P.6 - Rational Exponents & Radicals

Square Roots

- **Def**: The principal square root of a nonnegative real number a is a real number b such that

  ① $b^2 = a$ and ② $b \geq 0$

  The principal square root of a is denoted $\sqrt{a}$, and we write $\sqrt{a} = b$.

  Ex: $\sqrt{36} = 6$, because ① $6^2 = 36$ and ② $6 \geq 0$

  - Even though $(-6)^2 = 36$, we don't have $-6 \geq 0$.

- When we write $\sqrt{a}$, a is called the radicand, and $\sqrt{\ }$ is the radical sign. Together, the radicand and the radical sign form a radical, $\sqrt{a}$.

Square Root Property: If $a \geq 0$, and $x^2 = a$, then

$$x = \pm\sqrt{a}$$

Ex: $x^2 = 16$, $x = \pm\sqrt{16} = \pm 4$.

Note: For any real number x, $\sqrt{x^2} = |x|$, not x.

Ex: ① $\sqrt{(-3)^2} = \sqrt{9} = 3$.

② $\sqrt{121} = \sqrt{(11)^2} = |11| = 11$

③ $\sqrt{\frac{4}{25}} = \frac{2}{5}$

$$\sqrt{\frac{4}{25}} = \sqrt{\frac{2^2}{5^2}} = \sqrt{\left(\frac{2}{5}\right)^2} = \left|\frac{2}{5}\right| = \frac{2}{5}.$$

④ $\sqrt{\frac{9}{64}} = \frac{3}{8}$

Simplifying Square Roots

Let a, b be real numbers and let $a \geq 0$ and $b \geq 0$.
Then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Ex ① $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$\vdots$

② $\sqrt{6} \cdot \sqrt{12} = \sqrt{6 \cdot 12} = \sqrt{72} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}.$

③ $\sqrt{48x^2} = \sqrt{16 \cdot 3 \cdot x^2} = \sqrt{16} \cdot \sqrt{3} \cdot \sqrt{x^2}$
$= 4\sqrt{3}\,|x|$

④ $\sqrt{\frac{25y^3}{9x^2}} = \frac{\sqrt{25y^3}}{\sqrt{9x^2}} = \frac{\sqrt{25} \cdot \sqrt{y^2 \cdot y}}{\sqrt{9} \cdot \sqrt{x^2}}$

$= \frac{5\sqrt{y^2} \cdot \sqrt{y}}{3\sqrt{x^2}}$

$= \frac{5|y|\sqrt{y}}{3|x|}.$

* Caution:* $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$

Other Roots

- Let n be a positive integer (whole number)

The Principal n^{th} Root of a Real Number a:

① If $a>0$, $\sqrt[n]{a}=b$ if ① $b^n=a$ and ② $b>0$.

② If $a<0$, and n is odd, then $\sqrt[n]{a}=b$ if $b^n=a$.

③ If $a<0$, and n is even, then $\sqrt[n]{a}$ is not a real number.

④ If $a=0$, $\sqrt[n]{a}=\sqrt[n]{0}=0$.

Note: • $\sqrt{a}=\sqrt[2]{a}$

- n is called the index

Ex ① $\sqrt[3]{27}$

- Radicand 27 is positive. So, we find a number b such that ① $b^3=27$ and ② $b>0$.

$\sqrt[3]{27}=3$.

② $\sqrt[3]{-64}$

- Radicand -64 is negative, so it only makes sense if index is odd. Find a number b such that $b^3=-64$.

$\sqrt[3]{-64}=-4$, because $(-4)^3=(-4)(-4)(-4)=16\cdot(-4)=-64$.

③ $-\sqrt[101]{-1}=-\left(\sqrt[101]{-1}\right)$

- Look for b such that $b^{101}=-1$.

Since $(-1)^{101}=-1$, then $\sqrt[101]{-1}=-1$.

So $-\sqrt[101]{-1}=-(-1)=1$.

④ $\sqrt[6]{-2} \rightarrow$ not a real number.

Radicand is negative, and index is even.

Rules for Radical

① If n is odd, then $\sqrt[n]{a^n} = a$.

② If n is even, then $\sqrt[n]{a^n} = |a|$.

③ Product Rule: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

④ Quotient Rule: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

⑤ Power Rule: $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Ex: ① $\sqrt[3]{135} = \sqrt[3]{27 \cdot 5} = \sqrt[3]{27} \cdot \sqrt[3]{5} = 3\sqrt[3]{5}$.

$2^3 = 8$ ② $\sqrt[4]{162a^4} = \sqrt[4]{81 \cdot 2 \cdot a^4} = \sqrt[4]{81} \cdot \sqrt[4]{2} \cdot \sqrt[4]{a^4}$

$3^3 = 27$ $= 3\sqrt[4]{2}\,|a|$

$4^3 = 64$ ③ $\sqrt[3]{\frac{5}{64}} = \frac{\sqrt[3]{5}}{\sqrt[3]{64}} = \frac{\sqrt[3]{5}}{4}$

$5^3 = 125$

⋮

$2^4 = 16$ ④ $\sqrt[2]{\sqrt[3]{7}} = \sqrt[6]{7}$

$3^4 = 81$

⋮

Adding & Subtracting Like Radicals

- Radicals with the same index and radicand are like radicals.

Ex: $2x$ & $10x$ were "like terms". So $2x + 10x = 12x$

$2x^2$ & $10x$ not "like terms", so $2x^2 + 10x$ can't be simplified further.

① $\sqrt{45} + 7\sqrt{20} = \sqrt{9}\cdot\sqrt{5} + 7\sqrt{4}\sqrt{5}$

$= 3\sqrt{5} + 14\sqrt{5}$

$= 17\sqrt{5}$

② $5\sqrt[3]{80x} - 3\sqrt[3]{270x} = 5\sqrt[3]{8}\cdot\sqrt[3]{10}\cdot\sqrt[3]{x} - 3\sqrt[3]{27}\cdot\sqrt[3]{10}\cdot\sqrt[3]{x}$

$= 10\sqrt[3]{10}\sqrt[3]{x} - 9\sqrt[3]{10}\sqrt[3]{x}$

$= 10\sqrt[3]{10x} - 9\sqrt[3]{10x}$

$= \sqrt[3]{10x}$

③ $3\sqrt{12} + 7\sqrt{3} = 3\sqrt{4}\cdot\sqrt{3} + 7\sqrt{3}$

$= 6\sqrt{3} + 7\sqrt{3}$

$= 13\sqrt{3}$.

Rational Exponents

- For any real number a, and any integer $n > 1$,

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Ex ① $16^{1/2} = \sqrt[2]{16} = 4$

② $(-27)^{1/3} = \sqrt[3]{-27} = -3$

③ $\left(\frac{1}{16}\right)^{1/4} = \sqrt[4]{1/16} = \frac{\sqrt[4]{1}}{\sqrt[4]{16}} = \frac{1}{2}$

④ $(-5)^{1/4} = \sqrt[4]{-5}$ → not a real number.

- If m, n are nonzero integers, and $\frac{m}{n}$ is in lowest terms, then

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \quad \boxed{\text{or}} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

* if $\sqrt[n]{a}$ is defined, i.e. if $\sqrt[n]{a}$ makes sense *

· We know that $\sqrt{x^2} = |x|$.

$$\cancel{\sqrt{x^2} = (x^2)^{1/2} = x}$$

if $x > 0$.

$$\sqrt{x^2} = (x^2)^{1/2} = x^{\frac{2}{2}} = (\sqrt{x})^2 = x$$

if $\sqrt{x}$ makes sense.

If $x < 0$, then $\sqrt{x}$ is not a real number.

<u>Ex</u> ① $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

$8^{2/3} = (\sqrt[3]{8})^2 = (2)^2 = 4$

② $-16^{5/2} = -(16^{5/2}) = -((\sqrt{16})^5)$

$= -(4^5)$

$= -(1024)$

$= -1024$

③ $\left.\begin{array}{l} (-25)^{7/2} = (\sqrt[2]{-25})^7 \\ (-25)^{7/2} = \sqrt[2]{(-25)^7} \end{array}\right\} \rightarrow$ Not a real number.

<u>Simplifying Expressions with Rational Exponents</u>

<u>Ex</u> ① $2x^{1/3} \cdot 5x^{1/4} = 10x^{1/3 + 1/4} = 10x^{7/12}$

② $\dfrac{21x^{-2/3}}{7x^{1/5}} = \dfrac{21}{7} \cdot \dfrac{x^{-2/3}}{x^{1/5}} = 3x^{-2/3 - 1/5} = 3x^{-13/15} = \dfrac{3}{x^{13/15}}$

③ $(x^{3/5})^{-1/6} = x^{\frac{3}{5}\cdot(-\frac{1}{6})} = x^{\frac{-3}{30}} = x^{-\frac{1}{10}} = \dfrac{1}{x^{1/10}}$

④ $\sqrt[8]{x^2} = (x^2)^{1/8} = x^{2/8} = x^{1/4} = \sqrt[4]{x}$

⑤ $\sqrt{\sqrt[3]{x^{12}}} = \sqrt[6]{x^{12}} = (x^{12})^{1/6} = x^{\frac{12}{6}} = x^2.$