Engineering Math II

Integrating over non-rectangular regions

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If we have a function of two variables, $f(x, y)$, and we want to integrate over a rectangular region given by $\mathbb{D} = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then we would compute

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx \quad \text{or} \quad \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Example 1. Integrate $f(x, y) = y\sin(x)$ over the rectangle where $0 \leq x \leq \frac{\pi}{2}$ and $-2 \leq y \leq 1$, or in other words, $\mathbb{D} = \{(x, y) : 0 \leq x \leq \frac{\pi}{2}, -2 \leq y \leq 1\}$[1].

Solution:

$$\begin{aligned}
\iint_{\mathbb{D}} f(x,y)da = \int_0^{\pi/2} \left(\int_{-2}^1 y\sin(x)dy \right) dx &= \int_0^{\pi/2} \left(\frac{y^2}{2}\sin(x) \right) \Big|_{y=-2}^{y=1} dx \\
&= \int_0^{\pi/2} \left(\frac{1}{2}\sin(x) - 2\sin(x) \right) dx \\
&= \left(-\frac{1}{2}\cos(x) + 2\cos(x) \right) \Big|_{x=0}^{x=\pi/2} \\
&= \left(-\frac{1}{2}\cos\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right) \right) - \left(-\frac{1}{2}\cos(0) + 2\cos(0) \right) \\
&= -\frac{3}{2}.
\end{aligned}$$

We could have just as easily computed the integral with the variables in the other order:

$$\iint_{\mathbb{D}} f(x,y)da = \int_{-2}^1 \left(\int_0^{\frac{\pi}{2}} y\sin(x)dx \right) dy.$$

That wasn't too bad! We just use the given ranges of our rectangle for x and y as our limits of integration. But it gets a bit more tricky when our region of integration is not a rectangle. For the rest of these notes, we will focus on explaining how to properly set up the integral when we integrate over non-rectangular regions.

Here are the steps I take when computing integrals over a region which is not a rectangle:

1. On an xy-plane, find the region we want to integrate over by plotting the graph(s) of the function(s) which describe the region. (We are NOT plotting the function we want to integrate, but instead, we are plotting the graph(s) of the function(s) which describe the region over which we want to integrate).

2. Fix an x-value (and get a vertical line) (or fix a y-value (and get a horizontal line); just pick one! While the choice doesn't really matter, it is sometimes easier to fix one variable over another. A comparison of Example 2 and Example 3 will illustrate this point.)

3. Find limits (i.e., the lowest and highest values) for y, going from bottom to top, *in terms of* x (and if we chose to fix y first in Step 2, we would find limits for x, going from left to right, *in terms of* y).

4. Find limits for x. These *must* be numbers! No variables allowed here! This will be the outermost integral. (and if we found limits for x in Step 3, we would find numerical limits for y).

5. Put it all together. Fill these expressions/numbers in for the limits of integration, and integrate.

$$\int_{x=\text{left}}^{x=\text{right}} \left(\int_{y=\text{bottom}}^{y=\text{top}} f(x,y)dy \right) dx \text{ and in the second case, } \int_{y=\text{bottom}}^{y=\text{top}} \left(\int_{x=\text{left}}^{x=\text{right}} f(x,y)dx \right) dy.$$

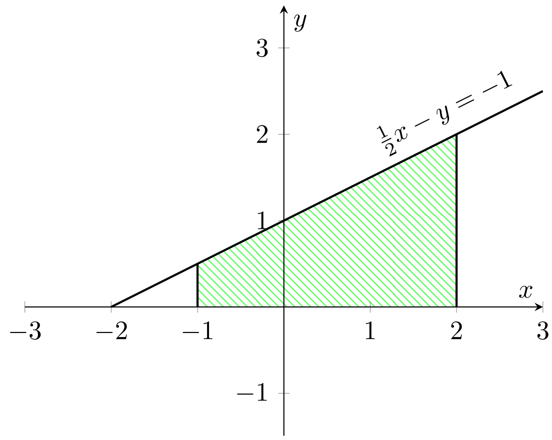
[1]We read this: " $\mathbb{D}$ is the set of all ordered pars (x, y) such that x is between 0 and $\frac{\pi}{2}$, and y is between -2 and 1."

You probably just thought "blah blah blah, just show me an example. k thx". Chill bruh, I got plenny.

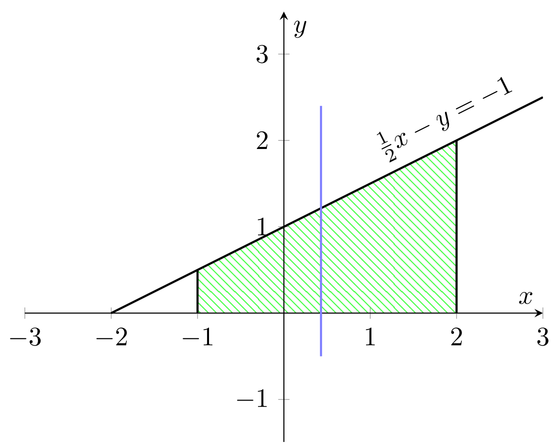
Example 2. Integrate the function $f(x, y)$ over the region between the line $\frac{1}{2}x - y = -1$ and the x-axis, for x between -1 and 2. The region is $\mathbb{D} = \{(x, y) : -1 \leq x \leq 2, y \geq 0, \frac{1}{2}x - y \leq -1\}$.

Solution:

1. Find the region. We graph the line, and only consider points between the line and the x-axis, and only for x-values between -1 and 2.



2. Fix an x-value. We draw a vertical line at the fixed x-value. Here, we are just picking a random x-value. It doesn't matter which x-value we pick; this is just a helpful way to think through the limits for y, which we will find in Step 3.



3. Find limits for y, going from *bottom* to *top*, in terms of x. Looking at the vertical line we drew in Step 2, the smallest that y can be alone the vertical line in our shaded region is 0. Again looking at our vertical line, the largest that y can be is a point on the line $\frac{1}{2}x - y = -1$. However, we are looking for a value for y, which means that the value we're looking for can't have y in it. We need to describe the highest value for y *in terms of* x. So we solve for y to get our top limit for y to be $y = \frac{1}{2}x + 1$.

4. Find limits for x. This one is easy, since we are given explicit limits for x in the problem.

5. Put it all together:

$$\int_{x=-1}^{x=2} \left(\int_{y=0}^{y=\frac{1}{2}x+1} f(x, y) dy \right) dx.$$

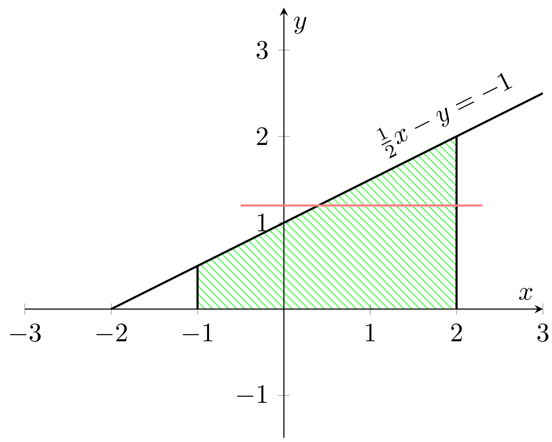
Let's do the same example, but do it the other way.

Example 3. Integrate the function $f(x, y)$ over the region between the line $\frac{1}{2}x - y = -1$ and the x-axis, for x between -1 and 2. The region is $\mathbb{D} = \{(x, y) : -1 \leq x \leq 2, y \geq 0, \frac{1}{2}x - y \leq -1\}$.

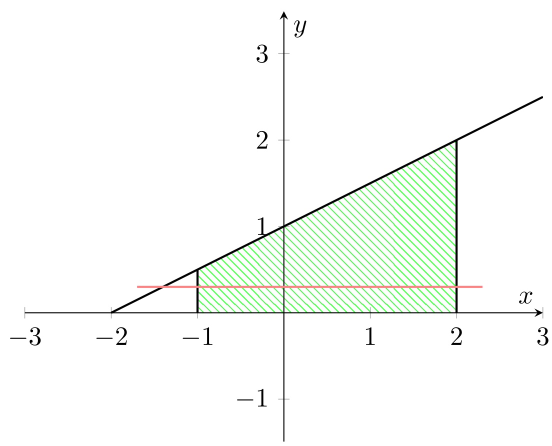
Solution:

1. Find the region.

2. Fix a y-value.



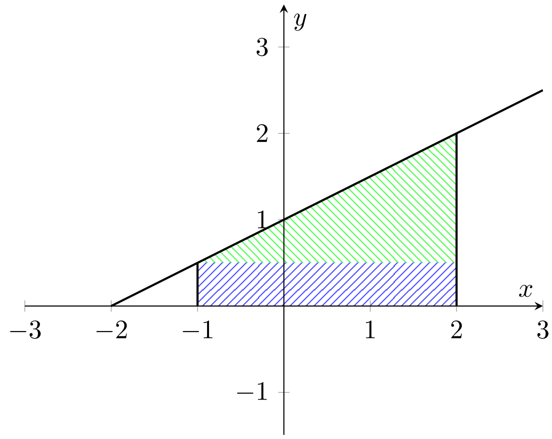
3. Find limits for x, going from *left* to *right*, in terms of y. Looking at the horizontal line we drew in Step 2, the smallest that x can be in our shaded region is described by the line $\frac{1}{2}x - y = -1$. Solving for x, we get $x = 2y - 2$. But wait. What if we had fixed a smaller y-value, and got a horizontal line like this:



In this case, the smallest that x can be in our region along the horizontal line is -1, which has nothing to do with the line $\frac{1}{2}x - y = -1$. In both cases, the largest that x can be is 2. So we have now two different sets of limits of integration for x: We have $x = -1, x = 2$ and we also have $x = 2y - 2, x = 2$. We need to somehow get around the issue that different choices for fixed y give us different limits of integration for x. The way to remedy this issue is to split up the integral into a sum, considering each set of limits of integration for x:

$$\int_{y=\text{bottom}}^{y=\text{top}} \left(\int_{x=2y-2}^{x=2} f(x, y) dx \right) dy + \int_{y=\text{bottom}}^{y=\text{top}} \left(\int_{x=-1}^{x=2} f(x, y) dx \right) dy$$

We are now integrating the function over two regions, and then adding them up:



The top region shaded above corresponds to the first integral in the sum. Any horizontal line drawn in this region has limits for x given by $x = 2y - 2$ and $x = 2$. Similarly, the bottom region corresponds to the second integral in the sum, and it has limits for x given by -1 and 2.

4. Find limits for y. From the graph, we see that the smallest y can be is 0 and the highest y can be is when the line $\frac{1}{2}x - y = -1$ intersects the vertical line $x = 2$. To find where they intersect, plug in $x = 2$ into the equation of the line, to obtain $y = 2$. (Another way to think about this step is: "What are all the possible values for y that I could have used in Step 2 when I fixed a y value? ")

   Notice that we are not considering the given line or any vertical/horizontal line or anything else in this step. We dealt with all that in the previous steps, and we just want numerical values for y to finish off the setup of the integral.

5. Put it all together:

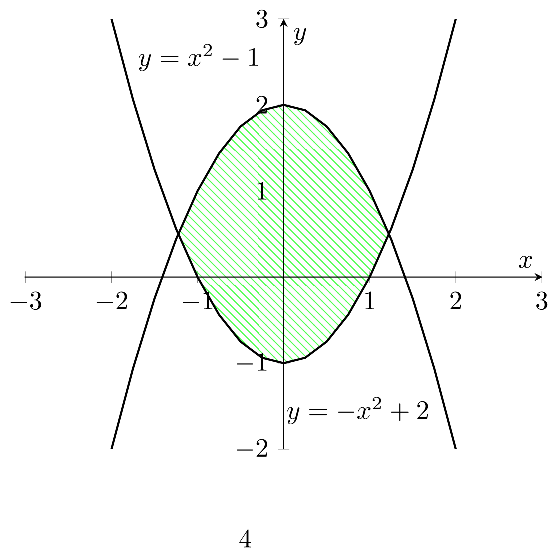
$$\int_{y=0}^{y=2} \left(\int_{x=2y-2}^{x=2} f(x,y)dx \right) dy + \int_{y=0}^{y=2} \left(\int_{x=-1}^{x=2} f(x,y)dx \right) dy.$$

Let's do a more challenging example:

Example 4 (The Egg Problem)**.** Integrate $f(x, y)$ over the region between the curves $y = x^2 - 1$ and $y = -x^2 + 2$.

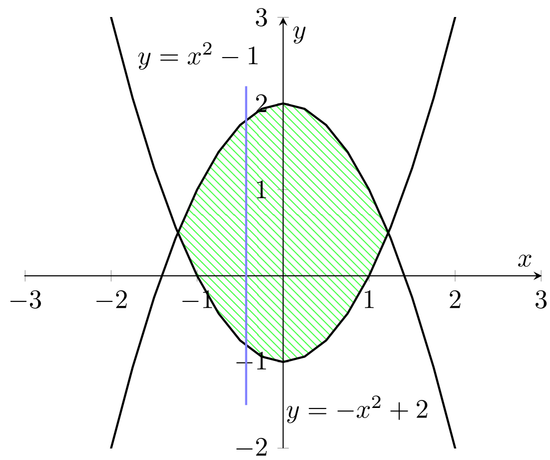
Solution:

1. Find the region by graphing the curves.



2. Which variable should we fix? Which choice would make the problem easier, so that we don't have to split up the region? If we fix y, the range of x-values must be given in terms of y. But the range of x-values in terms of y will change about halfway up the egg. So if we fix y, then we would not only have to solve for x in the equations of the curves (which would be especially nasty since a square root would pop up), but it would also require us to split the egg into two regions, and then add two integrals together. So let's not crack the egg in two.

   Instead, fix an x to get a vertical line for our helpful visual aid:



3. Find limits for y from bottom to top, in terms of x. The smallest y can be on our vertical line in the region in terms of x is given by the curve $y = x^2 - 1$, and the largest y can be on our vertical line in the region in terms of x is given by the curve $y = -x^2 + 2$. So far, our double integral is

$$\int_{x=\text{left}}^{x=\text{right}} \left(\int_{y=x^2-1}^{y=-x^2+2} f(x,y)dy \right) dx$$

4. Find limits for x. This will take a little work. Remember, we need numbers now; we don't have to describe the limits for x in terms of y. Reading values off a graph is not always possible nor accurate. We can see that the leftmost and rightmost values x can take in our region are precisely the two values for x that make the functions $y = -x^2 + 2$ and $y = x^2 - 1$ take on the same y-value (in other words, where the graphs intersect). So let's do some algebra to find what these numbers are:

$$\begin{aligned} x^2 - 1 &= -x^2 + 2 \\ 2x^2 &= 3 \\ x^2 &= \frac{3}{2} \\ x &= \pm\sqrt{\frac{3}{2}}. \end{aligned}$$

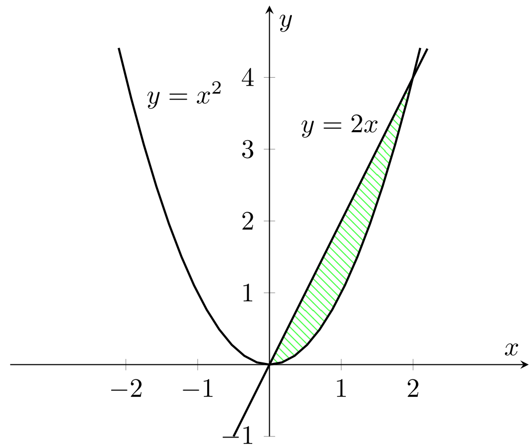
5. Put it all together:

$$\int_{x=-\sqrt{\frac{3}{2}}}^{x=\sqrt{\frac{3}{2}}} \left(\int_{y=x^2-1}^{y=-x^2+2} f(x,y)dy \right) dx.$$

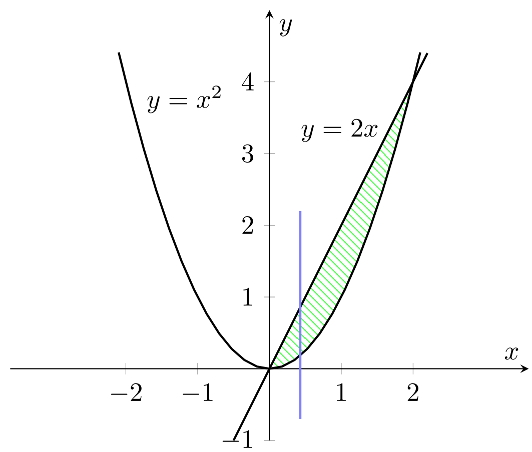
Let's do one more example, both ways:

Example 5. Integrate $f(x, y)$ over the region which is between the two curves $y = x^2$ and $y = 2x$ for x between 0 and 2.

1. Find the region



2. Fix x.



3. Find limits for y from bottom to top, in terms of x. The lowest y can be on our vertical line in our region is determined by the curve $y = x^2$. The highest y can be on our vertical line is determined by the line $y = 2x$.

4. Find limits for x. Numerical values. We can tell from the picture that the graphs intersect when $x = 0$ and when $x = 2$. And also the problem statement gives us the limits for x. But let's make sure we know how to find these values algebraically: [2]

$$\begin{aligned} 2x &= x^2 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0. \end{aligned}$$

   When the product of two things equals zero, either both or one of those things is zero, so $x = 0$ and $x - 2 = 2$, and so we get $x = 0$ and $x = 2$, as we guessed.

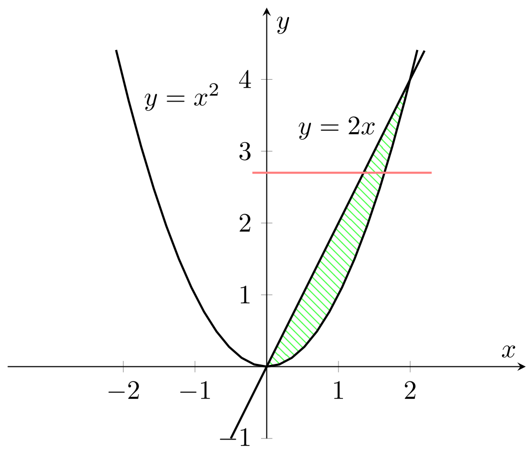
[2] Some problems in the future may only have a picture of the region along with the equations of the graphs, leaving us to find limits for x on our own. And we may not always be able to tell from the picture.

5. Put it all together:

$$\int_{x=0}^{x=2} \left(\int_{y=x^2}^{y=2x} f(x,y)dy \right) dx.$$

Now, the other way.

1. Find region.
2. Fix y.



3. Find limits for x from left to right, in terms of y. The leftmost that x can be on the horizontal line in our region is on the line $y = 2x$. But we need x *in terms of* y, so we solve for x and get the left limit $x = y/2$. The rightmost that x can be on the horizontal line in our region is on the line $y = x^2$. Again we solve for x and get $x = \pm\sqrt{y}$. So, which do we pick? $x = \sqrt{y}$ or $x = -\sqrt{y}$? Well, since x needs to be positive, we pick $x = \sqrt{y}$.
4. Find limits for y. We did the work in the previous case and found that the graphs intersect when $x = 0$ and when $x = 2$. But we need y-values now. So we plug in these numbers to either one of our equations (since the graphs intersect at these x-values) to get $y = 0$ and $y = 4$ to get the y-values which correspond to the intersection points.
5. Put it all together:

$$\int_{y=0}^{y=4} \left(\int_{x=\frac{y}{2}}^{x=\sqrt{y}} f(x,y)dx \right) dy.$$