PARTIAL DIFFERENTIAL EQUATIONS REVIEW
BOOK: *INTRO TO PDE'S, J. COOPER*

NICHOLAS CAMACHO

PDE + Initial Data + Boundary Conditions	**Solution**
Transport Equation (Version 1) $u_t + c(x)u_x = 0,\ x \in \mathbb{R}, t > 0,$ $u(x,0) = f(x),\ x \in \mathbb{R}$	$u(x,t) = f(x_0),$ where one obtains x_0 by first solving the IVP $\frac{dx}{dt} = c(x), x(0) = x_0$, and then then by solving for x_0 in terms of x and t.
Transport Equation (Version 2) $u_t + F(u)_x = 0,\ \ x \in \mathbb{R}, t > 0,$ $u(x,0) = f(x),\ \ x \in \mathbb{R}$	**Riemann Problem:** Suppose $f(x) = \begin{cases} u_l & x < x_0 \\ u_r & x > x_0 \end{cases}$. Define $s := \frac{F(u_l) - F(u_r)}{u_l - u_r}$. • If $F'(u_l) > s > F'(u_r)$, then $u(x,t) = \begin{cases} u_\ell & x - x_0 < st \\ u_r & x - x_0 > st \end{cases}$. • If $F'(u_l) \not> s \not> F'(u_r)$, then $u(x,t) = \begin{cases} u_\ell & x - x_0 < F'(u_l)t \\ (F')^{-1}\left(\frac{x-x_0}{t}\right) & F'(u_l)t + x_0 \le x \le F'(u_r)t + x_0 \\ u_r & x - x_0 > F'(u_r)t \end{cases}$.
	Other initial data: If $f(x)$ is not as above, and continuous, then $u(x,t) = f(x_0)$, where one solves for x_0 in the equation of the characteristic line: $x = x_0 + c(f(x_0))t$. In the case that f is given piecewise, x_0 will vary depending on these piecewise conditions, and so $u(x,t)$ will also be piecewise.
	Numerical Schemes: • Using *forward* difference quotients, $u_t(x,t) \approx \frac{u_{j,n+1} - u_{j,n}}{\Delta t}$. $F(u)_x(x,t) \approx \frac{F(u_{j+1,n}) - F(u_{j,n})}{\Delta x}$. • Using a *backward* difference quotient for $F(u)_x$, $F(u)_x(x,t) \approx \frac{F(u_{j,n}) - F(u_{j-1,n})}{\Delta x}$. • Always use the forward difference scheme to approximate u_t. • If $c(u(x,0)) > 0$ for all x, then with $\rho := \frac{\Delta x}{\Delta t}$, the scheme is $u_{j,n+1} = u_{j,n} - \frac{1}{\rho}[F(u_{j,n}) - F(u_{j-1,n})]$. • If $c(u(x,0)) < 0$ for all x, then the scheme is $u_{j,n+1} = u_{j,n} - \frac{1}{\rho}[F(u_{j+1,n}) - F(u_{j,n})]$. • (CFL condition) We insist that the ratio ρ is chosen so that $\frac{\max_{x\in\mathbb{R}} \lvert c(u(x,0))\rvert}{\rho} \le 1$. Putting $c_{\max} := \max_{x\in\mathbb{R}} c(u(x,0))$, $\Delta t\ \lvert c_{\max}\rvert \le \Delta x$.

PDE + Initial Data + Boundary Conditions	**Solution**
Homogeneous Heat Equation IVP $u_t - ku_{xx} = 0,\ x \in \mathbb{R}, t > 0$ $u(x,0) = f(x),\ x \in \mathbb{R}.$	$u(x,t) = \int_{\mathbb{R}} S(x-y,t)f(y)dy$ where $S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$ **Numerical Schemes:** • Always use *forward* difference quotient for u_t, $u_t(x,t) \approx \frac{u_{j,n+1} - u_{j,n}}{\Delta t}$. • Use *centered* difference quotient for u_{xx} at $t + \Delta t$, $u_{xx}(x,t) \approx \frac{u_{j+1,n+1} - 2u_{j,n+1} + u_{j-1,n+1}}{\Delta x^2}$. • Then the scheme is $u_{j,n} = (1+2s)u_{j,n+1} - s(u_{j+1,n+1} + u_{j-1,n+1})$, where $s = k\Delta t / \Delta x^2$.
Inhomogeneous Heat Equation IVP $u_t - ku_{xx} = q(x,t),\ x \in \mathbb{R}, t > 0,$ $u(x,0) = f(x),\ x \in \mathbb{R}.$	$u(x,t) = \int_{\mathbb{R}} S(x-y,t)f(y)dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)q(y,s)dyds$
Homogeneous Heat Equation IBVP on the Half Line (Dirichlet) $u_t - ku_{xx} = 0,\ x > 0, t > 0,$ $u(x,0) = f(x),\ x > 0,$ $u(0,t) = 0,\ t > 0$	(1) Define an odd extension of $f(x)$: $\tilde{f}(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$. (2) The solution to $\tilde{u}_t - k\tilde{u}_{xx} = 0,\ t > 0, x \in \mathbb{R},$ $\tilde{u}(x,0) = \tilde{f}(x),\ x \in \mathbb{R}$ is given by $\tilde{u}(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]f(y)dy.$ (3) Then $u(x,t) = \tilde{u}(x,t)\big\vert_{x>0}$.
Homogeneous Heat Equation IBVP on the Half Line (Neumann) $u_t - ku_{xx} = 0,\ x > 0, t > 0,$ $u(x,0) = f(x),\ x > 0,$ $u_x(0,t) = 0,\ t > 0$	(1) Define an even extension of $f(x)$: $\tilde{f}(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$. (2) The solution to $\tilde{u}_t - k\tilde{u}_{xx} = 0,\ x \in \mathbb{R}, t > 0,$ $\tilde{u}(x,0) = \tilde{f}(x),\ x \in \mathbb{R}$ is given by $\tilde{u}(x,t) = \int_0^\infty [S(x-y,t) + S(x+y,t)]f(y)dy.$ (3) Then $u(x,t) = \tilde{u}(x,t)\big\vert_{x>0}$.
Homogeneous Heat Equation IBVP on a Finite Interval (Dirichlet) $u_t - ku_{xx} = 0,\ 0 < x < L, t > 0,$ $u(x,0) = f(x),\ 0 < x < L,$ $u(0,t) = u(L,t) = 0,\ t > 0$	$u(x,t) = \sum_{n=1}^\infty A_n \varphi_n(x)\psi_n(t)$, where • $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ • $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx$ • $\psi_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 kt}$

<table>
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IBVP on a Finite Interval (Neumann)
$$u_t - ku_{xx} = 0,\ 0 < x < L, t > 0,$$ $$u(x,0) = f(x),\ 0 < x < L,$$ $$u_x(0,t) = u_x(L,t) = 0,\ t > 0$$</td><td>$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \varphi_n(x)\psi_n(t), \quad \text{where}$$
• $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$
• $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx$
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IBVP on a Finite Interval (Dirichlet)
$$u_t - ku_{xx} = q(x,t),\ 0 < x < L, t > 0,$$ $$u(x,0) = f(x),\ 0 < x < L,$$ $$u(0,t) = u(L,t) = 0,\ t > 0$$</td><td>$$u(x,t) = \sum_{n=1}^{\infty} A_n \varphi_n(x)\psi_n(t) + \varphi_n(x)\int_0^t q_n(s)\psi_n(t-s)ds, \quad \text{where}$$
• $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
• $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx$
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• $q_n(t) = \frac{\langle q, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L q(x,t)\varphi_n(x)dx$</td></tr>
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• $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$
• $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx$
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on a Finite Interval with
Inhomogeneous Boundary Data (Dirichlet)
$$u_t - ku_{xx} = 0,\ 0 < x < L, t > 0,$$ $$u(x,0) = f(x),\ 0 < x < L,$$ $$u(0,t) = l(t), u(L,t) = r(t),\ t > 0$$</td><td>(1) Define $g(x,t) = l(t) + \frac{x}{L}(r(t) - l(t))$ and $v = u - g$.
(2) Then v satisfies $$v_t - kv_{xx} = -g_t(x,t),\ 0 < x < L, t > 0,$$ $$v(x,0) = f(x) - g(x,0),\ 0 < x < L,$$ $$v(0,t) = v(L,t) = 0,\ t > 0$$
(3) Use the previous solution to find v, and then $u = v + g$.</td></tr>
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(2) Then v satisfies $$v_t - kv_{xx} = -g_t(x,t) - kg_{xx}(x,t),\ 0 < x < L, t > 0,$$ $$v(x,0) = f(x) - g(x,0),\ 0 < x < L,$$ $$v_x(0,t) = v_x(L,t) = 0,\ t > 0$$
(3) Use the previous solution to find v, and then $u = v + g$.</td></tr>
<tr><td>Homogeneous Wave Equation IVP
$$u_{tt} - c^2u_{xx} = 0,\ x \in \mathbb{R}, t \in \mathbb{R}$$ $$u(x,0) = f(x),\ x \in \mathbb{R}$$ $$u_t(x,0) = g(x),\ x \in \mathbb{R}$$</td><td>$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$</td></tr>
<tr><td>Inhomogeneous Wave Equation IVP
with Initial Data zero
$$u_{tt} - c^2u_{xx} = q(x,t),\ x \in \mathbb{R}, t \in \mathbb{R}$$ $$u(x,0) = 0,\ x \in \mathbb{R}$$ $$u_t(x,0) = 0,\ x \in \mathbb{R}$$</td><td>$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} q(y,s)dyds$$</td></tr>
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PDE + Initial Data + Boundary Conditions	Solution
Homogeneous Wave Equation IBVP on the Half Line (Dirichlet) $u_{tt} - c^2 u_{xx} = 0,\ x > 0, t \in \mathbb{R}$ $u(x,0) = f(x),\ x \geq 0$ $u_t(x,0) = g(x),\ x \geq 0$ $u(0,t) = 0,\ t \in \mathbb{R}$	• On $\mathcal{Q} = \{(x,t) \in \mathbb{R}^2 \mid x > ct, t \geq 0\}$, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$ • On $\mathcal{R} = \{(x,t) \in \mathbb{R}^2 \mid 0 \leq x \leq ct, t \geq 0\}$, $u(x,t) = \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(y)dy - \int_0^{ct-x} g(y)dy\right]$
Homogeneous Wave Equation IBVP on the Half Line (Neumann) $u_{tt} - c^2 u_{xx} = 0,\ x > 0, t \in \mathbb{R}$ $u(x,0) = f(x),\ x \geq 0$ $u_t(x,0) = g(x),\ x \geq 0$ $u_x(0,t) = 0,\ t \in \mathbb{R}$	• On $\mathcal{Q} = \{(x,t) \in \mathbb{R}^2 \mid x > ct, t \geq 0\}$, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$ • On $\mathcal{R} = \{(x,t) \in \mathbb{R}^2 \mid 0 \leq x \leq ct, t \geq 0\}$, $u(x,t) = \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(y)dy + \int_0^{ct-x} g(y)dy\right]$
Homogeneous Wave Equation IBVP on a Finite Interval (Dirichlet) $u_{tt} - c^2 u_{xx} = 0,\ 0 < x < L, t \in \mathbb{R}$ $u(x,0) = f(x),\ 0 < x < L$ $u_t(x,0) = g(x),\ 0 < x < L$ $u(0,t) = u(L,t) = 0,\ t \in \mathbb{R}$	$u(x,t) = \sum_{n=1}^{\infty}[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\varphi_n(x)$ • $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ • $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx.$ • $\omega_n = \frac{cn\pi}{L}$ • $\omega_n B_n = \frac{2}{L}\int_0^L g(x)\varphi_n(x)dx.$
Homogeneous Wave Equation IBVP on a Finite Interval (Neumann) $u_{tt} - c^2 u_{xx} = 0,\ 0 < x < L, t \in \mathbb{R}$ $u(x,0) = f(x),\ 0 < x < L$ $u_t(x,0) = g(x),\ 0 < x < L$ $u_x(0,t) = u_x(L,t) = 0,\ t \in \mathbb{R}$	$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty}[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\varphi_n(x)$ • $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ • $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx.$ • $\omega_n = \frac{cn\pi}{L}$ • $\omega_n B_n = \frac{2}{L}\int_0^L g(x)\varphi_n(x)dx.$ • $B_0 = \frac{2}{L}\int_0^L g(x)dx$
Inhomogeneous Wave Equation IBVP on a Finite Interval (Dirichlet) $u_{tt} - c^2 u_{xx} = q(x,t),\ 0 < x < L, t \in \mathbb{R}$ $u(x,0) = f(x),\ 0 < x < L$ $u_t(x,0) = g(x),\ 0 < x < L$ $u(0,t) = u(L,t) = 0,\ t \in \mathbb{R}$	$u(x,t) = \sum_{n=1}^{\infty}\Big[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) + \frac{1}{\omega_n}\int_0^t q_n(s)\sin(\omega_n(t-s))ds\Big]\varphi_n(x).$ • $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ • $A_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx.$ • $\omega_n = \frac{cn\pi}{L}$ • $\omega_n B_n = \frac{2}{L}\int_0^L g(x)\varphi_n(x)dx.$ • $q_n(t) = \frac{\langle q, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L q(x,t)\varphi_n(x)dx$

PDE + Initial Data + Boundary Conditions	Solution
Inhomogeneous Wave Equation IBVP on a Finite Interval (Neumann) $u_{tt} - c^2 u_{xx} = q(x,t),\ 0 < x < L, t \in \mathbb{R}$ $u(x,0) = f(x),\ 0 < x < L$ $u_t(x,0) = g(x),\ 0 < x < L$ $u_x(0,t) = u_x(L,t) = 0,\ t \in \mathbb{R}$	$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \Big[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) + \frac{1}{\omega_n}\int_0^t q_n(s)\sin(\omega_n(t-s))ds \Big] \varphi_n(x).$ • $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ • $A_n = \frac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} = \frac{2}{L}\int_0^L f(x)\varphi_n(x)dx.$ • $\omega_n = \frac{cn\pi}{L}$ • $\omega_n B_n = \frac{2}{L}\int_0^L g(x)\varphi_n(x)dx.$ • $B_0 = \frac{2}{L}\int_0^L g(x)dx$ • $q_n(t) = \frac{\langle q, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} = \frac{2}{L}\int_0^L q(x,t)\varphi_n(x)dx$
Homogeneous Wave Equation IBVP on a Finite Interval with Inhomogeneous Boundary Data $u_{tt} - c^2 u_{xx} = 0,\ 0 < x < L, t \in \mathbb{R}$ $u(x,0) = f(x),\ 0 < x < L$ $u_t(x,0) = g(x),\ 0 < x < L$ $u_x(0,t) = 0, u_x(L,t) = h(t),\ t \in \mathbb{R}$	(1) Define $v := u - \frac{xh(t)}{L}$. (2) Then v satisfies $v_{tt} - c^2 v_{xx} = -\frac{xh''(t)}{L},\ 0 < x < L, t \in \mathbb{R}$ $v(x,0) = f(x) - \frac{xh(0)}{L},\ 0 < x < L$ $u_t(x,0) = g(x) - \frac{xh'(0)}{L},\ 0 < x < L$ $u(0,t) = u(L,t) = 0,\ t \in \mathbb{R}$ (3) Use previous solution to find v, and then $u = v + \frac{xh(t)}{L}$.

Fourier Series:

- A piecewise continuous, real-valued function $f(x)$ on $[-L, L]$ has Fourier series

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right),$$

  where

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{and} \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

- A complex-valued function $f(z)$ has Fourier series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi z/L} \quad \text{where} \quad c_n = \frac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} = \frac{1}{2L}\int_{-L}^{L} f(z)e^{-in\pi z/L}dz.$$

- Let $f(x)$ be real-valued with $f(x) \to 0$ as $x \to \pm\infty$. The Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx. \qquad \text{We can write } f \text{ as} \quad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

- **Fourier Transformation Rules**
  (1) Let $a > 0$ and $f_a(x) := f(ax)$. Then $\hat{f}_a(\xi) = \frac{1}{a}\hat{f}(\frac{\xi}{a})$.
  (2) $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$.
  (3) $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.
  (4) $\int_{\mathbb{R}} |f(x)|^2 dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$.
  (5) Let $a \in \mathbb{R}$ and $\tau_a f(x) := f(x-a)$. Then $\widehat{\tau_a f}(\xi) = e^{-ia\xi}\hat{f}(\xi)$.

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