INTRODUCTION TO BRAUER THEORY

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We closely follow Serre’s exposition in Part III of Linear Representations of Finite Groups.

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Throughout, we use the following notation and and make the following assumptions.

- $G$ is a finite group and $m$ denotes the exponent of $G$.
- A field is said to be sufficiently large (relative to $G$) if it contains the $m$th roots of unity.
- All modules are assumed to be finitely generated.
- $K$ denotes a complete field with respect to a discrete valuation $v$ with valuation ring $O$. $m$ is the maximal ideal of $O$ and $k = O/m$ is the residue field.
- We assume $K$ has characteristic 0 and $k$ has characteristic $p > 0$.
1. The groups $R_K(G), R_k(G), \text{ and } P_k(G)$

1.1. The rings $R_K(G)$ and $R_k(G)$.

Definition 1.1. If $L$ is any field, we denote by $R_L(G)$ the Grothendieck group of the category of finitely generated $L[G]$-modules, which is defined as follows:

- Generators of $R_L(G)$: For every finitely generated $L[G]$-module $E$, let $[E]$ be a generator of $R_L(G)$.
- Relations of $R_L(G)$: For every short exact sequence $0 \to E \to E' \to E'' \to 0$ of finitely generated $L[G]$-modules, we have a relation

$$[E'] = [E] + [E''].$$

For a finitely generated $L[G]$-module $E$, we have short exact sequences $0 \to E \to E \to 0$ and $0 \to 0 \to E \to E \to 0$, giving $[E] + [0] = [E] = [0] + [E]$. So $[0]$ is the additive identity of $R_L(G)$.

If $E \cong E'$, the short exact sequence $0 \to E \to E' \to 0 \to 0$ gives $[E'] = [E] + [0] = [E]$. For two finitely generated $L[G]$-modules $E, F$, we have a short exact sequence $0 \to E \to E \oplus F \to F \to 0$, and so

$$[E \oplus F] = [E] + [F].$$

We can define a multiplication on $R_L(G)$ via the tensor product

$$[E] \cdot [F] := [E \otimes_L F],$$

and since the tensor product commutes with direct sums, this multiplication distributes over addition, making $R_L(G)$ into a commutative ring. Moreover, the $L[G]$-module $L$ with the trivial $G$-action (the trivial representation) provides a unit element for $R_L(G)$, since $L \otimes_L E \cong E \cong E \otimes_L L$ for all finitely generated $L[G]$-modules $E$.

Definition 1.2. Let $R^+_L(G)$ collection of all $[E]$ for $E$ a finitely generated $L[G]$-module (so $R^+_L(G)$ is the set of generators for $R_L(G)$). Let $S_L$ denote the set of isomorphism classes of simple $L[G]$-modules.

Lemma 1.3. For any $L[G]$-module $F$ of finite length, we have an expression $[F] = \sum_i n_i[E_i]$ in $R_L(G)$, where the $E_i$ are the distinct simple modules appearing in a composition series for $F$ (i.e. the composition factors of $F$), and the $n_i$ are positive integers indicating the number of times the simple module $E_i$ appears as a composition factor for $F$.

Proof. By induction on the length of $F$. If $F$ has length 0 or 1, the claim is obvious. Suppose the claim is true for all modules of length less than the length of $F$; say $F$ has length $s$. Suppose $F$ has composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_s = F$.

By induction, we can write $[M_{s-1}] = \sum_i n_i[E_i]$ where $E_i \in S_L$ and $n_i \in \mathbb{Z}_{\geq 0}$. The short exact sequence of $L[G]$-modules $0 \to M_{s-1} \to M_s \to M_s/M_{s-1} \to 0$ then yields

$$[M_s] = [M_{s-1}] + [M_s/M_{s-1}] = \sum_i n_i[E_i] + [M_s/M_{s-1}],$$

which completes the induction. \qed

Proposition 1.4. The family of elements $\{[E]\}_{E \in S_L}$ is a basis for the group $R_L(G)$, i.e.,

$$R_L(G) = \bigoplus_{E \in S_L} \mathbb{Z}[E].$$
Proof. Let \( R = \bigoplus_{E \in S_L} \mathbb{Z}E \), the free \( \mathbb{Z} \)-module with basis \( S_L \). Define a map
\[
\alpha : R \rightarrow R_L(G)
\]
\[
\sum_{E \in S_L} n_E E \rightarrow \sum_{E \in S_L} n_E [E].
\]
Clearly \( \alpha \) is a group homomorphism.

Given an \( L[G] \)-module \( F \), and \( E \in S_L \), let \( \ell_E(F) \) denote the number of times which \( E \) appears as a composition factor in a composition series for \( F \). Suppose we have composition series
\[
0 = M_0 \subset M_1 \subset \cdots \subset M_s = F
\]
for \( F \). Then we have a composition series
\[
0 = M_0 \subset M_1 \subset \cdots \subset M_s = F = F \oplus M_0 \subset F \oplus N_1 \subset \cdots \subset F \oplus N_t = F \oplus F'
\]
for \( F \oplus F' \). Indeed,
\[
(F \oplus N_{i+1})/(F \oplus N_i) \cong N_{i+1}/N_i
\]
is simple for all \( 0 \leq i \leq t-1 \). In particular, the simple factors of \( F \oplus F' \) are those of \( F \) and \( F' \). Hence
\[
\ell_E(F \oplus F') = \ell_E(F) + \ell_E(F'),
\]
so \( \ell_E \) is a group homomorphism. Then the map
\[
\beta : R_L(G) \rightarrow R
\]
\[
[F_i] \mapsto \sum_{E \in S_L} \ell_E(F_i) E,
\]
(extend to all of \( R_L(G) \) by linearity), is also a group homomorphism. Then for \( E' \in S_L \)
\[
\beta \alpha (E') = \beta ([E']) = \sum_{E \in S_L} \ell_E(E') E = E'.
\]
We also have
\[
\alpha \beta ([F]) = \alpha \left( \sum_{E \in S_L} \ell_E(F) E \right) = \sum_{E \in S_L} \ell_E(F) [E] = [F],
\]
the last equality following from Lemma 1.3. Since \( \alpha \) and \( \beta \) are both additive, the equalities \( \alpha \beta = \beta \alpha = 1 \) extend to all of \( R \) and \( R_L(G) \). Therefore \( \alpha \) is an isomorphism which takes the basis \( \{E\}_{E \in S_L} \) to a basis of \( R_L(G) \), namely \( \{[E]\}_{E \in S_L} \).

Remark 1.5. Since \( K[G] \) is semisimple, \( K[G] \)-modules \( E, E' \) have decompositions \( E \cong \bigoplus_i n_i E_i \) and \( E' \cong \bigoplus_i n_i E_i \) where the \( n_i, n_i' \in \mathbb{Z}_{\geq 0} \) and the \( E_i \) are representatives of simple \( K[G] \)-modules (of which there are finitely many since \( K[G] \) is semisimple). If \( [E] = [E'] \), then
\[
\sum_i n_i [E_i] = \left[ \bigoplus_i n_i E_i \right] = [E] = [E'] = \left[ \bigoplus_j n_j' E_i \right] = \sum_i n_j' [E_i].
\]
By Proposition 1.4, the \([E_i]\) form a basis for \( R_K(G) \), and hence \( n_i = n_i' \) giving \( E \cong E' \).
1.2. The groups $\mathbf{P}_k(G)$ and $\mathbf{P}_{O}(G)$.

We define $\mathbf{P}_k(G)$ (resp. $\mathbf{P}_{O}(G)$) as the Grothendieck group of the category of finitely generated projective $k[G]$-modules (resp. projective $O[G]$-modules).

If $E$ (resp. $F$) is a $k[G]$-module (resp. a projective $k[G]$-module), then $E \otimes_k F$ is a projective $k[G]$-module. Indeed, since $F$ is projective, it is the direct summand of a free $k[G]$-module: $F \oplus F' = \bigoplus_{\alpha} k[G]$. Then

$$(E \otimes_k F) \oplus (E \otimes_k F') \cong E \otimes_k F \oplus F' \cong E \otimes_k \bigoplus_{\alpha} k[G] \cong \bigoplus_{\alpha} (E \otimes_k k[G]),$$

so $E \otimes_k F$ is a summand of a free $k[G]$ module and hence projective. In this way, we obtain an $R_k(G)$-module structure on $\mathbf{P}_k(G)$: For every $[E] \in R_k(G)$ and for every $[F] \in \mathbf{P}_k(G)$, define an action $[E].[F] := [E \otimes_k F]$.

1.3. Structure of $\mathbf{P}_k(G)$.

**Definition 1.6.** An epimorphism $\epsilon : P \twoheadrightarrow M$ is said to be essential if for every submodule $P' \subseteq P$, the equality $\epsilon(P') = M$ implies $P' = M$.

A module $M$ is said to have a projective cover if there exists a projective module $P$ and an essential epimorphism $\epsilon : P \twoheadrightarrow M$.

Since $k[G]$ is a finite dimensional algebra over $k$, any descending chain of $k[G]$ ideals of $k[G]$ also consists in a descending chain of $k$-vector spaces, since every ideal of $k[G]$ is a $k$-vector space. In particular, any descending chain of ideals (either left or right ideals) must stabilize, and hence $k[G]$ is an Artinian ring (both a left and right Artinian.)

Notice that a similar argument works for any ascending chain of ideals, and moreover, that both of these arguments work for any finite dimensional algebra over any field.

**Lemma 1.7.** Let $L$ be any field and let $A$ be a finite dimensional algebra over $L$. Then $A$ is both an Artinian and Noetherian ring. In particular, any finitely generated $A$-module is both Artinian and Noetherian, and hence any finitely generated $A$ module has a composition series.

Moreover, any (finitely generated) $k[G]$-module has a projective cover:

**Proposition 1.8.**

(a) Let $A$ be a left Artinian ring, and let $M$ be a finitely generated left $A$-module. Then $M$ has a projective cover, which is unique up to isomorphism.

(b) If $P_i$ is the projective cover for $M_i$ for all $1 \leq i \leq n$, then $\bigoplus_i P_i$ is the projective cover for $\bigoplus_i M_i$.

(c) If $P$ is a projective module, and if $E$ is the largest semisimple quotient of $P$, then the natural projection $\pi : P \twoheadrightarrow E$ is a projective cover for $E$.

**Proof.**

(a) Let $L$ be a projective module with an epimorphism $f : L \twoheadrightarrow M$. Let $S$ be the set of all submodules $N \subseteq \ker(f)$ so that $f_N : L/N \twoheadrightarrow M$ is an essential epimorphism. Now $S \neq \emptyset$ since $L/\ker(f) \cong M$. Since $A$ is Artinian and $L$ is finitely generated, then $L$ is Artinian and hence $S$ has a minimal element, say $X$. It remains to show that $L/X$ is projective.

If $\pi : L \twoheadrightarrow L/X$, then $f = \pi \circ f_X$. So if $\pi$ is essential, then $f$ is essential, and we are done. If not, let $Y \subsetneq L$ be a minimal submodule such that...
\(\pi(Y) = L/X\). Then \(\pi|_Y : Y \to L/X\) is essential, and since \(L\) is projective, we find a surjective map \(g : L \to Y\) so that \(\pi = \pi|_Y \circ g\).

Since \(g\) is surjective, then the induced map \(\tilde{g}\) is an isomorphism. So the composition \(f_X \circ \pi|_Y \circ \tilde{g}\) is an essential epimorphism.

If \(\ell \in \ker g\), then

\[
0 + X = \pi(0) = \pi|_Y (g(\ell)) = \pi(\ell) = \ell + X,
\]

so \(\ker g \subseteq X\). But \(f_X \circ \pi|_Y \circ \tilde{g}\) is essential, and by the minimality of \(X\) as an element of \(S\), it follows that \(X = \ker (g)\). Hence the map \(h : L/X \to L\) given by \(\ell + X \mapsto g(\ell)\) is a well-defined \(R\)-module homomorphism. Moreover

\[
(\pi \circ h)(\ell + X) = \pi(g(\ell)) = \pi|_Y (g(\ell)) = \pi(\ell) = \ell + X,
\]

hence \(\pi \circ h = \text{id}_{L/X}\), and so \(X \oplus L/X \cong L\), implying \(L/X\) is projective.

Now we show uniqueness. Suppose \(\epsilon : P \to M\) and \(\epsilon' : P' \to M\) are projective covers for \(M\). Since \(P\) is projective, there exists a map \(f : P \to P'\) that \(\epsilon' \circ f = \epsilon\). Now \(f(P) \subseteq P'\) and \(\epsilon'(f(P)) = \epsilon(P) = M\), so \(f(P) = P'\) since \(\epsilon'\) is essential, i.e. \(f\) is surjective.

Since \(P'\) is projective and \(f\) is surjective, there exists \(g : P' \to P\) so that \(fg = \text{id}_{P'}\) and \(P\) decomposes as \(P = g(P') \oplus \ker f\). But since

\[
\epsilon(g(P')) = \epsilon' (fg(P')) = \epsilon'(P') = M
\]

and \(\epsilon\) is essential, then \(g(P) = P'\) and hence \(\ker f = 0\). So \(f\) is an isomorphism.

(b) Let \(\epsilon_i : P_i \to M_i\) be a projective cover for \(M_i\) for all \(1 \leq i \leq n\). Define

\[
\epsilon := \oplus_i \epsilon_i : P := \bigoplus_i P_i \to M := \bigoplus_i M_i, \quad \sum_i p_i \mapsto \sum_i \epsilon_i(p_i).
\]

Clearly \(P\) is projective and \(\epsilon\) is surjective. Suppose \(P' \subseteq P\) is a submodule with \(\epsilon(P') = M\). Define \(P'_i := P' \cap P_i\), so \(P' = \bigoplus_i P'_i\). We show that \(P'_i = P_i\) for all \(i\), from which it follows that \(P = P'\).

If \(m_j \in M_j\), there exists \(p' = \sum_i p'_i \in P'\) with \(\epsilon(p') = m_j\). Then

\[
m_j = \epsilon(p') = \sum_i \epsilon_i(p'_i),
\]
and hence $\epsilon_i(p'_i) = 0$ for all $i \neq j$ and $\epsilon_j(p'_j) = m_j$. Then $\epsilon_j(P'_j) = M_j$, and so $P'_j = P_j$ since $\epsilon_j$ is essential, for all $j$.

(c) We only need to show that $\pi : P \to E$ is essential. Suppose $E = P/P'$ and that $\pi(Q) = P/P'$ for some proper submodule $Q \subsetneq P$.

Since $P$ is finitely generated, every proper submodule is contained in a maximal submodule. So there exists $N \subseteq P$ maximal with $Q \subseteq N$. Since $P/N$ is simple, $P' \subsetneq N$ and we have a well defined surjective map $\pi' : P/P' \to P/N, p + P' \mapsto p + N$. So $\pi' \circ \pi|_Q : Q \to P/N$ is surjective, but on the other hand

$$\pi'(\pi(q)) = \pi'(q + P') = q + N = 0 + N$$

for all $q \in Q$ since $Q \subseteq N$, which gives the desired contradiction.

□

Proposition 1.8 then gives us an explicit way to decompose any projective $k[G]$-module.

**Corollary 1.9.** Each projective $k[G]$-module is a direct sum of projective indecomposable $k[G]$-modules; this decomposition is unique up to isomorphism. The projective indecomposable $k[G]$-modules are the projective covers of the simple $k[G]$-modules.

**Proof.** Any (finitely generated) $k[G]$-module is both Artinian and Noetherian, and hence has a composition series. The Krull-Schmidt Theorem then gives the existence and uniqueness of a decomposition of such a module into indecomposables. In the case that $P$ is a finitely generated $k[G]$-module, every indecomposable summand of $P$ is itself projective, being a summand of a projective module.

However, we’d like to see more explicitly where the projective indecomposable summands of $P$ come from. They are the projective covers of the simple summands appearing in the largest semisimple quotient of $P$.

By Proposition 1.8(c), the natural quotient map $\pi : P \to E$ is a projective cover for $E$, where $E$ is the largest semisimple quotient of $P$. Write $E = \bigoplus_i S_i$, for simple modules $S_i$. By Proposition 1.8(a), there exists projective covers $\epsilon_i : P_i \to S_i$, and by Proposition 1.8(b), we obtain a projective cover for $E$:

$$\epsilon = \bigoplus_i \epsilon_i : \bigoplus_i P_i \longrightarrow \bigoplus_i S_i = E.$$ 

But Proposition 1.8(a) says that projective covers are unique up to isomorphism, meaning $P \cong \bigoplus_i P_i$. Moreover, each $P_i$ is indecomposable since any decomposition $P_i = A \oplus B$ of $P_i$ into submodules would mean $\epsilon_i(A)$ is a submodule of the simple module $S_i$ and so $\epsilon_i(A) = S_i$ or $\epsilon_i(A) = 0$. In the former case, $A = P_i$ since $\epsilon_i$ is essential, and since $A \cap B = 0$, then $B = 0$. The latter case gives $B = P_i$ and $A = 0$ by similar reasoning.

The above argument shows that the projective cover of a simple module is indecomposable. However, the last statement of the corollary is saying something more: *every* projective indecomposable $k[G]$-module is a projective cover of a simple $k[G]$-module.

Proposition 1.8(c) says that any projective module $P$ is a projective cover of its largest semisimple quotient $E$. In the case that $P$ that an *indecomposable* projective module $P$, $E$ is in fact simple. Using notation from the previous argument, we have
$P \cong \bigoplus P_i$, and since $P$ is indecomposable, only one of the $P_i$ is nonzero, and hence $E = S_i$ is simple. \hfill \Box

**Corollary 1.10.** For each $E \in S_k$, let $P_E$ be the projective cover of $E$. Then \{[$P_E$]$]_{E \in S_k}$ form a basis of $P_k(G)$, i.e.,

$$P_k(G) = \bigoplus_{E \in S_k} \mathbb{Z}[P_E].$$

**Proof.** We adapt the proof of Proposition 1.4. Let $R = \bigoplus_{E \in S_k} \mathbb{Z}P_E$ be the free abelian group with basis the projective covers of elements in $S_k$, and define

$$\alpha : R \rightarrow P_k(G) \quad \sum_{E \in S_k} n_E P_E \mapsto \sum_{E \in S_k} n_E [P_E].$$

A projective $k[G]$-module $P$ can be decomposed uniquely into a direct sum of indecomposable projective modules, which are projective covers of simple $k[G]$-modules by Corollary 1.9, so $\alpha$ is surjective.

For a projective module $P$ and an indecomposable projective module $P_E$, let $\ell_E(P)$ denote the number of times the projective cover $P_E$ of $E$ appears in the factorization of $P$ into indecomposables. Then

$$\beta : P_k(G) \rightarrow R \quad [P] \mapsto \sum_{E \in S_k} \ell_E(P)P_E$$

(extend by linearity to all of $P_k(G)$) is a group homomorphism, and for a simple module $E' \in S_k$,

$$\beta \alpha (P_{E'}) = \beta ([P_{E'}]) = \sum_{E \in S_k} \ell_E(P_{E'})P_E = P_{E'},$$

and hence $\alpha$ is injective. So $\alpha$ is an isomorphism and sends the basis $\{P_E\}_{E \in S_k}$ to the basis of $P_k(G)$, namely $\{[P_E]\}_{E \in S_k}$. \hfill \Box

**Corollary 1.11.** Two projective $k[G]$-modules $P$ and $P'$ are isomorphic if and only if $[P] = [P']$ in $P_k(G)$.

**Proof.** We’ve already seen that $P \cong P'$ implies $[P] = [P']$. On the other hand Corollary 1.9 gives unique decompositions of $P$ and $P'$ into projective indecomposables, and that these indecomposables are projective covers for simple $k[G]$ modules. Let $\{P_i\}_i$ be the collection of all projective indecomposable modules appearing in the decompositions for $P$ and $P'$. Then

$$P \cong \bigoplus_i n_i P_i \quad \text{and} \quad P \cong \bigoplus_i n'_i P_i$$

where $n_i, n'_i \in \mathbb{Z}_{\geq 0}$. Then

$$\sum_i n_i [P_i] = \left[ \bigoplus_i n_i P_i \right] = [P] = [P'] = \left[ \bigoplus_i n'_i P_i \right] = \sum_i n'_i [P_i],$$

and hence by Corollary 1.10 $n_i = n'_i$ for all $i$, i.e. $P \cong P'$. \hfill \Box
Lemma 1.12. Let \( R, S \) be commutative rings with 1 and \( f : R \to S \) be a ring homomorphism (sending 1 to 1). If \( P \) is a projective \( R \) module, then the \( S \)-module \( S \otimes_R M \) (obtained via extension of scalars along \( f \)) is also projective.

Proof. We show \( \text{Hom}_S(S \otimes_R P, -) \) is right exact. Let \( M \xrightarrow{h} N \to 0 \) be an exact sequence of \( S \)-modules. Since \( S \otimes_R - \) is left adjoint to \( \text{Hom}_S(S, -) \), we have a diagram

\[
\begin{array}{c}
\text{Hom}_S(S \otimes_R P, M) & \to & \text{Hom}_S(S \otimes_R P, N) & \to & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\text{Hom}_R(P, \text{Hom}_S(S, M)) & \to & \text{Hom}_R(P, \text{Hom}_S(S, N)) & \to & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\text{Hom}_R(P, M) & \to & \text{Hom}_R(P, N) & \to & 0
\end{array}
\]

If the diagram commutes, the exactness of the top row follows from exactness of the bottom row. We show the diagram does indeed commute.

For the top square: Let \( \alpha \in \text{Hom}_R(P, \text{Hom}_S(S, M)) \). Following the top square across the top and then down,

\[
\begin{array}{c}
\alpha \\
\downarrow \\
h \circ \alpha
\end{array}
\]

\[
\bar{h} \circ \alpha : p' \mapsto (s' \mapsto (h \circ \alpha)(s' \otimes p'))
\]

Following the top square down and then across the bottom,

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\bar{\alpha} : p \mapsto (s \mapsto \alpha(s \otimes p))
\end{array}
\]

\[
\bar{h} \circ \bar{\alpha} : p' \mapsto \bar{h}(\bar{\alpha}(p'))
\]

and since

\[
h_{\ast}(\bar{\alpha}(p')) = h_{\ast}(s' \mapsto \alpha(s' \otimes p')) = s' \mapsto h(\alpha(s' \otimes p')) = h \circ \alpha(s' \otimes p'),
\]

the top square commutes.

For the bottom square: Let \( \beta \in \text{Hom}_R(P, \text{Hom}_R(S, M)) \). Following the bottom square across the top and then down,

\[
\begin{array}{c}
\beta \\
\downarrow \\
(h_{\ast} \circ \beta) : p \mapsto h \circ \beta(p)
\end{array}
\]

\[
p' \mapsto ((h_{\ast} \circ \beta)(p'))(1) = (h \circ \beta(p'))(1)
\]
Following the bottom square down and then across the bottom,

\[
\begin{align*}
\beta \\
\tilde{\beta} : p \mapsto (\beta(p))(1) \quad \text{(1)}
\end{align*}
\]

\[
\begin{align*}
h \circ \tilde{\beta} : p' \mapsto h(\beta(p')(1))
\end{align*}
\]

and since \((h \circ \beta(p'))(1) = h(\beta(p')(1))\), the bottom square commutes. □

**Lemma 1.13.** Let \(R\) be any ring and \(P\) any \(R\)-module. Suppose there exists a projective \(R\)-module \(Q\) and a surjective \(R\)-module map \(f : Q \to P\). Then \(P\) is a projective \(R\)-module if and only if there exists an \(R\)-module map \(h : Q \to P\) such that \(f \circ h = 1_P\).

**Proof.** Obviously the condition is necessary (from the definition of projective), and it is sufficient since: If \(M\) is any \(R\)-module which surjects onto \(P\), say \(\varphi : M \to P\), then there exists \(\varphi' : Q \to M\) such that \(\varphi \circ \varphi' = f\) since \(Q\) is a projective \(R\)-module.

\[
\begin{align*}
Q &
\quad \psi
\downarrow
M \\
\quad \quad \psi
\quad \quad f
\downarrow
P
\end{align*}
\]

Letting \(\psi := \varphi' \circ h\), we get \(\varphi \circ \psi = \varphi \circ \varphi' \circ h = f \circ h = 1_P\), hence \(P\) is projective. □

**Lemma 1.14.** Let \(\Lambda\) be a commutative ring, and let \(P\) be an \(\Lambda[G]\) module. Then \(P\) is projective over \(\Lambda[G]\) if and only if \(P\) is projective over \(\Lambda\) and there exists an \(\Lambda\)-endomorphism \(u\) of \(P\) such that \(\sum_{s \in G} su(s^{-1}x) = x \quad \forall x \in P\).

**Proof.** If \(P\) is projective over \(\Lambda[G]\), then it is projective over \(\Lambda\), since \(\Lambda[G] = \bigoplus_{g \in G} \Lambda g\) and hence

\[
P \oplus P' \cong \bigoplus_i \Lambda[G] = \bigoplus_i \bigoplus_{g \in G} \Lambda g.
\]

Conversely, let suppose the underlying \(\Lambda\)-module \(P_0\) of \(P\) is projective, and set \(Q = \Lambda[G] \otimes_{\Lambda} P_0\), which is a projective \(\Lambda[G]\)-module by [Lemma 1.12](#). The identity map \(P_0 \to P\) extends to a surjective \(\Lambda[G]\)-homomorphism

\[
f : Q \longrightarrow P
\]

\[
\sum_{g \in G} a_g g \otimes_{\Lambda} p \longmapsto \sum_{g \in G} a_g g \cdot p.
\]

Since \(Q\) is a projective \(\Lambda[G]\)-module, it follows that \(P\) is a projective \(\Lambda[G]\)-module if and only if there exists an \(\Lambda[G]\)-module homomorphism \(h : P \to Q\) with \(f \circ h = 1_P\) by [Lemma 1.13](#).

Since \(\Lambda[G]\) is a free \(\Lambda\)-module with basis \(G\), an arbitrary element of \(Q\) has a **unique** expression the form

\[
\sum_{g \in G} g \otimes_{\Lambda} x_g.
\]
The expression is unique in the sense that \( \sum_{g \in G} g \otimes_A x_g = \sum_{g \in G} g \otimes_A x'_g \) implies \( x_g = x'_g \) for all \( g \in G \). So, any \( \Lambda[G] \)-homomorphism from \( P \) to \( Q \) takes the form

\[
p \mapsto \sum_{g \in G} g \otimes_A p_g.
\]

From the \( \Lambda[G] \)-homomorphism \( h : P \to Q \), we may define a map \( u : P_0 \to P_0 \) by the rule \( g^{-1}p \mapsto p_g \). In particular, \( p \mapsto p_1 \) via \( u \), where \( 1 \) is the identity of \( G \). Then \( u \) is in fact an element of \( \text{End}_A(P_0) \): For \( p, p' \in P_0 \) and \( r \in \Lambda \), we have

\[
\sum_{g \in G} g \otimes_A (p + rp') = \sum_{g \in G} g \otimes_A (p_g + r(p_g))
\]

since \( h \) is a \( \Lambda \)-homomorphism. Since these expressions are unique, we have \( (p + rp)_1 = p_1 + rp' \), and so \( u \) is a \( \Lambda \)-homomorphism from \( P_0 \) to itself. Then \( h \) is given by

\[
p \mapsto \sum_{g \in G} g \otimes_A u(g^{-1}p).
\]

So \( f \circ h = 1_P \) if and only if

\[
p = f(h(p)) = f \left( \sum_{g \in G} g \otimes_A u(g^{-1}p) \right) = \sum_{g \in G} gu(g^{-1}p)
\]

for all \( p \in P \). \( \square \)

**Lemma 1.15.** Let \( R \) be a commutative ring, let \( I \) an ideal of \( R \), and let \( M \) be an \( R \)-module. Then

\[
(R/I) \otimes_R M \cong M/IM
\]

as \( R/I \)-modules. In other words, extension of scalars of \( M \) from \( R \) to \( R/I \) along the natural projection yields the same \( R/I \)-module as the quotient \( M/IM \).

**Proof.** We provide only a sketch. The map \( f : (R/I) \otimes_R M \to M/IM \) given by \( \overline{r} \otimes_R m \mapsto \overline{rm} + IM \) is well-defined \( R/I \)-module homomorphism, and has well-defined \( R/I \)-homomorphism inverse \( h : M/IM \to (R/I) \otimes_R M \) given by \( \overline{m} \mapsto \overline{1} \otimes_R m \). \( \square \)

**Lemma 1.16.** Let \( \Lambda \) be a local ring with residue field \( k_\Lambda = \Lambda/m \).

(a) Let \( P \) be an \( \Lambda[G] \)-module which is free over \( \Lambda \). Then \( P \) is projective over \( \Lambda[G] \) if and only if the \( k_\Lambda[G] \)-module \( \overline{P} : = k_\Lambda \otimes_\Lambda P \) is projective.

(b) Two projective \( \Lambda[G] \)-modules \( P \) and \( P' \) are isomorphic if and only if the corresponding \( k_\Lambda[G] \)-modules \( \overline{P} : = k_\Lambda \otimes_\Lambda P \) and \( \overline{P'} : = k_\Lambda \otimes_\Lambda P' \) are isomorphic.

**Proof.**

(a) If \( P \) is projective over \( \Lambda[G] \), then \( P \) is projective over \( \Lambda \). The extension of scalars (via the natural map \( \Lambda \to k_\Lambda \)) of the projective \( \Lambda \)-module \( P \) to the \( k_\Lambda \)-module \( \overline{P} = k_\Lambda \otimes_\Lambda P \) is projective by **Lemma 1.12**.

Conversely, if the \( k_\Lambda \)-module \( \overline{P} = k_\Lambda \otimes_\Lambda P \) is projective, then **Lemma 1.14** shows that there exists a \( k_\Lambda \)-endomorphism \( \overline{u} \) of \( \overline{P} \) such that

\[
\sum_{g \in G} g\overline{u}g^{-1} = \text{id}_{\overline{P}}.
\]
By Lemma 1.15, we may consider $p_\Lambda$ as the $k\Lambda$-module $P/mP$. We have a diagram

$$P \xrightarrow{\pi} \mathcal{P} \cong P/mP$$

which can be completed by a map $u : P \rightarrow P$, i.e. $\pi \circ u = \pi \circ \pi$. So for all $p \in P$, we have $u(p) + mP = \pi(p + mP)$. Then

$$\sum_{g \in G} gu(g^{-1}p) \equiv \sum_{g \in G} g\pi(g^{-1}p + mP) = p + mP \pmod{mP},$$

i.e. we have $u' := \sum_{g \in G} gug^{-1} \equiv 1_P \pmod{mP}$. So $u'$ is an automorphism of $P$, and moreover, $u'$ commutes with $G$: For all $g' \in G$, we have

$$u'g' = \sum_{g \in G} gug^{-1}g' = \sum_{g \in G} (g'g^{-1}) u(g^{-1}g') = \sum_{g \in G} gug^{-1} = g'u.$$

So

$$\sum_{g \in G} guu'g^{-1}(g^{-1}p) = \left(\sum_{g \in G} gug^{-1}\right) u^{-1}(p) = u'u^{-1}(p) = p$$

for all $p \in P$. Since $P$ is also projective over $\Lambda$ (as it is $\Lambda$-free), then $P$ is a projective $\Lambda[G]$-module by Lemma 1.14.

(b) It's clear that $P \cong P'$ implies $\mathcal{P} \cong \mathcal{P}'$. Conversely, let $\pi : \mathcal{P} \rightarrow \mathcal{P}'$ be a $\Lambda[G]$-isomorphism. Let $\pi : P \rightarrow P/mP \cong \mathcal{P}$ and $\pi' : P' \rightarrow P'/mP' \cong \mathcal{P}'$ be the natural maps. We have a diagram

$$P \xrightarrow{\pi \circ \pi} \mathcal{P}$$

and since $P$ is projective, there exits a $G$-homomorphism $w : P \rightarrow P'$ such that $\pi' \circ w = \pi \circ \pi$. Notice that for any $x \in P$, there exists $y \in P'$ so that $w(y) + mP' = \pi(x + mP)$.

We now return to the ring $O$:

**Proposition 1.17.**

(a) Let $E$ be an $O[G]$-module. Then $E$ is projective over $O[G]$ if and only if $E$ is free over $O$ and that $\overline{E} = E/mE$ is a projective $k[G]$-module.

(b) If $F$ is a projective $k[G]$ module, there exists a unique (up to isomorphism) projective $O[G]$-module whose reduction mod $m$ is isomorphic to $F$.

**Proof.**

(a) Suppose $E$ is projective over $O[G]$. Then $E$ is projective over the principal ideal domain $O$, and hence free over $O$. By Lemma 1.16, $\overline{E} = E/mE$ is a projective $k[G]$-module. Conversely, if $\overline{E} = E/mE$ is a projective
$k[G]$-module which is also free over $O$, then Lemma 1.16 says that $E$ is a projective $O[G]$-module.

(b) See Serre for existence. Uniqueness follows from Lemma 1.16.

\[\square\]

**Corollary 1.18.** $P \cong Q$ as projective $O[G]$-modules if and only if $P/mP \cong Q/mQ$ as projective $k[G]$-modules.

**Proof.** If $P \cong Q$, then obviously $P/mP \cong Q/mQ$. Conversely, we clearly have that $P$ and $Q$ are $O[G]$-modules whose reductions mod $m$ are $P/mP$ and $Q/mQ$, respectively. So if $P/mP \cong Q/mQ$, then by the uniqueness in Proposition 1.17(b), we have $P \cong Q$.

\[\square\]

**Corollary 1.19.** Every projective $O[G]$-module is a direct sum of projective indecomposable $O[G]$-modules; this decomposition is unique up to isomorphism.

A projective indecomposable $O[G]$-module is characterized up to isomorphism by its reduction mod $m$ which is a projective indecomposable $k[G]$-module (i.e. the projective cover of a simple $k[G]$-module).

**Proof.** Let $P$ be a projective $O[G]$-module. Then $P$ is projective over the local ring $O$, and hence free over $O$ (since projective modules are free over such rings). Let $\mathcal{P} = P/mP$ be its reduction mod $m$ to a $k[G]$-module. By Proposition 1.17(a), $\mathcal{P}$ is a projective $k[G]$-module, and hence by Corollary 1.19 $\mathcal{P}$ has a unique decomposition into indecomposable projective $k[G]$-modules, $\mathcal{P} = \bigoplus_i P_i$.

For all $i$, there exists a unique projective $O[G]$-module $Q_i$ such that $Q_i/mQ_i = \widehat{Q}_i \cong P_i$ by Proposition 1.17(b). So

$$P/mP = \mathcal{P} \cong \bigoplus_i \widehat{Q}_i = \bigoplus_i Q_i/mQ_i \cong \bigoplus_i Q_i/mQ_i = \bigoplus_i Q_i,$$

so $P \cong \bigoplus Q_i$, and this decomposition is unique since the $Q_i$ are unique. Moreover, the $Q_i$ are indecomposable since any decomposition $Q_i = A \oplus B$ would give to a decomposition of

$$P_i \cong Q_i/mQ_i = \frac{A \oplus B}{mA \oplus mB} \cong \frac{A}{mA} \oplus \frac{B}{mB}$$

which is indecomposable.

The last statement of the corollary follows from Corollary 1.18 and the fact that if $P$ is an indecomposable $O[G]$-module and we write $P \cong \bigoplus Q_i$, then $P \cong Q_i$ for some $i$ since $P$ is indecomposable, and hence $P/mP \cong \widehat{Q}_i$, i.e. $P/mP$ is indecomposable.

\[\square\]

**Corollary 1.20.** Two projective $O[G]$-modules $P, Q$ are isomorphic if and only if $[P] = [Q]$ in $P_O(G)$.

**Proof.** We have already seen that $P \cong Q$ implies $[P] = [Q]$.

We may argue in essentially the same way as in the proof for Corollary 1.10 to see that the classes $\{[P]\}$ of projective indecomposable $O[G]$-modules form a basis for $P_O(G)$: Indeed, the proof of Corollary 1.10 depends only on the fact that each projective $k[G]$-module has a unique decomposition into projective indecomposable $k[G]$-modules, which we have now for $O[G]$-modules by Corollary 1.19. The proof now follows in essentially the same way as in that of Corollary 1.10.

\[\square\]
Corollary 1.21. Reduction mod \( m \) defines an isomorphism from \( P_{\mathcal{O}}(G) \) to \( P_k(G) \); this isomorphism maps \( P^+_\mathcal{O}(G) \) to \( P^+_k(G) \).

Proof. Indeed, both \( P_{\mathcal{O}}(G) \) and \( P_k(G) \) have as a basis the classes of indecomposable projective modules, and since reduction mod \( m \) sends a projective indecomposable \( \mathcal{O}[G] \)-module to a projective indecomposable \( k[G] \)-module, reduction mod \( m \) sends the basis of \( P_{\mathcal{O}}(G) \) to that of \( P_k(G) \). The last statement of the corollary is clear. \( \square \)

2. The cde triangle

We will define group homomorphisms which form a commutative triangle:

\[
\begin{array}{ccc}
P_k(G) & \xrightarrow{c} & R_k(G) \\
\downarrow{c} & & \downarrow{d} \\
R_k(G) & & \\
\end{array}
\]

2.1. Definition of \( c : P_k(G) \to R_k(G) \).

This map is essentially inclusion, sending the class of projective \( k[G] \)-module in \( P_k(G) \) to its class in \( R_k(G) \). Evidently \( c \) is an additive homomorphism, and is called the Cartan homomorphism.

Since \( P_k(G) \) and \( R_k(G) \) have bases of size \( S_k \), we obtain a matrix \( C \in \mathbb{M}_{|S_k| \times |S_k|}(\mathbb{Z}_{\geq 0}) \) corresponding to \( c \), called the Cartan matrix. In particular, for an indecomposable projective \( k[G] \)-module \( P_T \) (say \( P_T \) is the projective envelope for a simple \( k[G] \)-module \( T \)), we can write

\[
[P_T] = \sum_{S \in S_k} \ell_S(P_T)[S],
\]

where \( \ell_S(P_T) \in \mathbb{Z}_{\geq 0} \) is the number of times the simple \( k[G] \)-module \( S \) appears in a composition series for \( P_T \). Then \( C_{ST} := \ell_S(P_T) \) is the \((S,T)\) entry of the Cartan matrix \( C \).

2.2. Definition of \( d : R_k(G) \to R_k(G) \).

Definition 2.1. Let \( E \) be a \( K[G] \)-module. A finitely generated \( \mathcal{O} \)-submodule \( E_1 \) of \( E \) such that \( E_1 \) generates \( E \) as a \( K \)-vector space is called a lattice in \( E \). In other words, \( E_1 \subseteq E \) is an \( \mathcal{O} \)-submodule and

\[
E = \left\{ \sum_{\text{finite}} \lambda_i e_i \mid \lambda_i \in K, e_i \in E_1 \right\}.
\]

We may assume that \( E_1 \) is stable under the action of \( G \), because

\[
E'_1 := \sum_{g \in G} gE_1
\]

is a \( G \)-stable subset of \( E \), and \( E'_1 \subseteq E \) is an \( \mathcal{O} \)-submodule:

\[
aE'_1 = \sum_{g \in G} gaE_1 = \sum_{g \in G} gE_1,
\]

for all \( a \in \mathcal{O} \). Since \( E_1 \subseteq E'_1 \), then \( E'_1 \) generates \( E \) as a \( K \)-vector space. Hence we may assume that \( E_1 \) is a \( \mathcal{O}[G] \)-submodule of \( E \), and so the reduction \( \overline{E}_1 = E_1/mE_1 \)

is a \( \mathcal{O}[G] \)-submodule of \( E \).
is then a $k[G]$-module. We may define a map
\[ d : R_k(G) \longrightarrow R_k(G) \]
\[ E \mapsto [E_1], \]
called the \textit{decomposition homomorphism}. Of course, we need to check that $d$ is well-defined:

\textbf{Theorem 2.2.} If $E_2$ is another $G$-stable lattice for $E$, then $[E_1] = [E_2]$.

\textit{Proof.} First note that we may replace $E_2$ by $mE_2$ if we begin by proving the special case where $mE_1 \subseteq E_2 \subseteq E_1$. Let $T$ be the $k[G]$-module $E_1/E_2$. Let $\pi$ be a generator for $m$. Consider the sequence of $k[G]$-modules
\[ 0 \rightarrow T \xrightarrow{\pi} \bar{E}_2 \rightarrow \bar{E}_1 \rightarrow T \rightarrow 0, \]
where $\pi : T \rightarrow \bar{E}_2$ is given by $e + E_2 \mapsto \pi e + mE_2$, the middle map is inclusion, and the last map is $e + mE_1 \mapsto e + E_2$. If
\[ 0 + mE_2 = \pi \cdot (e + E_2) = \pi e + mE_2, \]
then $e \in E_2$ so $\pi \cdot$ is injective and hence the sequence is exact at the first $T$. Now $\pi e + mE_1 = 0 + mE_1$ so the image of $\pi \cdot$ is in the kernel of the middle map, and conversely, an element $e' + mE_2$ in the kernel of the middle map must have $e' \in mE_1$, so
\[ e' = \sum_i \pi a_i e_i = \pi \left( \sum_i a_i e_i \right) \quad \text{where} \quad a_i \in O, e_i \in E_1. \]
Hence $e' + mE_2$ is in the image of $\pi \cdot$, and the sequence is exact at $\bar{E}_2$. Since $e + E_2 = 0 + E_2$ for all $e \in E_2$, then the image of $\bar{E}_2 \rightarrow \bar{E}_1$ is contained in the kernel of $\bar{E}_1 \rightarrow T$. Conversely, and element $e + E_2$ is in the kernel of $\bar{E}_1 \rightarrow T$ if and only if $e \in E_2$, so the sequence is exact at $\bar{E}_1$. Finally, $\bar{E}_1 \rightarrow T$ is obviously surjective. Hence the sequence is exact, and so passing to $R_k(G)$,
\[ [T] - [\bar{E}_2] + [\bar{E}_1] - [T] = 0, \]
thus $[\bar{E}_1] = [\bar{E}_2]$ in this case. \hfill $\Box$

2.3. \textbf{Definition of $e : P_k(G) \rightarrow R_k(G)$.}

2.4. \textbf{Basic properties of the cde triangle.}

3. \textbf{Theorems}

3.1. \textbf{Properties of the cde triangle.}

3.2. \textbf{Characterization of the image of $e$.}

3.3. \textbf{Characterization of $O[G]$-modules by their characters.}

4. \textbf{Proofs}

4.1. \textbf{Change of groups.}

4.2. \textbf{Brauer's theorem in the modular case.}
5. Modular characters

5.1. The modular character of a representation.

5.2. Independence of module characters.

5.3. Reformulations.

5.4. A section for \( d \).

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