When can we swap the limit with the integral sign?

Nicholas Camacho

In the table below, we are assuming that the conclusion of each theorem/lemma under the listed assumptions is $\lim_{n\to\infty} \int_E f_n = \int_E f$, unless otherwise specified in the "Comments" column. This column also contains additional assumptions and/or additional conclusions as needed. Page references are to *Real Analysis* (4th Edition) by Royden & Fitzpatrick.

Theorem	Assumptions			Comments
———	**E**	$\{\mathbf{f_n}\}$ **on E**	$\{\mathbf{f_n}\} \to$ **f on E**	———
Proposition 8 (§4.2, page 77)	-Finite measure	1.Measurable functions 2.Each f_n is bounded	-Uniform	Notice that the function f here (and in the BCT) is measurable, bounded, and defined on a set of finite measure, since it is the uniform limit of such functions. Combining Theorem 4 and the definition of the integrability of f in this section allows us to conclude that f is integrable and $\int_E f < \infty$.
Bounded Convergence Theorem (BCT), page 78	-Finite measure	1.Measurable functions 2.Uniformly pointwise bounded, i.e., $\exists M \geq 0$ s.t. $\lvert f_n \rvert \leq M \ \forall n$.	-Pointwise	Notice the difference between BCT and Prop 8: Proposition 8 requires that each f_n is bounded (i.e., there may be different bounds for each n), while the BCT requires a single bound for all f_n. Also, the convergence $\{f_n\} \to f$ only needs to be pointwise in BCT.
Fatou's Lemma (page 82)	-Measurable	1.Measurable functions 2.Non-negative	-Pointwise, almost everywhere	In this Lemma, we actually do *NOT* have $\lim_{n\to\infty} \int_E f_n = \int_E f$, but instead the conclusion is that $\int_E f \leq \liminf \int_E f_n$.
Monotone Convergence Theorem (MCT), page 83	-Measurable	1.Measurable functions 2.Increasing 3.Non-negative	-Pointwise, almost everywhere	Note that in the MCT (and Fatou's Lemma), we make no assumption about the integrability of f. In other words, it may be the case that $\int_E f = \infty$.

Theorem	**Assumptions**			**Comments**
———	**E**	**$\{f_n\}$ on E**	**$\{f_n\} \to f$ on E**	———
Beppo Levi's Lemma, page 84	-Measurable	1.Measurable functions 2.Increasing 3.Non-negative	-No assumption here, but the conclusion of the lemma gives us the existence of a function f that is: 1.measurable, 2.finite almost everywhere on E, and, 3.the pointwise limit of $\{f_n\}$ on E.	<u>An extra assumption:</u> The sequence $\{\int_E f_n\}$ is bounded. <u>Conclusion:</u> $\{f_n\}$ converges pointwise on E to a measurable function f that is finite almost everywhere on E and $\lim_{n\to\infty} \int_E f_n = \int_E f < \infty$.
Lebesgue Dominated Convergence Theorem (LDCT), page 88	-Measurable	-Measurable functions	-Pointwise, almost everywhere	<u>An extra assumption:</u> There exists a function g which is integrable over E and for which $\|f_n\| \leq g$ for all n. <u>An extra conclusion:</u> f is integrable over E. The conclusion here tells us that, unlike Fatou's Lemma and MCT, $\int_E f < \infty$.

Theorem	Assumptions			Comments
———	**E**	$\{\mathbf{f_n}\}$ **on E**	$\{\mathbf{f_n}\} \rightarrow \mathbf{f}$ **on E**	———
General Lebesgue Dominated Convergence Theorem (GLDCT), page 89	-Measurable	-Measurable functions	-Pointwise, almost everywhere	Extra assumptions: 1.There exits a sequence $\{g_n\}$ of non-negative measurable functions on E that converge pointwise almost everywhere on E to g. 2.$\lvert f_n\rvert \leq g_n$ for all n. 3. $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$. An extra conclusion: Even though it's not explicitly stated in the book, we have again that f is integrable over E because $\int_E f_n \leq \int_E \lvert f_n\rvert \leq \int_E g_n \ \forall n \implies \int_E f = \lim_{n\to\infty} \int_E f_n \leq \lim_{n\to\infty} \int_E g_n = \int_E g < \infty$
Vitali Convergence Theorem (Version 1, page 94)	-Finite measure	-Uniformly integrable	-Pointwise, almost everywhere	An extra conclusion: f is integrable over E.
Vitali Convergence Theorem (Version 2, page 98)	-Measurable	1.Uniformly integrable 2.Tight	-Pointwise, almost everywhere	An extra conclusion: f is integrable over E.