x_1, \ldots, x^m)$ and $(\tilde{V}, \tilde{\psi}) = (V, y^1, \ldots, y^n)$ be charts centered[3] around p and $F(p)$, respectively. Since F is an immersion at p, then $m \leq n$ and map $g = \tilde{\psi} \circ F \circ \varphi^{-1}$ is an immersion at 0. Hence, by an adjustment of the coordinates of $\tilde{\psi}$ if necessary, we have
   $$g_{*,0} = \begin{pmatrix} I_m \\ 0 \end{pmatrix},$$

[3] $\varphi(p) = 0, \psi(F(p)) = 0$

Define $G : \varphi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by $G(a,b) = (g(a), b)$. Then

$$G_{*,0} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

Hence by the Inverse Function Theorem, G is a local diffeomorphism at 0, i.e., there exists a neighborhood $W \subseteq \mathbb{R}^n$ of 0 such that $G|_W : W \to G(W)$ is a diffeomorphism.

So, define $V := \tilde{V} \cap \tilde{\psi}^{-1}(W)$ and $\psi = G^{-1} \circ \tilde{\psi}$. Then $F(p) \in V$ since $F(p) \in \tilde{V}$, and, $\tilde{\psi}(F(p)) = 0 \in W$. Moreover, ψ is a diffeomorphism on V since both G^{-1} and $\tilde{\psi}$ are. Hence (V, ψ) is a chart in N about $F(p)$. Notice that $g = G \circ i$, and so

$$\psi \circ F \circ \varphi^{-1} = G^{-1} \circ \tilde{\psi} \circ F \circ \varphi^{-1} = G^{-1} \circ g = G^{-1} \circ G \circ i = i.$$

□

We also state and prove the local submersion theorem:

Theorem (Local Submersion Theorem)**.** *Let $F : M^m \to N^n$ be a smooth map of smooth manifolds. If F is an submersion at $p \in M$, then there exists charts $(U, \varphi), (V, \psi)$ about p and $F(p)$, respectively so that $\psi \circ F \circ \varphi^{-1} = \pi : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$, where π is projection.*

Proof. Let $(\tilde{U}, \tilde{\varphi}) = (\tilde{U}, x_1, \ldots, x^m)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be charts centered around p and $F(p)$, respectively. Since F is a submersion at p, then $m \geq n$ and map $g = \psi \circ F \circ \tilde{\varphi}^{-1}$ is a submersion at 0. Hence, by an adjustment of the coordinates of $\tilde{\varphi}$ if necessary, we have

$$g_{*,0} = \begin{pmatrix} I_n & 0 \end{pmatrix},$$

Define $G : \tilde{\varphi}(U) \to \mathbb{R}^m$ by $G(a) = G(a_1, \ldots, a_m) = (g(a), a_{n+1}, \ldots, a_m)$. Then

$$G_{*,0} = \begin{pmatrix} I_n & 0 \\ 0 & I_{m-n} \end{pmatrix}.$$

Hence by the Inverse Function Theorem, G is a local diffeomorphism at 0, i.e., there exists a neighborhood $W \subseteq \mathbb{R}^m$ of 0 such that $G|_W : W \to G(W)$ is a diffeomorphism.

So, define $U := \tilde{U} \cap \tilde{\varphi}^{-1}(W)$ and $\varphi = G \circ \tilde{\varphi}$. Then $p \in U$ since $p \in \tilde{U}$, and, $\tilde{\varphi}(p) = 0 \in W$. Moreover, φ is a diffeomorphism on U since both G and $\tilde{\varphi}$ are. Hence (U, φ) is a chart in M about p. Notice that $g = \pi \circ G$, and so

$$\psi \circ F \circ \varphi^{-1} = \psi \circ F \circ \tilde{\varphi}^{-1} \circ G^{-1} = g \circ G^{-1} = \pi \circ G \circ G^{-1} = \pi.$$

□

3. **Iowa Qual, Fall 2007**

Suppose that M and N are smooth manifolds. Give $M \times N$ the structure of a smooth manifold by producing a compatible atlas. Prove that your atlas is compatible.

Proof. Suppose that $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are smooth atlases for M^m and N^n, respectively. We show that $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$ defines a smooth structure on $M \times N$, where $\varphi_\alpha \times \psi_\beta : M \times N \to \mathbb{R}^m \times \mathbb{R}^n$ is given by $(\varphi_\alpha \times \psi_\beta)(m,n) = (\varphi_\alpha(m), \psi_\beta(n))$.

Evidently the collection of open sets $\{(U_\alpha \times V_\beta)\}$ covers $M \times N$. Now suppose $(U_{\alpha_1} \times V_{\beta_1}, \varphi_{\alpha_1} \times \psi_{\beta_1})$ and $(U_{\alpha_2} \times V_{\beta_2}, \varphi_{\alpha_2} \times \psi_{\beta_2})$ are overlapping charts in $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$. Then the map

$$(\varphi_{\alpha_1} \times \psi_{\beta_1}) \circ (\varphi_{\alpha_2} \times \psi_{\beta_2})^{-1} : \varphi_{\alpha_2}(U_{\alpha_1\alpha_2}) \times \psi_{\beta_2}(V_{\beta_1\beta_2}) \to \varphi_{\alpha_1}(U_{\alpha_1\alpha_2}) \times \psi_{\beta_1}(V_{\beta_1\beta_2})$$

is smooth at $(x, y) \in \varphi_{\alpha_2}(U_{\alpha_1\alpha_2}) \times \psi_{\beta_2}(V_{\beta_1\beta_2}) \subset \mathbb{R}^m \times \mathbb{R}^n$ because

$$\begin{aligned}(\varphi_{\alpha_1} \times \psi_{\beta_1}) \circ (\varphi_{\alpha_2} \times \psi_{\beta_2})^{-1}(x, y) &= \Big((\varphi_{\alpha_1} \times \psi_{\beta_1})(\varphi_{\alpha_2}^{-1}(x), \psi_{\beta_2}^{-1}(y))\Big) \\ &= \Big(\varphi_{\alpha_1}(\varphi_{\alpha_2}^{-1}(x)), \psi_{\beta_1}(\psi_{\beta_2}^{-1}(y))\Big) \\ &= \Big((\varphi_{\alpha_1} \circ \varphi_{\alpha_2}^{-1})(x), (\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(y)\Big),\end{aligned}$$

and $(\varphi_{\alpha_1} \circ \varphi_{\alpha_2}^{-1})$ and $(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})$ are smooth at x and y, respectively. Hence our atlas is compatible. □

4. **Iowa Qual, Spring 2008**

Prove that the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a smooth manifold by exhibiting charts. Carefully calculate <u>one</u> transition map and its derivative to explain why the transition map is a diffeomorphism.

Proof. We have charts

$$\begin{aligned}X^+ &= \{(x, y, z) \in S^2 : x > 0\},\ \varphi_{X^+}(x, y, z) = (y, z), \\ X^- &= \{(x, y, z) \in S^2 : x < 0\},\ \varphi_{X^-}(x, y, z) = (y, z), \\ Y^+ &= \{(x, y, z) \in S^2 : y > 0\},\ \varphi_{Y^+}(x, y, z) = (x, z), \\ Y^+ &= \{(x, y, z) \in S^2 : y < 0\},\ \varphi_{Y^-}(x, y, z) = (x, z), \\ Z^+ &= \{(x, y, z) \in S^2 : z > 0\},\ \varphi_{Z^+}(x, y, z) = (x, y), \\ Z^+ &= \{(x, y, z) \in S^2 : z < 0\},\ \varphi_{Z^-}(x, y, z) = (x, y).\end{aligned}$$

The inverse maps are,

$$\begin{aligned}\varphi_{X^+}^{-1}(y, z) &= \left(\sqrt{1 - y^2 - z^2}, y, z\right), \\ \varphi_{X^-}^{-1}(y, z) &= \left(-\sqrt{1 - y^2 - z^2}, y, z\right), \\ \varphi_{Y^+}^{-1}(x, z) &= \left(x, \sqrt{1 - x^2 - z^2}, z\right), \\ \varphi_{Y^-}^{-1}(x, z) &= \left(x, -\sqrt{1 - x^2 - z^2}, z\right), \\ \varphi_{Z^+}^{-1}(x, y) &= \left(x, y, \sqrt{1 - x^2 - y^2}\right), \\ \varphi_{Z^-}^{-1}(x, y) &= \left(x, y, -\sqrt{1 - x^2 - y^2}\right).\end{aligned}$$

We compute the transition map $\varphi_{X^-} \circ \varphi_{Z^+}^{-1} : \varphi_{Z^+}(X^- \cap Z^+) \to \varphi_{X^-}(X^- \cap Z^+)$.

$$\left(\varphi_{X^-} \circ \varphi_{Z^+}^{-1}\right)(x, y) = \varphi_{X^-}\left(x, y, -\sqrt{1 - x^2 - y^2}\right) = \left(y, -\sqrt{1 - x^2 - y^2}\right).$$

The derivative of $\varphi_{X^-} \circ \varphi_{Z^+}^{-1}$ at a point $(x_0, y_0) \in \varphi_{Z^+}(X^- \cap Z^+)$ is

$$\left(\varphi_{X^-} \circ \varphi_{Z^+}^{-1}\right)_{*,(x_0,y_0)} = \begin{pmatrix} \left.\dfrac{\partial y}{\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial y}{\partial y}\right|_{(x_0,y_0)} \\ \left.\dfrac{\partial\left(-\sqrt{1-x^2-y^2}\right)}{\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial\left(-\sqrt{1-x^2-y^2}\right)}{\partial y}\right|_{(x_0,y_0)} \end{pmatrix}$$

$$= \begin{pmatrix} 0\big|_{(x_0,y_0)} & 1\big|_{(x_0,y_0)} \\ x(1-x^2-y^2)^{-1/2}\Big|_{(x_0,y_0)} & y(1-x^2-y^2)^{-1/2}\Big|_{(x_0,y_0)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ x_0(1-x_0^2-y_0^2)^{-1/2} & y_0(1-x_0^2-y_0^2)^{-1/2} \end{pmatrix}.$$

The above matrix fails to have full rank precisely when $x_0 = 0$ or $(1-x_0^2-y_0^2)^{-1/2} = 0$. The latter case is never true. Notice that

$$\varphi_{Z^+}(X^- \cap Z^+) = \left\{(x,y) \in \mathbb{R}^2 : -1 < y < 1,\ -\sqrt{1-y^2} < x < 0\right\},$$

and so $x_0 \neq 0$. Hence $\left(\varphi_{X^-} \circ \varphi_{Z^+}^{-1}\right)_{*,(x,y)}$ has full rank. Then by the Inverse Function Theorem, $\varphi_{X^-} \circ \varphi_{Z^+}^{-1}$ is a local diffeomorphism at (x, y). Since (x, y) was arbitrary, $\varphi_{X^-} \circ \varphi_{Z^+}^{-1}$ is a diffeomorphism. □

5. **Iowa Qual, Spring 2008**

Suppose X, Y, Z are smooth manifolds, $f : X \to Y$ is a diffeomorphism, and $g : Y \to Z$ is a smooth map such that for some $y \in Y$, $g_{*,y} : T_yY \to T_{g(y)}Z$ is injective. Prove that for each $x \in f^{-1}(y)$, $(g \circ f)_{*,x}$ is injective.

Proof. Let $x \in f^{-1}(y)$. By the Chain Rule,

$$(g \circ f)_{*,x} = g_{*,f(x)} \circ f_{*,x} = g_{*,y} \circ f_{*,x}.$$

Since $g_{*,y}$ is injective, we need only to show that $f_{*,x}$ is injective to obtain the desired result. Since f is a diffeomorphism $f^{-1} \circ f = \mathbb{1}_X$ and so again by the Chain Rule,

$$f^{-1}_{*,f(x)} \circ f_{*,x} = (f^{-1} \circ f)_{*,x} = (\mathbb{1}_X)_{*,x},$$

from which it follows that $f_{*,x}$ is injective. □

6. **Iowa Qual, January 2011**

Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ be the standard unit sphere of dimension 2. Let $F : S^2 \to S^2$ be defined by restricting the linear map $T(x,y,z) = (-y,x,-z)$

to S^2, that is $F = T|_{S^2}$. The stereographic projection ϕ given below defines a chart, $(S^2 \setminus (0,0,1), \phi)$.

$$\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (u,v) \text{ and}$$

$$\phi^{-1}(u,v) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1)$$

Let $p = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$ and $q = F(p)$. Calculate the matrix of $F_* : T_pS^2 \to T_qS^2$ with respect to the bases $\{\frac{\partial}{\partial u}|_p, \frac{\partial}{\partial v}|_p\}$ and $\{\frac{\partial}{\partial u}|_q, \frac{\partial}{\partial v}|_q\}$.

Solution:

Let u, v denote the standard coordinates in $\mathbb{R}^2$, and let $\phi = (\phi_1, \phi_2)$.Then notice that $\phi_1 = u \circ \phi$ and $\phi_2 = u \circ \phi$. So the component functions of F with respect to the chart $(S^2 \setminus (0,0,1), \phi)$ are

$$F^1 = \phi_1 \circ F = u \circ \phi \circ F \text{ and } F^2 = \phi_2 \circ F = v \circ \phi \circ F.$$

So,

$$F_{*,p} = \begin{pmatrix} \frac{\partial F^1}{\partial u}(p) & \frac{\partial F^1}{\partial v}(p) \\ \frac{\partial F^2}{\partial u}(p) & \frac{\partial F^2}{\partial v}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial(u\circ\phi\circ F)}{\partial u}(p) & \frac{\partial(u\circ\phi\circ F)}{\partial v}(p) \\ \frac{\partial(v\circ\phi\circ F)}{\partial u}(p) & \frac{\partial(v\circ\phi\circ F)}{\partial v}(p) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial(u\circ\phi\circ F\circ\phi^{-1})}{\partial u}(\phi(p)) & \frac{\partial(u\circ\phi\circ F\circ\phi^{-1})}{\partial v}(\phi(p)) \\ \frac{\partial(v\circ\phi\circ F\circ\phi^{-1})}{\partial u}(\phi(p)) & \frac{\partial(v\circ\phi\circ F\circ\phi^{-1})}{\partial v}(\phi(p)) \end{pmatrix}.$$

Now,

$$\begin{aligned}
(\phi\circ F\circ\phi^{-1})(u,v) &= (\phi\circ F)\left(\frac{1}{1+u^2+v^2}\left(2u, 2v, u^2+v^2-1\right)\right) \\
&= \phi\left(\frac{1}{1+u^2+v^2}(-2v, 2u, -(u^2+v^2-1))\right) \\
&= \frac{1}{1+u^2+v^2}\left(\frac{-2v}{1+\frac{u^2+v^2-1}{1+u^2+v^2}}, \frac{2u}{1+\frac{u^2+v^2-1}{1+u^2+v^2}}\right) \\
&= \frac{1}{1+u^2+v^2}\left(\frac{-2v}{\frac{2u^2+2v^2}{1+u^2+v^2}}, \frac{2u}{\frac{2u^2+2v^2}{1+u^2+v^2}}\right) \\
&= \left(\frac{-2v}{2u^2+2v^2}, \frac{2u}{2u^2+2v^2}\right).
\end{aligned}$$

So,

$$\frac{\partial(u\circ\phi\circ F\circ\phi^{-1})}{\partial u} = \frac{\partial\left(\frac{-2v}{2u^2+2v^2}\right)}{\partial u} = \frac{8uv}{(2u^2+2v^2)^2},$$

$$\begin{aligned}\frac{\partial(u\circ\phi\circ F\circ\phi^{-1})}{\partial v} &= \frac{\partial\left(\frac{-2v}{2u^2+2v^2}\right)}{\partial v} = \frac{(2u^2+2v^2)(-2)-(-2v)(4v)}{(2u^2+2v^2)^2}\\ &= -\frac{1}{u^2+v^2} + \frac{8v^2}{(2u^2+2v^2)^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial(v\circ\phi\circ F\circ\phi^{-1})}{\partial u} &= \frac{\partial\left(\frac{2u}{2u^2+2v^2}\right)}{\partial u} = \frac{(2u^2+2v^2)(2)-(2u)(4u)}{(2u^2+2v^2)^2}\\ &= \frac{1}{u^2+v^2} - \frac{8u^2}{(2u^2+2v^2)^2},\end{aligned}$$

$$\frac{\partial(v\circ\phi\circ F\circ\phi^{-1})}{\partial v} = \frac{\partial\left(\frac{2u}{2u^2+2v^2}\right)}{\partial v} = \frac{-8uv}{(2u^2+2v^2)^2}.$$

Then $\phi(p) = \left(\sqrt{2}, 1\right)$, and so finally, we get

$$F_{*,p} = \begin{pmatrix} \frac{\partial F^1}{\partial u}(p) & \frac{\partial F^1}{\partial v}(p) \\ \frac{\partial F^2}{\partial u}(p) & \frac{\partial F^2}{\partial v}(p) \end{pmatrix} = \begin{pmatrix} \frac{8\sqrt{2}}{36} & -\frac{1}{3}+\frac{8}{36} \\ \frac{1}{3}-\frac{16}{36} & -\frac{-8\sqrt{2}}{36} \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{2}}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{-2\sqrt{2}}{9} \end{pmatrix}.$$

7. **Iowa Qual, Fall 2014**

Recall that $\mathbb{R}P(2)$ is the quotient space of $\mathbb{R}^3 \setminus \{\vec{0}\}$ by the equivalence relation $(x,y,z) = (x',y',z')$ if there exists $\lambda \in \mathbb{R}\setminus\{0\}$ so that $\lambda(x,y,z) = (x',y',z')$. Denote the equivalence class of (x,y,z) by $[x,y,z]$. Give $\mathbb{R}P(2)$ the standard smooth structure. Consider the map $f : \mathbb{R}P(2) \to \mathbb{R}^2$ given by

$$f([x,y,z]) = \left(\frac{xy}{x^2+y^2+z^2}, \frac{yz}{x^2+y^2+z^2}\right).$$

- **a.** Prove that f is smooth.
- **b.** Find the set where f has rank 1.

Proof. • **a.** Define $\tilde{f} : \mathbb{R}^3 \setminus \{\vec{0}\} \to \mathbb{R}^2$ by

$$(x,y,z) \mapsto \left(\frac{xy}{x^2+y^2+z^2}, \frac{yz}{x^2+y^2+z^2}\right).$$

As a rational expression, $\tilde{f}$ is smooth. Suppose $\lambda(x,y,z) = (x',y',z')$ for $\lambda \in$

$\mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned}
\tilde{f}((x', y', z')) &= \left(\frac{x'y'}{x'^2 + y'^2 + z'^2}, \frac{y'z'}{x'^2 + y'^2 + z'^2} \right) \\
&= \left(\frac{(\lambda x)(\lambda y)}{(\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2}, \frac{(\lambda y)(\lambda z)}{(\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2} \right) \\
&= \left(\frac{xy}{x^2 + y^2 + z^2}, \frac{yz}{x^2 + y^2 + z^2} \right) \\
&= f(x, y, z),
\end{aligned}$$

and hence, $\tilde{f}$ is constant on the fibers above points in the quotient. Therefore, $\tilde{f}$ descends to a smooth map $f : \mathbb{R}P(2) \to \mathbb{R}^2$ such that $\tilde{f} = f \circ p$, where $p : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P(2)$ is the quotient map. Evidently the map f is the one described in the problem.

- **b.** Recall that the standard smooth structure on $\mathbb{R}P(2)$ is given by the atlas $\{(U_i, \phi_i)\}_{i=0}^{2}$, where

$$\begin{aligned}
U_0 &= \{[x, y, z] \in \mathbb{R}P(2) : x \neq 0\},\ \phi_0([x, y, z]) = \left(\frac{y}{x}, \frac{z}{x} \right),\ \phi_0^{-1}(x, y) = [1, x, y], \\
U_1 &= \{[x, y, z] \in \mathbb{R}P(2) : y \neq 0\},\ \phi_1([x, y, z]) = \left(\frac{x}{y}, \frac{z}{y} \right),\ \phi_1^{-1}(x, y) = [x, 1, y], \\
U_2 &= \{[x, y, z] \in \mathbb{R}P(2) : z \neq 0\},\ \phi_2([x, y, z]) = \left(\frac{x}{z}, \frac{y}{z} \right),\ \phi_2^{-1}(x, y) = [x, y, 1].
\end{aligned}$$

Let $q_0 = [x_0, y_0, z_0] \in U_0$. Then

$$f_{*,q_0} = \begin{pmatrix} \frac{\partial f^1}{\partial x}(q_0) & \frac{\partial f^1}{\partial y}(q_0) \\ \frac{\partial f^2}{\partial x}(q_0) & \frac{\partial f^2}{\partial y}(q_0) \end{pmatrix} = \begin{pmatrix} \frac{x^2}{}(q_0) & \frac{\partial f^1}{\partial y}(q_0) \\ \frac{\partial f^2}{\partial x}(q_0) & \frac{\partial f^2}{\partial y}(q_0) \end{pmatrix}$$

□

8. **Iowa Qual, Fall 2015**

- (a) Recall $S^1 = \{z \in \mathbb{C} \| z| = 1\}$. Consider that map $\mu : \mathbb{Z} \times S^1 \to S^1$ given by $\mu(n, z) = e^{2\pi i n \sqrt{2}} z$. Prove that μ defines a C^∞ action of $\mathbb{Z}$ on the smooth manifold S^1.
- (b) Is $S^1/\mathbb{Z}$ a manifold? Prove or disprove.

Proof.
- (a)
- (b)

□

9. **Iowa Qual, Winter 2016**

Recall that for any point $x \in S^n$, where $S^n = S^n_1$, the space $S^n \setminus \{x\}$ is diffeomorphic to $\mathbb{R}^n$. Find a map $\varphi : S^n \setminus \{x\} \to \mathbb{R}^n$. Prove that the map φ is a diffeomorphism.

Proof. Note that $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 \cdots + x_{n+1}^2 = 1\}$. By possibly rotating S^n as needed, we may assume that $x = N = (0, \ldots, 0, 1)$. We view $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ as all points with last coordinate 0; that is, $\mathbb{R}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$.

Consider the line from N through a point $p = (x_1, \ldots, x_n, x_{n+1}) \in S^n$,

$$L_p : [0, \infty) \to \mathbb{R}^{n+1},\ t \mapsto (1-t)N + tp = (tx_1, \ldots, tx_n, tx_{n+1} + (1-t)).$$

Then if $t_p = \frac{1}{1-x_{n+1}}$, $L_p(t_p)$ has last coordinate 0, and so $L_p(t_p) \in \mathbb{R}^n$. So we define $\varphi : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$t \mapsto L_p(t_p) = \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n, 0).$$

On the other hand, if $y = (x_1, \ldots, x_n, 0) \in \mathbb{R}^n$, consider the line from N to y,

$$L_y : [0, \infty) \to \mathbb{R}^{n+1},\ t \mapsto (1-t)N + ty = (tx_1, \ldots, tx_n, 1-t).$$

Then $L_y(t_y) \in S^n$ if $t_y = \frac{2}{x_1 + \cdots + x_n + 1}$, and so we have an inverse map $\varphi^{-1} : \mathbb{R}^n \to S^n \setminus \{N\}$ given by

$$(x_1, \ldots, x_n, 0) \mapsto L_y(t_y) = \frac{1}{x_1^2 + \cdots + x_n^2 + 1}(2x_1, \ldots, 2x_n, x_1^2 + \cdots + x_n^2 - 1).$$

We argue why φ and φ^{-1} are smooth. First, φ^{-1}, is smooth map on all of $\mathbb{R}^n$ since its component functions are given in the global coordinates of $\mathbb{R}^n$ and are therein smooth.

Consider $A = \mathbb{R}^{n+1} \setminus \{\text{pts w/ last coord. } 1\}$. If $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ is projection onto the last coordinate, then $A = \mathbb{R}^{n+1} \setminus f^{-1}(\{1\})$, and so A is open since $f^{-1}(\{1\})$ is closed. So, as a function on the open subset A of $\mathbb{R}^{n+1}$, the map φ is smooth since it is given the the global coordinates of $\mathbb{R}^{n+1}$ and is indeed smooth with respect to those coordinates. Since $S^n \setminus \{N\} \subset A$ is a regular submanifold of $\mathbb{R}^{n+1}$, then φ restricts to a smooth map on $S^n \setminus \{N\}$. □

10. **Iowa Qual, Winter 2016**

Suppose that M is a C^∞-smooth manifold.

- **a.** Define the ring $C(M)$.
- **b.** Define the ring $C^\infty(M)$.
- **c.** Define the rings $C^r(M)$, for $r = 1, 2, \ldots$.
- **d.** Prove that there are injective ring homomorphisms:

$$C^0(M) \subset \cdots \subset C^r(M) \subset C^{r+1}(M) \subset \cdots \subset C^\infty(M).$$

- **e.** Prove that at least two of the homomorphisms in the previous part are not surjective.

2.2 Regular Submanifolds

1. **Iowa Qual, Fall 2005**

   - **a.** Prove that the hyperboloid H of points in $\mathbb{R}^3$ that satisfy

   $$x^2 + y^2 - z^2 = 1$$

   is a smooth [i.e., regular] submanifold of $\mathbb{R}^3$.

   - **b.** Let $p : H \to \mathbb{R}^2$ be the resriction of orthogonal projection to the xz-plane to H. What are the regular values and critical values of p? Justify your answer.

Proof. • **a.** We exhibit H as a regular level set of a smooth map from $\mathbb{R}^3$ to $\mathbb{R}$, showing that H is a regular submanifold of $\mathbb{R}^3$ of dimension 2. Consider the smooth map $f(x, y, z) = x^2 + y^2 - z^2 - 1$. Evidently $H = f^{-1}(\{0\})$, and so it remains to show that 0 is a regular value of f. Observe that

$$f_{*,(x,y,z)} = \begin{pmatrix} 2x & 2y & -2z \end{pmatrix},$$

and so f_* fails to be surjective precisely at the origin; hence the only critical value of f is $f(0, 0, 0) = 1$, giving that 0 is a regular value of f.

• **b.** To find the critical values of p, we want to find the points $q \in H$ for which the differential

$$p_{*,q} : T_qH \to T_{p(f)}\mathbb{R}^2 \cong \mathbb{R}^2$$

fails to be surjective. Since p_* is a linear map between 2-dimensional vector spaces, this happens precisely when $\dim \operatorname{Ker}(p_*) > 0$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote projection onto the xz-plane, so that $p = \pi|_H$. Note that since π is a linear map, $\pi_* = \pi$. For $q \in H$, we have

$$\operatorname{Ker} p_{*,q} = \operatorname{Ker} (\pi|_H)_{*,q} = \operatorname{Ker} (\pi_{*,q})\,|_{TqH} = \operatorname{Ker} (\pi)\,|_{TqH} = \operatorname{Ker} \pi \cap T_qH$$

Recall that since H is a regular level set of f, then for any $q \in H$, we have $T_qH = \operatorname{Ker} f_{*,q}$, and so the above becomes

$$\operatorname{Ker} p_{*,q} = \operatorname{Ker} \pi \cap \operatorname{Ker} f_{*,q}.$$

We have

$$\operatorname{Ker} \pi = \{(a, b, c) \in \mathbb{R}^3 : (a, c) = (0, 0)\},$$

and letting $q = (x_0, y_0, z_0)$, we have

$$\begin{aligned} \operatorname{Ker} f_{*,q} &= \left\{ (a, b, c) \in T_qH : \begin{pmatrix} 2x_0 & 2y_0 & -2z_0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \right\} \\ &= \{(a, b, c) \in T_qH : 2x_0a + 2y_0b - 2z_0c = 0\}. \end{aligned}$$

Hence

$$\operatorname{Ker} p_{*,q} = \{(a, b, c) \in T_qH : 2y_0b = 0\}$$

For an arbitrary q, there certainly exists $(a,b,c) \in T_q H$ with $b \neq 0$, and so q is a critical point of p if and only if $y_0 = 0$. Therefore, the critical points of p are

$$C = \{(x_0, y_0, z_0) \in H : y_0 = 0\} = \{(x,y,z) \in \mathbb{R}^3 : x^2 - z^2 = 1\}$$

and so the critical values of p are $p(C) = \{(x,z) \in \mathbb{R}^2 : x^2 - z^2 = 1\}$. Graphically, $p(C)$ is a hyperbola in the xz-plane. The regular values of p are therefore

$$R_v\{(x,z) \in \mathbb{R}^2 : x^2 - z^2 \neq 1\} = \{(x,z) \in \mathbb{R}^2 : x^2 - z^2 < 1\},$$

where the last set equality follows from the fact that no point $(x,y,z) \in H$ is such that $x^2 - z^2 > 1$, because then $1 = x^2 + y^2 - z^2 > 1 + y^2$, which is a contradiction for *any* value of y. So R is the region inside of the hyperbola $p(C)$.

□

2. **Iowa Qual, Fall 2006**

Let $W_c = \{(x,y,z,w) \in \mathbb{R}^4 : xyz = c\}$ and $Y_c = \{(x,y,z,w) \in \mathbb{R}^4 : xzw = c\}$. For what real numbers c is Y_c a three-manifold? For what pairs (c_1, c_2) is $W_{c_1} \cap Y_{c_2}$ a two-manifold?

Proof. Define a map $f : \mathbb{R}^4 \to \mathbb{R}$, $(x,y,z,w) \mapsto xzw - c$. Then $Y_c = f^{-1}(\{0\})$, and

$$f_{*,(x,y,z,w)} = \begin{pmatrix} zw & 0 & xw & xz \end{pmatrix}.$$

Therefore, $f_{*,(x,y,z,w)}$ fails to be surjective – i.e., we obtain critical points of f – when any two of the coordinates x, z, w is zero. Therefore, the regular points of f are

$$R = \{(x,y,z,w) \in \mathbb{R}^4 : y \in \mathbb{R}, \text{ and, at most one coordinate among } x, y, w \text{ is } 0\}.$$

Suppose $c \neq 0$. Then in fact *none* of the coordinates among x, z, w is zero. In other words, $Y_c \subset R$ when $c \neq 0$. Hence by the Regular Level Set Theorem, Y_c is a three-manifold when $c \neq 0$. On the other hand, we claim that Y_0 is not a three-manifold. In this case, $Y_0 = \{(x,y,z,w) \in \mathbb{R}^4 : xzw = 0\}$, or equivalently,

$$Y_0 = \{(x,y,z,w) \in \mathbb{R}^4 : x = 0\} \cup \{(x,y,z,w) \in \mathbb{R}^4 : z = 0\} \cup \{(x,y,z,w) \in \mathbb{R}^4 : w = 0\}.$$

In other words, Y_c is the union of the yzw, xyw, and xyz hyperplanes in $\mathbb{R}^4$. Recall the the dimension of the tangent space at any point in a manifold is the same as that of said manifold. We show that $\dim T_{(0,0,0,0)} Y_0 > 3$, showing that Y_0 cannot be a three-manifold.

Consider the following curves, which all start at $(0,0,0,0)$:

$$\begin{aligned}
\alpha_1 &: (-\epsilon, \epsilon) \to Y_0,\ t \mapsto (t,0,0,0) \\
\alpha_2 &: (-\epsilon, \epsilon) \to Y_0,\ t \mapsto (0,t,0,0) \\
\alpha_3 &: (-\epsilon, \epsilon) \to Y_0,\ t \mapsto (0,0,t,0) \\
\alpha_4 &: (-\epsilon, \epsilon) \to Y_0,\ t \mapsto (0,0,0,t)
\end{aligned}$$

The vectors $X_i = \alpha_i'(0)$ are therefore elements of $T_{(0,0,0,0)} Y_0$, and are linearly independent.

For the second question, first observe

$$W_{c_1} \cap Y_{c_2} = \{(x,y,z,w) \in \mathbb{R}^4 : xyz = c_1 \text{ and } xzw = c_2\}.$$

Define a map $g : \mathbb{R}^4 \to \mathbb{R}^2$ by $(x,y,z,w) \mapsto (xyz, xzw)$. Then $W_{c_1} \cap Y_{c_2} = g^{-1}(\{(c_1,c_2)\})$, and

$$g_{*,(x,y,z,w)} = \begin{pmatrix} yz & xz & xy & 0 \\ zw & 0 & xw & xz \end{pmatrix}.$$

Then g_* fails to have maximal rank when all of its 2×2–minors have determinant zero:

$$0 = \begin{vmatrix} yz & xz \\ zw & 0 \end{vmatrix} = -xz^2w, \quad 0 = \begin{vmatrix} yz & xy \\ zw & xw \end{vmatrix} = xyzw - xyzw, \quad 0 = \begin{vmatrix} yz & 0 \\ zw & xz \end{vmatrix} = xyz^2,$$
$$0 = \begin{vmatrix} xz & xy \\ 0 & xw \end{vmatrix} = x^2zw, \qquad 0 = \begin{vmatrix} xz & 0 \\ 0 & xz \end{vmatrix} = x^2z^2, \qquad 0 = \begin{vmatrix} xy & 0 \\ xw & xz \end{vmatrix} = x^2yz.$$

From here we see immediately that if $x = 0$ or $y = 0$, then g_* fails to have maximal rank. If y and/or w is 0, the fifth determinant above shows that g_* can still have maximal rank if both x and z are nonzero. Hence the g_* has maximal rank if and only if $x \neq 0$ and $z \neq 0$. In other words, the regular points of g are

$$R' = \{(x,y,z,w) \in \mathbb{R}^4 : x \neq 0, z \neq 0\}.$$

If c_1 and c_2 are nonzero, then $W_{c_1} \cap Y_{c_2} \subset R'$. If $c_1 = 0$ and $c_2 \neq 0$ or if $c_1 \neq 0$ and $c_2 = 0$, then x, z are nonzero and so $W_{c_1} \cap Y_{c_2} \subset R'$. Hence $W_{c_1} \cap Y_{c_2}$ is a two-manifold when at most one of c_1, c_2 is 0 by the Regular Lever Set Theorem.

Now suppose $c_1 = c_2 = 0$, then we have

$$\begin{aligned} W_0 \cap Y_0 &= \{(x,y,z,w) \in \mathbb{R}^4 : xyz = 0 \text{ and } xzw = 0\} \\ &= \{x = 0\} \cup \{y = 0\} \cup \{z = 0\} \cup \{w = 0\} \end{aligned}$$

In other words, $W_0 \cap Y_0$ is the union of the yzw, xzw, xyw, and xyz hyperplanes in $\mathbb{R}^4$. An analogous argument with curves as before shows that $\dim T_{(0,0,0,0)} W_0 \cap Y_0 > 2$. Hence $W_0 \cap Y_0$ is not a two-manifold. $\square$

3. **Iowa Qual, Fall 2007**

Let T^2 be the subset of $\mathbb{R}^3$ that is the result of rotating the circle in the yz plane of radius 1 centered at $(0,2,0)$ about the z-axis. It may be useful to note that T^2 is the set of points in 3-space satisfying the equation $((x^2+y^2)^{1/2} - 2)^2 + z^2 = 1$.

- **a.** Prove that T^2 is a smooth 2-manifold.
- **b.** Let $p : T^2 \to \mathbb{R}^2$ be the restriction of $p : \mathbb{R}^3 \to \mathbb{R}^2$ given by $p(x,y,z) = (x,y)$. Identify the regular values of $p : T^2 \to \mathbb{R}^2$.

Proof. • **a.** By the Regular Level Set Theorem: Define a smooth map $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto ((x^2+y^2)^{1/2} - 2)^2 + z^2 - 1$. Then $T^2 = f^{-1}(\{0\})$ and so we must show that 0 is a regular value of f, or equivalently, that $f^{-1}(\{0\})$ is contained in the set of regular points of f. For $(x,y,z) \in T^2$, we have

$$f_{*,(x,y,z)} = \begin{pmatrix} \frac{2x((x^2+y^2)^{1/2}-2)}{(x^2+y^2)^{1/2}} & \frac{2y((x^2+y^2)^{1/2}-2)}{(x^2+y^2)^{1/2}} & 2z \end{pmatrix}.$$

Notice that if $x^2+y^2=0$ then $z^2=-3 \implies x \notin \mathbb{R}$. Hence $f_{*,(x,y,z)}$ is well-defined for all $(x,y,z) \in T^2$. Now, f_* fails to have maximal rank when all of its components are 0; that is, when

$$0 = \frac{2x((x^2+y^2)^{1/2}-2)}{(x^2+y^2)^{1/2}} = \frac{2y((x^2+y^2)^{1/2}-2)}{(x^2+y^2)^{1/2}} = 2z,$$

i.e., when $x=0$, $y=0$, and $z=0$, or when, $x^2+y^2=4$ and $z=0$. The former case is not true for any $(x,y,z) \in T^2$ and the latter case also never happens in T^2 since that would imply $0=1$. Hence f_* has full rank on all of T^2, and so T^2 is a regular submanifold of $\mathbb{R}^3$ of dimension 2.

- **b.** To find the critical values of p, we want to find the points $q \in T^2$ for which the differential

$$p_{*,q} : T_qT^2 \to T_{p(f)}\mathbb{R}^2 \cong \mathbb{R}^2$$

fails to be surjective. Since p_* is a linear map between 2-dimensional vector spaces, this happens precisely when $\dim \operatorname{Ker}(p_*) > 0$.
Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote projection onto the xy-plane, so that $p = \pi|_{T^2}$. Note that since π is a linear map, $\pi_* = \pi$. For $q \in T^2$, we have

$$\operatorname{Ker} p_{*,q} = \operatorname{Ker}\left(\pi|_{T^2}\right)_{*,q} = \operatorname{Ker}\left(\pi_{*,q}\right)|_{T_qT^2} = \operatorname{Ker}(\pi)|_{T_qT^2} = \operatorname{Ker}\pi \cap T_qT^2$$

Recall that since T^2 is a regular level set of f, then for any $q \in T^2$, we have $T_qT^2 = \operatorname{Ker} f_{*,q}$, and so the above becomes

$$\operatorname{Ker} p_{*,q} = \operatorname{Ker}\pi \cap \operatorname{Ker} f_{*,q}.$$

We have

$$\operatorname{Ker}\pi = \{(a,b,c) \in \mathbb{R}^3 : (a,b) = (0,0)\},$$

and letting $q = (x_0, y_0, z_0)$, we have

$$\begin{aligned}\operatorname{Ker} f_{*,q} &= \left\{ (a,b,c) \in T_qT^2 : f_{*,q}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \right\} \\ &= \left\{ (a,b,c) \in T_qH : \frac{(2ax_0 + 2by_0)((x_0^2+y_0^2)^{1/2}-2)}{(x_0^2+y_0^2)^{1/2}} + 2cz_0 = 0 \right\}.\end{aligned}$$

Hence

$$\operatorname{Ker} p_{*,q} = \left\{(a,b,c) \in T_qT^2 : 2cz_0 = 0\right\}$$

For an arbitrary q, there certainly exists $(a,b,c) \in T_qH$ with $c \neq 0$, and so q is a critical point of p if and only if $z_0 = 0$. Therefore, the critical points of p are

$$\begin{aligned} C &= \{(x_0,y_0,z_0) \in T^2 : z_0 = 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : ((x^2+y^2)^{1/2}-2)^2 = 1\} \\ &= \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1 \text{ or } x^2+y^2 = 9\right\} \end{aligned}$$

and so the critical values of p are

$$p(C) = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1 \text{ or } x^2+y^2 = 9\right\}.$$

Graphically, $p(C)$ is two circles in the the xy-plane; one of radius 1, and the other of radius 3. The regular values of p are therefore

$$\begin{aligned} R_v &= \{(x,z) \in \mathbb{R}^2 : ((x^2+y^2)^{1/2}-2)^2 \neq 1\} \\ &\overset{*}{=} \{(x,z) \in \mathbb{R}^2 : ((x^2+y^2)^{1/2}-2)^2 < 1\} \\ &= \{(x,z) \in \mathbb{R}^2 : 1 < x^2+y^2 < 9\} \end{aligned}$$

where $*$ follows from the fact that no points of T^2 are such that $((x^2+y^2)^{1/2} - 2)^2 > 1$. For then, $1 = (x^2+y^2)^{1/2}-2)^2 + z^2 > 1 + z^2$, a contradiction for any $z \in \mathbb{R}$. Graphically, R_v is an open annulus in the xy-plane between the circles of radii 1 and 3.

□

4. **Iowa Qual, Spring 2008**

Let $A = \{(x,y,z) \in \mathbb{R}^3 : z = 2x+3y\}$ and let $B =$ the z-axis. Prove A and B meet transversally.

Solution 1:

Since A and B are submanifolds of $\mathbb{R}^3$, we need to show that for all $p \in A \cap B$, we have

$$T_pA + T_pB = T_p\mathbb{R}^3 \cong \mathbb{R}^3.$$

Now,

$$\begin{aligned} A \cap B = \{(x,y,z) \in \mathbb{R}^3 : z = 2x+3y, x = y = 0\} &= \{(x,y,z) \in \mathbb{R}^3 : z = x = y = 0\} \\ &= \{(0,0,0)\}. \end{aligned}$$

and so we let $p = (0,0,0)$. We need to describe elements in T_pA and T_pB. Define smooth maps $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto 2x+3y-z$, $g : \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$. Then $A = f^{-1}(\{0\})$ and $B = g^{-1}(\{(0,0)\})$, and

$$f_{*,(x,y,z)} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}, \; g_{*,(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So f and g have no critical values, giving that A and B are regular submanifolds of $\mathbb{R}^3$ of dimensions 2 and 1, respectively. So we have

$$T_pA = \mathrm{Ker}(f_{*,p}) = \{(x,y,z) \in \mathbb{R}^3 : 2x+3y = z\}$$

and

$$T_pB = \mathrm{Ker}(g_{*,p}) = \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\}.$$

Now evidently $T_pA + T_pB \subseteq \mathbb{R}^3$ and conversely if $(x_0,y_0,z_0) \in \mathbb{R}^3$, then pick $(x_0/2 - (3/2)y_0, y_0, 0) \in T_pA$ and $(0,0,z_0) \in T_pB$ so that

$$(x_0,y_0,z_0) = (x_0/2 - (3/2)y_0, y_0, 0) + (0,0,z_0) \in T_pA + T_pB.$$

Solution 2: We exhibit two vectors in T_pA and one vector in T_pB which are linearly independent for $p = (0,0,0)$.

5. **Iowa Qual, Fall 2015**

Let $\Delta = \{(\vec{v}, \vec{w}) \in \mathbb{R}^3 \times \mathbb{R}^3 | \vec{v} = \vec{w}\}, S^2 = \{\vec{x} \in \mathbb{R}^3 | ||\vec{x}|| = 1\}$ and $C = \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta$. Defined the map $g : C \to S^2$ by

$$g(\vec{v}, \vec{w}) = \frac{\vec{v} - \vec{w}}{||\vec{v} - \vec{w}||},$$

where $||\vec{v} - \vec{w}||$ means the norm of the difference. (This problem is computational.)

- **a.** Prove that g is a smooth mapping.
- **b.** Let $N = g^{-1}(0, 0, 1)$. Prove that N is a regular submanifold of C.

6. **Introduction to Smooth Manifolds, Spring 2017, Final Review**

Let $H = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 1\}$.

- **a.** Prove that H is a regular submanifold of $\mathbb{R}^3$ and describe its tangent space at each point.
- **b.** Let $r > 0$. Let $S_r = \{(x, y, z) | x^2 + y^2 + z^2 = r^2\}$. Prove that S_r is a regular submanifold of $\mathbb{R}^3$ and identify its tangent space at each point.
- **c.** For which r is S_r transverse to H?

Proof.

- **a.** Define a smooth map $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2 + y^2 - z^2 - 1$. Then $H = f^{-1}(\{0\})$, ans so we show that 0 is a regular value of f and apply the Regular Level Set Theorem to conclude that H is a regular submanifold of $\mathbb{R}^3$ of dimension 2. For $(x, y, z) \in H$, we have

$$f_{*(x,y,z)} = \begin{pmatrix} 2x & 2y & -2z \end{pmatrix},$$

and so $f_{*,(x,y,z)}$ fails to be surjective if and only if $(x, y, z) = (0, 0, 0)$. Since $(0, 0, 0) \notin H$, then every point in H is a regular point of f, and so 0 is a regular value of f.

- **b.** Just as above, 0 is a regular value of a map $g(x, y, z) = z^2 + y^2 + z^2 - r^2$, since

$$g_{*(x,y,z)} = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$$

fails to be surjective if and only if $(x, y, z) = (0, 0, 0)$ and since $(0, 0, 0) \notin g^{-1}(\{0\}) = S_r$. So S_r is a regular submanifold of $\mathbb{R}^3$ of dimension 2. The tangent space at any point in $(x, y, z) \in S_r$ can be identified with the kernel of $g_{*,(x,y,z)}$; that is

$$\operatorname{Ker} g_{*(x,y,z)} = \{(a, b, c) \in \mathbb{R}^3 : 2ax + 2by - 2cz = 0\}.$$

- **c.** A normal vector to H is $\begin{pmatrix} 2x & 2y & -2z \end{pmatrix}$, and a normal vector to S_r is $\begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$. So H and S_r will fail to be transverse precisely when these normal vectors are parallel, which happens if and only if $z = 0$. In this case, $x^2 + y^2 = 1$ on H and so $1 = x^2 + y^2 = r^2$ on S_r. Hence H and S_r are transverse if and only if $r \neq 1$.

□

2.3 Lie Groups

1. **Iowa Qual, Fall 2005**

   - **a.** Define *Lie group*.
   - **b.** Let $SL_n\mathbb{R}$ denote $n \times n$ matrices with real entries of determinant 1. Prove that $SL_n\mathbb{R}$ is a Lie group.
   - **c.** What is the tangent space of $SL_n\mathbb{R}$ at the identity?

   Solution:

   - **a.** A *Lie group* is a manifold G which is also a group so that the multiplication and inversion maps are smooth.
   - **b.** $\underline{SL_n\mathbb{R}\text{ is a group:}}$ Since the identity matrix has determinant 1, $SL_n\mathbb{R} \neq \varnothing$. If $A, B \in SL_n\mathbb{R}$, then $\det(AB^{-1}) = \det(A)\det(B^{-1}) = 1$ and so $SL_n\mathbb{R} \leq GL_n\mathbb{R}$.
     $\underline{SL_n\mathbb{R}\text{ is a manifold:}}$ Since $GL_n\mathbb{R} = \det^{-1}((-\infty, 0) \cup (0, \infty))$, then $GL_n\mathbb{R}$ is an open subset of $\mathbb{R}^{n^2}$; hence $GL_n\mathbb{R}$ inherits a manifold structure from $\mathbb{R}^{n^2}$. Let $f : GL_n\mathbb{R} \to \mathbb{R}$ be the determinant function. Evidently, $f^{-1}(\{1\}) = SL_n\mathbb{R}$, and so we show that 1 is a regular value of f, giving that $SL_n\mathbb{R}$ is a regular submanifold of $GL_n\mathbb{R}$ of dimension $n^2 - 1$ by the Regular Level Set Theorem.
     Let $A = (a_{ij}) \in SL_n\mathbb{R}$. Denote by $m_{i,j}$ the determinant of the matrix obtained from A by deleting its ith row and jth column. Then, expanding along the ith row of A, we have $f(A) = \det(A) = (-1)^{i+1}a_{i1}m_{i1} + \cdots + (-1)^{i+n}a_{in}m_{in}$, and so
     $$f_{*,A} = \left(\frac{\partial f}{\partial a_{i1}} \cdots \frac{\partial f}{\partial a_{in}}\right) = \left((-1)^{i+1}m_{i1} \cdots (-1)^{i+n}m_{in}\right).$$
     Hence $f_{*,A}$ fails to be surjective precisely when all m_{ij} are zero, but this occurs if and only if $\det(A) = 0$. Since $A \in SL_n\mathbb{R}$, $\det(A) = 1$, and so $f_{*,A}$ is surjective.
     $\underline{SL_n\mathbb{R}\text{ has smooth multiplication and inversion maps:}}$ Since matrix multiplication in $GL_n\mathbb{R}$ are polynomials in the coordinates of $\mathbb{R}^{n^2}$, it is smooth. The inverse of a matrix $A = (a_{ij}) \in GL_n\mathbb{R}$ has (i, j)-entry
     $$(A^{-1})_{(i,j)} = \frac{1}{\det A}(-1)^{i+j}((i, j)\text{-minor of } A),$$
     which is smooth in a_{ij} if $\det A \neq 0$. Hence inversion in $GL_n\mathbb{R}$ is smooth. Since $SL_n\mathbb{R}$ is a regular submanifold of $GL_n\mathbb{R}$, the inclusion map $\iota : SL_n\mathbb{R} \hookrightarrow GL_n\mathbb{R}$ is smooth. Since restriction of the multiplication and inversion maps of $GL_n\mathbb{R}$ to $SL_n\mathbb{R}$ is composition of said maps with ι, we get that multiplication and inversion in $SL_n\mathbb{R}$ is smooth.
   - **c.** Since $SL_n\mathbb{R}$ is a regular level set of the determinant function, then $T_I SL_n\mathbb{R} = \operatorname{Ker} \det_{*,I}$. Pick $X \in T_I SL_n\mathbb{R}$, and define a curve $c : (-\epsilon, \epsilon) \to SL_n\mathbb{R}$ by $t \mapsto e^{tX}$. Then $c(0) = I$ and $c'(0) = X$, and we get
     $$0 = \det{}_{*,I}(X) = \frac{d}{dt}\bigg|_{t=0} \det(e^{tX}) = \frac{d}{dt}\bigg|_{t=0} e^{t \operatorname{Tr} X} = \operatorname{Tr} X e^{t \operatorname{Tr} X}\bigg|_{t=0} = \operatorname{Tr} X.$$
     Hence $T_I SL_n\mathbb{R}$ is contained in the vector space V which consists of all those matrices with trace 0. But V has dimension $n^2 - 1$, as does $T_I SL_n\mathbb{R}$; hence $T_I SL_n\mathbb{R} = V$.

2. **Iowa Qual, Fall 2007**

Let $O(n)$ denote the set of $n \times n$ matrices with real entries, A so that $AA^T = Id$. Prove that $O(n)$ is a Lie group, and identify the tangent space at the identity.

Proof. $O(n)$ is a group: $O(n) \neq \varnothing$ since $I \in O(n)$, and if $A, B \in O(n)$, then

$$(AB^{-1})(AB^{-1})^T = AB^{-1}(B^T)^{-1}(A^T)^{-1} = A(B^TB)^{-1}(A^T)^{-1} = (A^TA)^{-1} = I.$$

and so $O(n) \leq GL_n\mathbb{R}$.

$O(n)$ is a manifold: Let $f : GL_n\mathbb{R} \to GL_n\mathbb{R}$ be given by $A \mapsto AA^T$. Notice that $(AA^T)^T = AA^T$, and so in fact $f(GL_n\mathbb{R}) = S_n = \{n \times n \text{ symmetric matrices}\}$. Note that a matrix in S_n is determined by its values on the diagonal and upper triangle, and so $\dim S_n = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$.

Obviously, $O(n) = f^{-1}(I)$, and so we show that I is a regular value of f, which gives that $O(n)$ is a regular submanifold of $GL_n\mathbb{R}$ of dimension $\dim GL_n\mathbb{R} - \dim S_n = n^2 - \frac{n^2+n}{2} = \frac{n^2-n}{2}$ by the Regular Level Set Theorem.

To that end, pick $A \in f^{-1}(I) = O(n)$ and $X \in T_AGL_n\mathbb{R} \cong GL_n\mathbb{R}$. Then we can find a curve $c : (\epsilon, \epsilon) \to GL_n\mathbb{R}$ starting at A with initial velocity X. Then

$$f_{*,A}(X) = \frac{d}{dt}\bigg|_{t=0} f(c(t)) = \frac{d}{dt}\bigg|_{t=0} c(t)c(t)^T = c(t)c'(t)^T + c'(t)c(t)^T\Big|_{t=0} = AX^T + XA^T.$$

We argue why $f_{*,A}$ is surjective. If $B \in S_n$, pick $X = \frac{1}{2}BA$, and then the above computation gives

$$f_{*,A}(X) = A\left(\frac{1}{2}BA\right)^T + \left(\frac{1}{2}BA\right)A^T = \frac{1}{2}AA^TB^T + \frac{1}{2}BAA^T = \frac{1}{2}(B^T + B) = B,$$

and so $f_{*,A}$ is surjective.

$O(n)$ has smooth multiplication and inversion maps: Since $O(n)$ is a regular submanifold of $GL_n\mathbb{R}$, the inclusion map $\iota : O(n)\mathbb{R} \hookrightarrow GL_n\mathbb{R}$ is smooth. Since restriction of the multiplication and inversion maps of $GL_n\mathbb{R}$ to $O(n)$ is composition of said maps with ι, we get that multiplication and inversion in $O(n)$ is smooth.

Since $O(n)$ is a regular level set of f, then $T_IO(n) = \operatorname{Ker} f_{*,I}$. The computation earlier gives that if $X \in T_IO(n)$,

$$0 = f_{*,I}(X) = X^T + X,$$

and so $T_IO(n)$ is contained in the vector space V of skew symmetric matrices. But every element in V is determined by its values on the upper triangle, and so $\dim V = (n-1) + (n-2) + \ldots 1 = \frac{(n-1)n}{2} = \frac{n^2-n}{2}$, and since also $\dim T_IO(n) = \dim O(n) = \frac{n^2-n}{2}$, then in fact $T_IO(n) = V$. $\square$

3. **Iowa Qual, Fall 2014 (Balz)**

Let

$$B = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in M_{3,3}(\mathbb{R}) | adf \neq 0 \right\}$$

be the set of invertible upper triangular matrices.

- **a.** Prove that B equipped with matrix multiplication is a Lie group.
- **b.** Compute the tangent space to B at the identity as a linear subspace of $M_{3,3}(\mathbb{R}) = \mathbb{R}^9$.

4. **Iowa Qual, Fall 2015 (Sanadhya)**

- **a.** Define $\mathfrak{g}$ is a **Lie algebra**.
- **b.** Recall the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, and prove that $\mathbb{R}^3$ equipped with the cross product is a Lie algebra.

Solution:

- **a.** A *Lie algebra* $\mathfrak{g}$ is a vector space together with a product $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that
  - is bilinear: $[aX+bY, Z] = a[X,Z]+b[X,Z]$, $[X, aY+bZ] = a[X,Z]+b[X,Y]$.
  - is anticommutative: $[X,Y] = -[Y,X]$.
  - satisfies the Jacobi Identity: $0 = [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]]$.
- **b.** $\mathbb{R}^3$ is a real vector space, and so we show that $\times$ is bilinear, anticommutative and satisfies the Jacobi Identity. The cross product is given by

$$[X,Y] = \times(X,Y) = \times\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

So

$$\begin{aligned} [aX+bY, Z] &= \begin{pmatrix} (ax_2+by_2)z_3 - (ax_3+by_3)z_2 \\ (ax_3+by_3)z_1 - (ax_1+by_1)z_3 \\ (ax_1+by_1)z_2 - (ax_2+by_2)z_1 \end{pmatrix} \\ &= \begin{pmatrix} ax_2z_3 + by_2z_3 - ax_3z_2 + by_3z_2 \\ ax_3z_1 + by_3z_1 - ax_1z_3 + by_1z_3 \\ ax_1z_2 + by_1z_2 - ax_2z_1 + by_2z_1 \end{pmatrix} \\ &= a\begin{pmatrix} x_2z_3 - x_3z_2 \\ x_3z_1 - x_1z_3 \\ x_1z_2 - x_2z_1 \end{pmatrix} + b\begin{pmatrix} y_2z_3 - y_3z_2 \\ y_3z_1 - y_1z_3 \\ y_1z_2 - y_2z_1 \end{pmatrix} \\ &= a[X,Z] + b[Y,Z], \end{aligned}$$

and similarly $[X, aY+bZ] = a[X,Z]+b[X,Y]$. Hence the cross product is bilinear. Now,

$$[X,Y] = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = -\left(-\begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}\right) = -\begin{pmatrix} x_3y_2 - x_2y_3 \\ x_1y_3 - x_3y_1 \\ x_2y_1 - x_1y_2 \end{pmatrix} = -[Y,X],$$

and hence the cross product is anticommutative. Finally,

$$\begin{aligned}
[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]] &= \left[X, \begin{pmatrix} y_2z_3-y_3z_2 \\ y_3z_1-y_1z_3 \\ y_1z_2-y_2z_1 \end{pmatrix}\right] + \left[Z, \begin{pmatrix} x_2y_3-x_3y_2 \\ x_3y_1-x_1y_3 \\ x_1y_2-x_2y_1 \end{pmatrix}\right] \\
&\quad + \left[Y, \begin{pmatrix} z_2x_3-z_3x_2 \\ z_3x_1-z_1x_3 \\ z_1x_2-z_2x_1 \end{pmatrix}\right] \\
&= \begin{pmatrix} x_2(y_1z_2-y_2z_1)-x_3(y_3z_1-y_1z_3) \\ x_3(y_2z_3-y_3z_2)-x_1(y_1z_2-y_2z_1) \\ x_1(y_3z_1-y_1z_3)-x_2(y_2z_3-y_3z_2) \end{pmatrix} \\
&\quad + \begin{pmatrix} z_2(x_1y_2-x_2y_1)-z_3(x_3y_1-x_1y_3) \\ z_3(x_2y_3-x_3y_2)-z_1(x_1y_2-x_2y_1) \\ z_1(x_3y_1-x_1y_3)-z_2(x_2y_3-x_3y_2) \end{pmatrix} \\
&\quad + \begin{pmatrix} y_2(z_1x_2-z_2x_1)-y_3(z_3x_1-z_1x_3) \\ y_3(z_2x_3-z_3x_2)-y_1(z_1x_2-z_2x_1) \\ y_1(z_3x_1-z_1x_3)-y_2(z_2x_3-z_3x_2) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

and hence the cross product satisfies the Jacobi Identity.

5. **Introduction to Smooth Manifolds, Spring 2017, Final Review**

   Let
   $$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

   Let $O(3,1)$ be the set of all four by four matrices A with real entries so that $ABA^t = B$. Prove that $O(3,1)$ is a Lie group. Calculate its tangent space at the identity and determine its dimension as a manifold.

2.4 Vector Fields

1. **Iowa Qual, Fall 2006**

   Define the notion of a smooth action of a Lie group G on a smooth manifold M. Give an example of S^1 acting smoothly on S^2. Prove that in general an action of S^1 yields a flow on M and therefore a vector field on M. Must this vector field be never zero?

2. **Iowa Qual, Fall 2007**

   Let M be a smooth manifold.

   - **a.** Given $P \in M$ define T_PM.
   - **b.** Define smooth vector field on M.

- **c.** If X and Y are smooth vector fields on M define $[X, Y]$.
- **d.** Compute $[X, Y]$ where $X = x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ are vector fields on the plane.

Solution:

- **a.** For $p \in M$, we have

$$T_pM = \left\{ D : C^\infty(M) \to \mathbb{R} : \begin{array}{l} D \text{ is } \mathbb{R}\text{-linear and } D \text{ satisfies the Leibniz rule:} \\ (D(fg))(p) = (Df)g(p) + f(p)(Dg) \end{array} \right\}.$$

- **b.** A *smooth vector field* on M is a smooth section X of the tangent bundle, $\pi : TM \to M$. In other words, X is a map $X : M \to TM$ such that $\pi \circ X$ is smooth. In local coordinates, if $(U, \varphi) = (U, x^1, \ldots, x^n)$ is a chart of M containing p, then $X(p) = X_p = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x^i}\Big|_p$, and we say that X is a smooth vector field if for every p and every chart about p, the coordinate functions a_i are smooth on U.
- **c.** Let $f \in C^\infty(M)$. The *Lie Bracket*, $[X, Y]$ of X and Y is a smooth vector field on M given by

$$[X, Y]f = (XY - YX)f = X(Yf) - Y(Xf).$$

- **d.**

$$\begin{aligned}
[X, Y] &= \left(x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) \\
&= \left\{\left(x\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right) + \left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right)\right\} \\
&\quad - \left\{\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\right) + \left(\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\right)\right\} \\
&= \left\{x\left(\frac{\partial^2}{\partial y\partial x} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\partial y}\right)\right\} \\
&\quad - \left\{\left(\left(\frac{\partial x}{\partial x}\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y}\right) + \frac{\partial^2}{\partial x^2}\right) + \left(\left(\frac{\partial x}{\partial y}\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial y\partial x}\right)\right\} \\
&= \left\{x\frac{\partial^2}{\partial y\partial x} + x\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\partial y}\right\} \\
&\quad - \left\{\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y\partial x}\right\} \\
&= -\frac{\partial}{\partial y}.
\end{aligned}$$

3. **Iowa Qual, Fall 2014**

   Suppose that $N \subset M$ is a regular submanifold of the smooth manifold M.

   - (a) Let X be a smooth vector field on M, so that for every $p \in N$, $X_p \in T_pN$. Show that $X|_N$ is a smooth vector field on N.

Proof. Let $\iota : N^n \hookrightarrow M^m$ be inclusion. Since N is a regular submanifold, ι is smooth: Let $p \in N$ and $(U \cap N, \varphi|N) = (U, x^1, \dots, x^n, 0, \dots, 0)$ be an adapted chart about p. Then

$$\varphi \circ \iota \circ \varphi^{-1} : \varphi(U \cap N) \to \varphi(U \cap N)$$

is smooth at $\varphi(p)$ since

$$(\varphi \circ \iota \circ \varphi^{-1})(\varphi(p)) = \varphi(\iota(\varphi^{-1}(\varphi(p))) = \varphi(\iota(p)) = \varphi(p),$$

and φ is smooth at p. Therefore, $X|_N = X \circ \iota : N \to TM$ is a smooth map since both X and ι are smooth. Moreover, if $p \in N$, then

$$(X|_N)(p) = (X \circ \iota)(p) = X(p) = X_p \in T_pN$$

and so in fact $X|_N : N \to TN$. Now, to show that $X|_N$ is a smooth *vector field*, we show that it is a section of the tangent bundle, $\pi : TM \to M$

$$\pi \circ X|_N = \pi \circ X \circ \iota = \mathbb{1}_M \circ \iota = \mathbb{1}_N.$$

□

- (b) Suppose that $\tilde{f} : M \to \mathbb{R}$ is smooth and $f : N \to \mathbb{R}$ is the restriction of $\tilde{f}$ to N. Prove or disprove,

$$X|_N(f) = X(\tilde{f}).$$

  [I'm pretty sure this last equation should read "$X|_N(f) = X(\tilde{f})|_N$", for otherwise the equals sign is meaningless. I've transcribed it as it was written on the qual. JH]

Proof. First note that $f = \tilde{f} \circ \iota$ □

2.5 Vector Bundles

1. **Iowa Qual, Fall 2006 (Balz)**

   Suppose M is a Lie group. Sketch the proof that M is parallelizable.

2. **Iowa Qual, Fall 2016 (Sanadhya)**

   - (a) Define the notion of a vector bundle $E \to X$ over X.
   - (b) Assume that X is a smooth compact manifold. Prove that every vector bundle $E \to X$ is homotopic to $X : E \simeq X$.

3. **Iowa Qual, Winter 2017 (Singh)**

   Recall that *cocycle data* is a collection of maps $\{g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R})\}_{(i,j) \in \Lambda \times \Lambda}$ from the pairwise intersections $U_i \cap U_j$ of an open cover $\{U_i\}_i \in \Lambda$ of M which satisfy:

$$g_{ij}g_{ji} = 1_{U_i \cap U_j} \qquad \text{and} \qquad g_{ij}g_{jk}g_{ki} = 1_{U_i \cap U_j \cap U_k}.$$

   - (a) Prove that a real vector bundle $E \to M$ determines a choice of cocycle data.
   - (b) Prove that a choice of cocycle data allows one to construct a real vector bundle $E \to M$.

2.6 Differential Forms

1. **Iowa Qual, Spring 2008**

   Consider the following 1-form on $\mathbb{R}^3$.

$$\omega = xy\ dx + x\ dy + x\ dz$$

- **(i)** Calculate the 2-form $d\omega$.
- **(ii)** Show by explicit calculation that $d(d\omega) = 0$.
- **(iii)** Does there exist a 1-form α on $\mathbb{R}^3$ such that $dd\alpha = (x^5y^3z^9)dx \wedge dy \wedge dz$? (Do not try to find such α; just state in one or two sentences why such a 1-form does or does not exist.)

Solution:

- **(i)**

$$\begin{aligned} d\omega &= d(xy) \wedge dx + dx \wedge dy + dx \wedge dz \\ &= \left(\frac{\partial(xy)}{\partial x} dx + \frac{\partial(xy)}{\partial y} dy \right) \wedge dx + dx \wedge dy + dx \wedge dz \\ &= (ydx + xdy) \wedge dx + dx \wedge dy + dx \wedge dz \\ &= ydx \wedge dx + xdy \wedge dx + dx \wedge dy + dx \wedge dz \\ &= xdy \wedge dx + dx \wedge dy + dx \wedge dz \\ &= -xdx \wedge dy + dx \wedge dy + dx \wedge dz \\ &= (1-x)dx \wedge dy + dx \wedge dz \end{aligned}$$

- **(ii)**

$$\begin{aligned} d(d\omega) &= d((1-x)dx \wedge dy + dx \wedge dz) \\ &= d(1-x) \wedge dx \wedge dy + d(1) \wedge dx \wedge dz \\ &= \left(\frac{\partial(1-x)}{\partial x} dx \right) \wedge dx \wedge dy + 0 \wedge dx \wedge dz \\ &= -xdx \wedge dx \wedge dy \\ &= 0. \end{aligned}$$

- **(iii)** Such a 1 form does not exist. The exterior derivative d has square zero: $d^2 \equiv 0$. To see this, let $\omega = f dx^I$ be a k form. Then

$$d(d\omega) = d\left(\sum_I \sum_j \frac{\partial f}{\partial x^j} dx^j \wedge dx^I \right) = \sum_I \sum_i \sum_j \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j \wedge dx^I.$$

  If $i = j$ in the last sum, then $dx^i \wedge dx^j = 0$, getting rid of that term. If $i \neq j$, then for each term

$$\frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i,$$

we have a corresponding term

$$\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j.$$

Since mixed partials are the same (f is assumed to be smooth), and $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we get

$$\frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = -\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j.$$

Hence all the terms in $d(d\omega)$ are zero. Since d is $\mathbb{R} linear$, we can extend this to the case when $\omega = \sum f_I dx^I$.

2. **Iowa Qual, Fall 2013**

Let (x, y, z, w) be the standard coordinates on the Euclidean space $\mathbb{R}^4$. Let $X = \frac{\partial}{\partial z}$ be a vector field. Let $\omega = (xyz)dz \wedge dx$ be a 2-form. Compute the Lie derivative $\mathcal{L}_X \omega$.

Solution:

$$\begin{aligned}
\mathcal{L}_X \omega &= [\mathcal{L}_X((xyz)dz) \wedge dx] + (xyz)dz \wedge \mathcal{L}_X(dx) \\
&= [((\mathcal{L}_X xyz)\, dz + xyz\mathcal{L}_X dz) \wedge dx] + (xyz)dz \wedge \mathcal{L}_X(dx) \\
&= [(\mathcal{L}_X(xyz)dz + xyz(d\mathcal{L}_X z)) \wedge dx] + (xyz)dz \wedge (d\mathcal{L}_X x) \\
&= [((Xxyz)dz + xyz(dXz)) \wedge dx] + (xyz)dz \wedge (dXx) \\
&= [((xy)dz + xyz(d(1))) \wedge dx] + (xyz)dz \wedge (d(0)) \\
&= (xy)dz \wedge dx
\end{aligned}$$

3. **Iowa Qual, Fall 2013**

Let M be an n-manifold. Let $\alpha \in \Omega^1(M)$. Is $\alpha \wedge \alpha = 0$? Give reasons.

Solution:

Indeed, $\alpha \wedge \alpha = 0$. This follows from the fact that the wedge product is anticommutative; that is, if $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$, then $\omega \wedge \tau = (-1)^{k\ell} \tau \wedge \omega$. Here's a proof: Let $\rho \in S_{k+\ell}$ be given by

$$\rho(i) = \begin{cases} k+i & \text{if } 1 \le i \le k \\ i-\ell & \text{if } k+1 \le i \le k+\ell \end{cases}$$

Then

$$
\begin{aligned}
(\omega \wedge \tau)(v_1, \ldots, v_{k+\ell}) &= \sum_{\substack{(k,\ell)\text{-shuffles} \\ \sigma \in S_{k+\ell}}} \operatorname{sgn} \sigma \ \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&= \sum_{\substack{(k,\ell)\text{-shuffles} \\ \sigma \in S_{k+\ell}}} \operatorname{sgn} \sigma \ \omega(v_{\sigma\rho(k+1)}, \ldots, v_{\sigma\rho(k+\ell)}) \cdot \tau(v_{\sigma\rho(1)}, \ldots, v_{\sigma\rho(k)}) \\
&= \sum_{\substack{(k,\ell)\text{-shuffles} \\ \sigma \in S_{k+\ell}}} \operatorname{sgn} \sigma (\operatorname{sgn} \rho)^2 \ \tau(v_{\sigma\rho(1)}, \ldots, v_{\sigma\rho(k)}) \cdot \omega(v_{\sigma\rho(k+1)}, \ldots, v_{\sigma\rho(k+\ell)}) \\
&= \operatorname{sgn} \rho \sum_{\substack{(k,\ell)\text{-shuffles} \\ \mu \in S_{k+\ell}}} \operatorname{sgn} \mu \ \tau(v_{\mu(1)}, \ldots, v_{\mu(k)}) \cdot \omega(v_{\mu(k+1)}, \ldots, v_{\mu(k+\ell)}) \\
&= (-1)^{k\ell} (\tau \wedge \omega)(v_1, \ldots, v_{k+\ell})
\end{aligned}
$$

4. **Iowa Qual, Winter 2017**

Suppose that (x_1, y_1, x_2, y_2) are standard coordinates in $\mathbb{R}^4$. Let $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \in \Omega^2(\mathbb{R}^4)$ be the standard symplectic form. Let $X = x_1 \partial / \partial x_1$.

- **a.** Compute $\mathcal{L}_X(\omega)$.
- **b.** Prove that ω determines a map

$$\alpha_p : T_p(\mathbb{R}^4) \to T_p^*(\mathbb{R}^4)$$

from the tangent space to the cotangent space at $p \in \mathbb{R}^4$.
- **c.** Compute the covector $\alpha_p(X_p)$ for each $p \in \mathbb{R}^4$.

Proof.

- **a.**

$$
\begin{aligned}
\mathcal{L}_X \omega &= \mathcal{L}_X(dx_1) \wedge dy_1 + dx_1 \wedge \mathcal{L}_X dy_1 + \mathcal{L}_X(dx_2) \wedge dy_2 + dx_2 \wedge \mathcal{L}_X dy_2 \\
&= d\mathcal{L}_X(x_1) \wedge dy_1 + dx_1 \wedge d\mathcal{L}_X y_1 + d\mathcal{L}_X(x_2) \wedge dy_2 + dx_2 \wedge d\mathcal{L}_X y_2 \\
&= d(Xx_1) \wedge dy_1 + dx_1 \wedge d(Xy_1) + d(Xx_2) \wedge dy_2 + dx_2 \wedge d(Xy_2) \\
&= d\left(x_1 \frac{\partial x_1}{\partial x_1}\right) \wedge dy_1 + dx_1 \wedge d\left(x_1 \frac{\partial y_1}{\partial x_1}\right) \\
&\qquad + d\left(x_1 \frac{\partial x_2}{\partial x_1}\right) \wedge dy_2 + dx_2 \wedge d\left(x_1 \frac{\partial y_2}{\partial x_1}\right) \\
&= dx_1 \wedge dy_1 + dx_1 \wedge d(0) + d(0) \wedge dy_2 + dx_2 \wedge d(0) \\
&= dx_1 \wedge dy_1
\end{aligned}
$$

- **b.** Consider the map $Z_p \mapsto \iota_{Z_p} \omega_p$, where ι is interior multiplication. Then for $Y_p \in T_p(\mathbb{R}^4)$,

$$
\begin{aligned}
(\alpha_p(Z_p))(Y_p) &= (\iota_{Z_p} \omega_p)(Y_p) \\
&= \omega_p(Z_p, Y_p) \\
&= dx_1 \wedge dy_1(Z_p, Y_p) + dx_2 \wedge dy_2(Z_p, Y_p) \\
&= dx_1(Z_p) dy_1(Y_p) - dx_1(Y_p) dy_1(Z_p) \\
&\qquad + dx_2(Z_p) dy_2(Y_p) - dx_2(Y_p) dy_2(Z_p).
\end{aligned}
$$

- **c.** By **b.**, we have

$$\begin{aligned}(\alpha_p(X_p))(Y_p) &= dx_1(X_p)dy_1(Y_p) - dx_1(Y_p)dy_1(X_p) \\ &\qquad + dx_2(X_p)dy_2(Y_p) - dx_2(Y_p)dy_2(X_p) \\ &= (X_p x_1)(Y_p y_1) - (Y_p x_1)(X_p y_1) + (X_p x_2)(Y_p y_2) - (Y_p x_2)(X_p y_2) \\ &= \left(x_1 \frac{\partial x_1}{\partial x_1}\right)(Y_p y_1) - (Y_p x_1)\left(x_1 \frac{\partial y_1}{\partial x_1}\right) \\ &\qquad + \left(x_1 \frac{\partial x_2}{\partial x_1}\right)(Y_p y_2) - (Y_p x_2)\left(x_1 \frac{\partial y_2}{\partial x_1}\right) \\ &= (x_1)(Y_p y_1).\end{aligned}$$

□

5. **Introduction to Smooth Manifolds, Spring 2017, Final Review (Burke)**

Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, let $Y = xy\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$. Let $\alpha = dz - y\ dx$, $\beta = x\ dx \wedge dy + y\ dx \wedge dz + z\ dx \wedge dy$. Let $f(x,y,z) = xyz$. Compute $\mathcal{L}_X f, \mathcal{L}_X \alpha, \mathcal{L}_X \beta, \mathcal{L}_X Y$. What is $\alpha \wedge d\alpha$? Write out the maximal flow underlying X.

Solution:

$$\mathcal{L}_X f = Xf = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) xyz = x\frac{\partial xyz}{\partial x} + y\frac{\partial xyz}{\partial y} + z\frac{\partial xyz}{\partial z} = 3xyz.$$

$$\begin{aligned}\mathcal{L}_X \alpha &= \mathcal{L}_X dz - \mathcal{L}_X y = d\mathcal{L}_X z - \mathcal{L}_X y = dXz - Xy = dz - y. \\ \mathcal{L}_X \beta &= \mathcal{L}_X(xdx \wedge dy) + \mathcal{L}_X(ydx \wedge dz) + \mathcal{L}_X(zdx \wedge dy) \\ &= \mathcal{L}_X(xdx) \wedge dy + xdx \wedge \mathcal{L}_X(dy) \\ &\qquad + \mathcal{L}_X(ydx) \wedge dz + ydx \wedge \mathcal{L}_X(dz) \\ &\qquad\qquad + \mathcal{L}_X(zdx) \wedge dy + zdx \wedge \mathcal{L}_X(dy) \\ &= (\mathcal{L}_X x \wedge dx + x\mathcal{L}_X(dx)) \wedge dy + xdx \wedge \mathcal{L}_X(dy) \\ &\qquad + (\mathcal{L}_X(y) \wedge dx + y\mathcal{L}_X(dx)) \wedge dz + ydx \wedge \mathcal{L}_X(dz) \\ &\qquad\qquad + (\mathcal{L}_X(z) \wedge dx + z\mathcal{L}_X(dx)) \wedge dy + zdx \wedge \mathcal{L}_X(dy) \\ &= (\mathcal{L}_X x \wedge dx + xd\mathcal{L}_X(x)) \wedge dy + xdx \wedge d\mathcal{L}_X(y) \\ &\qquad + (\mathcal{L}_X(y) \wedge dx + yd\mathcal{L}_X(x)) \wedge dz + ydx \wedge d\mathcal{L}_X(z) \\ &\qquad\qquad + (\mathcal{L}_X(z) \wedge dx + zd\mathcal{L}_X(x)) \wedge dy + zdx \wedge d\mathcal{L}_X(y) \\ &= (Xx \wedge dx + xd(Xx)) \wedge dy + xdx \wedge d(X(y)) \\ &\qquad + (Xy \wedge dx + yd(Xx)) \wedge dz + ydx \wedge d(Xz) \\ &\qquad\qquad + (Xz \wedge dx + zd(Xx)) \wedge dy + zdx \wedge d(Xy) \\ &= (xdx + xdx) \wedge dy + xdx \wedge dy \\ &\qquad + (ydx + ydx) \wedge dz + ydx \wedge dz \\ &\qquad\qquad + (zdx + zdx) \wedge dy + zdx \wedge dy \\ &= 3xdx \wedge dy + 3ydx \wedge dz + 3zdx \wedge dy \\ &= (3x + 3z)dx \wedge dy + 3ydx \wedge dz.\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_X Y = [X,Y] &= XY - YX \\
&= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(xy\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right) \\
&\quad - \left(xy\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) \\
&= x\left(\frac{\partial xy}{\partial x}\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial x^2} + \frac{\partial y}{\partial x}\frac{\partial}{\partial z} + y\frac{\partial^2}{\partial x \partial z}\right) \\
&\qquad + y\left(\frac{\partial xy}{\partial y}\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial y \partial x} + \frac{\partial y}{\partial y}\frac{\partial}{\partial z} + y\frac{\partial^2}{\partial y \partial z}\right) \\
&\qquad\quad + z\left(\frac{\partial xy}{\partial z}\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial z \partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial z} + y\frac{\partial^2}{\partial z^2}\right) \\
&\quad - xy\left(\frac{\partial x}{\partial x}\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2} + \frac{\partial y}{\partial x}\frac{\partial}{\partial y} + y\frac{\partial^2}{\partial x \partial y} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + z\frac{\partial^2}{\partial x \partial z}\right) \\
&\quad - y\left(\frac{\partial x}{\partial z}\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial z \partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} + y\frac{\partial^2}{\partial z \partial y} + \frac{\partial z}{\partial z}\frac{\partial}{\partial z} + z\frac{\partial^2}{\partial z \partial z}\right) \\
&= x\left(y\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x \partial z}\right) \\
&\qquad + y\left(x\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial y \partial x} + \frac{\partial}{\partial z} + y\frac{\partial^2}{\partial y \partial z}\right) \\
&\qquad\quad + z\left(xy\frac{\partial^2}{\partial z \partial x} + y\frac{\partial^2}{\partial z^2}\right) \\
&\quad - xy\left(\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x \partial y} + z\frac{\partial^2}{\partial x \partial z}\right) \\
&\quad - y\left(x\frac{\partial^2}{\partial z \partial x} + y\frac{\partial^2}{\partial z \partial y} + \frac{\partial}{\partial z} + z\frac{\partial^2}{\partial z \partial z}\right) \\
&= xy\frac{\partial}{\partial x}.
\end{aligned}
$$

2.7 Integration

1. **Iowa Qual, January 2011**

   Let $\omega = xy\, dx \wedge dz + 3x\, dy \wedge dz + xz\, dx \wedge dy$ be a 2-form on $\mathbb{R}^3$ (with the standard coordinates (x, y, z)).

   - **a.** Calculate $d\omega$.
   - **b.** Is ω exact? Justify your answer.
   - **c.** Calculate $\int_{S^2} i^*\omega$ by using Stokes' Theorem, where $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is given the standard (outward-pointing) orientation as the boundary of the unit ball, and $i : S^2 \to \mathbb{R}^3$ is the standard embedding.

   Solution:

- **a.**

$$\begin{aligned} d\omega &= d(xy) \wedge dx \wedge dz + d(3x) \wedge dy \wedge dz + d(xz) \wedge dx \wedge dy \\ &= (ydx + xdy) \wedge dx \wedge dz + 3dx \wedge dy \wedge dz + (zdx + xdz) \wedge dx \wedge dy \\ &= xdy \wedge dx \wedge dz + 3dx \wedge dy \wedge dz + xdz \wedge dx \wedge dy \\ &= 3dx \wedge dy \wedge dz. \end{aligned}$$

- **b.** No, ω is not exact. For if it were, and $d\tau = \omega$, then $d\omega = dd\tau = 0$ since $d^2 \equiv 0$, but $d\omega \neq 0$.
- **c.** Recall that $i^*\omega = \omega|_{S^2}$, and so by Stokes Theorem,

$$\int_{S^2} i^*\omega = \int_{S^2} \omega|_{S^2} = \int_{S^2} \omega|_{\partial B^3} = \int_{B^3} d\omega$$

We use the following parametrization of B^3:

$$F : [0,1] \times [0,\pi] \times [0,2\pi] \to \mathbb{R}^3, \; F(\rho,\varphi,\theta) = (\rho \sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

Let's check to see if F is an orientation preserving diffeomorphism:

$$\begin{aligned} \det(J(F)) &= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} \\ &= \cos\varphi(\rho^2\cos^2\theta\sin\varphi\cos\varphi + \rho^2\sin^2\theta\sin\varphi\cos\varphi) \\ &\quad + \rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta) \\ &= \rho^2\cos^2\varphi\sin\varphi(\cos^2\theta + \sin^2\theta) \\ &\quad + \rho^2\sin^3\varphi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2\sin\varphi(\cos^2\varphi + \sin^2\varphi) \\ &= \rho^2\sin\varphi \end{aligned}$$

Since $\det(J(F))$ is almost everywhere positive, we get that F is orientation preserving. So

$$\begin{aligned} \int_{B^3} d\omega = \int_0^{2\pi}\int_0^{\pi}\int_0^1 F^*d\omega &= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 F^*dx \wedge F^*dy \wedge F^*dz \\ &= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \det(J(F)) \; d\rho \wedge d\varphi \wedge d\theta \\ &= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\varphi \; d\rho d\varphi d\theta \\ &= \int_0^{2\pi} (-\cos(\pi) - \cos(0)) \; d\theta \\ &= 2\int_0^{2\pi} d\theta \\ &= 4\pi. \end{aligned}$$

2. **Iowa Qual, Fall 2014**

- **a.** Prove that
$$S = \{(x,y,z) \in \mathbb{R}^3 | x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1\}$$
is a regular submanifold of $\mathbb{R}^3$, and compute its tangent space as a subspace of the tangent space of $\mathbb{R}^3$ at the point $(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}})$.
- (b) Note that S is the boundary of the domain
$$R = \{(x,y,z) \in \mathbb{R}^3 | x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1\}.$$
Give R the standard orientation from $\mathbb{R}^3$ and give S the induced boundary orientation. Let $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$. Compute both $\int_S \omega$ and $\int_R d\omega$ and check that they are equal.

Solution:

- **a.**
- **b.** We can parametrize S by
$$F : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3,\ F(\varphi,\theta) = (\cos\theta \sin\varphi, 2\sin\theta\sin\varphi, 3\cos\varphi).$$
A normal vector to S is $f_* = \begin{pmatrix} 2x & y/2 & 2z/9 \end{pmatrix}$, and so a smooth outward pointing vector field on S is $X := f_*$. In terms of F,
$$X = \left(2\cos\theta\sin\varphi, \sin\theta\sin\varphi, \frac{2\cos\varphi}{3}\right)$$
So if $\tau = dx \wedge dy \wedge dz$ is the standard orientation form on $\mathbb{R}^3$, then $\iota_X \tau$ is an induced boundary orientation form for S. To see that the parametrization F coincides with this orientation, we compute $\iota_X\tau(F_\varphi, F_\theta)$:
$$\begin{aligned}
\iota_X\tau(F_\varphi, F_\theta) &= (dx \wedge dy \wedge dz)(X, F_\varphi, F_\theta) \\
&= \det \begin{vmatrix} 2\cos\theta\sin\varphi & 2\cos\theta\cos\varphi & -2\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta\sin\varphi \\ 2\cos\varphi/3 & -2\sin\varphi/3 & 0 \end{vmatrix} \\
&= 2\cos\varphi/3(2\cos^2\theta\sin\varphi\cos\varphi + 2\sin^2\theta\cos\varphi\sin\varphi) \\
&\quad + 2\sin\varphi/3(2\cos^2\theta\sin^2\varphi + 2\sin^2\theta\sin^2\varphi) \\
&= 2\cos\varphi/3(2\sin\varphi\cos\varphi) + 2\sin\varphi/3(2\sin^2\varphi) \\
&= \frac{4}{3}\cos^2\varphi\sin\varphi + \sin^3\varphi,
\end{aligned}$$

which is almost everywhere positive for $\varphi \in [0, \pi]$. So

$$\begin{aligned}
\int_S \omega &= \int_0^\pi \int_0^{2\pi} F^*\omega \\
&= \int_0^\pi \int_0^{2\pi} F^*x\ F^*dy \wedge F^*dz \\
&\qquad + \int_0^\pi \int_0^{2\pi} F^*y\ F^*dz \wedge F^*dx \\
&\qquad\qquad + \int_0^\pi \int_0^{2\pi} F^*z\ F^*dx \wedge F^*dy \\
&= \int_0^\pi \int_0^{2\pi} \cos\theta \sin\varphi \begin{vmatrix} 2\sin\theta\cos\varphi & 2\cos\theta\sin\varphi \\ -2\sin\varphi & 0 \end{vmatrix} d\varphi d\theta \\
&\qquad + \int_0^\pi \int_0^{2\pi} 2\sin\theta\sin\varphi \begin{vmatrix} -2\sin\varphi & 0 \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} d\varphi d\theta \\
&\qquad\qquad + \int_0^\pi \int_0^{2\pi} 3\cos\varphi \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ 2\sin\theta\cos\varphi & 2\cos\theta\sin\varphi \end{vmatrix} d\varphi d\theta \\
&= \int_0^\pi \int_0^{2\pi} 4\cos^2\theta \sin^3\varphi\ d\varphi d\theta \\
&\qquad + \int_0^\pi \int_0^{2\pi} 4\sin^2\theta \sin^3\varphi\ d\varphi d\theta \\
&\qquad\qquad + \int_0^\pi \int_0^{2\pi} 6\cos^2\varphi \sin\varphi\ d\varphi d\theta
\end{aligned}$$

3. **Iowa Qual, Winter 2016 (Wickrama)**

   Let $\omega \in \Omega^1(\mathbb{R}^2)$ be a compactly supported 1-form such that $d\omega = dx \wedge dy$. Consider the inclusion map $i : S^1 \to \mathbb{R}^2$, $i(x) = x$ where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Compute the integral:

$$\int_{S^1} i^*(\omega).$$

4. **Iowa Qual, Fall 2016 (Oswald)**

   Consider the upper hemisphere $X = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\} \subset \mathbb{R}^3$ and let $i : X \hookrightarrow \mathbb{R}^3$ be the inclusion of X into $\mathbb{R}^3$. If $\omega = dxdy + dydz + dzdx \in \Omega^2(\mathbb{R}^3)$ [Yes, they neglected to include the "$\wedge$".] then compute the integral:

$$\int_X i^*(\omega).$$

5. **Iowa Qual, Fall 2016 (Singh)**

   Let $T = S^1 \times S^1$ be the torus and $H : S^1 \times [0, 1] \to S^1 \times S^1$ be a smooth embedding of an annulus into the torus. Each side of the annulus defines an embedding of a circle into the torus: $\alpha = H(t, 0)$ and $\beta = H(t, 1)$. Prove that if $\omega \in \Omega^1(T)$ is a 1-form which satisfies $d\omega = 0$ then integrating ω over α is the same as integrating ω over β up to sign:

$$\int_\alpha \omega = \pm \int_\beta \omega.$$

6. **Introduction to Smooth Manifolds, Spring 2017, Final Review (Poudel)**

   Realize the three sphere as the set of vectors in four space of length one, and give it the orientation from the outward normal. The volume form on the three sphere is $x\ dy \wedge dz \wedge dw - y\ dx \wedge dz \wedge dw + z\ dx \wedge dy \wedge dw - w\ dx \wedge dy \wedge dz$. Compute the volume of the three sphere.

2.8 De Rham Cohomology

1. **Iowa Qual, Fall 2006 (Aceves)**

   Let M be a connected smooth manifold. Prove that the De Rham cohomology group $H^0(M) = \mathbb{R}$. [This should be proved without just citing the result that says that H^0 measures the number of connected components of a smooth manifold.]

2. **UGA Qual, January 2015 [slightly modified] (Wood)**

   Let X be a manifold and $U, V \subset X$ be open subsets with $X = U \cup V$. Prove that the Euler characteristics of $U, V, U \cap V$ and X obey the relation

   $$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

   (You may assume that the De Rham cohomologies of $U, V, U \cap V$ and X are finite-dimensional so that their Euler characteristics are well-defined.)

3. **UGA Qual, Fall 2015 [slightly modified] (Poudel)**

   Express a Klein bottle as the union of two annuli. Use the Mayer-Vietoris sequence and this decomposition to compute its De Rham cohomology.

4. **Introduction to Smooth Manifolds, Spring 2017, Final Review (Wood)**

   Let $\Sigma_{g,k}$ denote the orientable surface of genus g with k boundary components. Starting with $\Sigma_{1,1}$ give an inductive proof of the cohomology of $\Sigma_{g,1}$. Use this to compute the cohomology of all closed orientable surfaces $\Sigma_{g,0}$.