College Algebra
Inequalities
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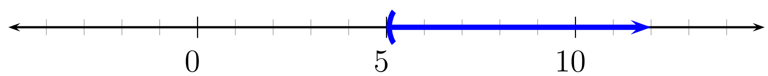
1 Linear Inequalities

1.1 Graphing Linear Inequalities

The first step in solving linear inequalities is converting a statement like $x > 5$ or $x \leq -10$ into an interval. Something that might help with this is graphing the inequality on a number line.

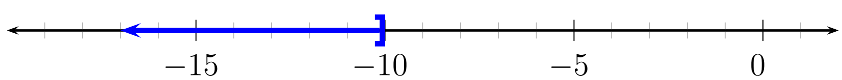
Example 1. $x > 5$ means that the variable x is *strictly* greater than 5.



Since the inequality is strict, we *do not* include 5 in our interval, so we use a parenthesis. Using the graph, we see that $x > 5$ means that we want our x-values to be in the interval

$$(5, \infty).$$

Example 2. $x \leq -10$ means that the variable x is less than *or equal to* -10.



Since the inequality is *not* strict, we *do* include -10 in our interval, so we use a bracket. Using the graph, we see that $x \leq -10$ means that we want our x-values to be in the interval

$$(-\infty, -10].$$

Hint: If you make sure that x is on the *left* side of the inequality, then we can use the direction that the inequality symbol is "pointing" to help us graph the inequality. In the first example, the " $>$ " symbol is "pointing" to the right, so we graph to the right of 5.

1.2 Solving Linear Inequalities

Solving linear inequalities is almost the same as solving linear equations. Most of the rules are the same, with some exceptions:

- When multiplying or dividing both sides of an inequality by a negative number, we must flip the inequality symbol.
- We cannot multiply/divide both sides of an inequality by expressions involving variables. (However, we *are* allowed to add/subtract expressions with variables on both sides of an inequality.)

Examples.

1. $3x - 2 \geq -6$.
   Solution:

$$\begin{aligned} 3x - 2 &\geq -8 & & \\ 3x &\geq -6 & &\text{(add 2 to both sides)} \\ x &\geq -2 & &\text{(divide both sides by 3)} \end{aligned}$$

   In interval notation, the final answer is $[-2, \infty)$.

2. $2(x+5) - 4x < 15 + 3x$.
   Solution:

$$\begin{aligned} 2(x+5) - 4x &< 15 + 3x & & \\ 2x + 10 - 4x &< 15 + 3x & &\text{(distribute)} \\ -2x + 10 &< 15 + 3x & &\text{(simplify left side)} \\ -5x + 10 &< 15 & &\text{(subtract } 3x \text{ from both sides)} \\ -5x &< 5 & &\text{(subtract 10 from both sides)} \\ x &> 1 & &\text{(divide both sides by } -5\text{, so flip the sign.)} \end{aligned}$$

   In interval notation, the final answer is $(1, \infty)$.

3. $3x - 5 < 3x + 1$.
   Solution:

$$\begin{aligned} 3x - 5 &< 3x + 1 & & \\ -5 &< 1. & &\text{(subtract } 3x \text{ from both sides)} \end{aligned}$$

   Since the final line is a *true* statement, our final answer is $(-\infty, \infty)$. In other words, *all* x values satisfy the original inequality.

4. $-2x + 10 \leq -2x + 5$.
   Solution:

$$\begin{aligned} -2x + 10 &\leq -2x + 5 & & \\ 10 &\leq 5. & &\text{(add } 2x \text{ to both sides)} \end{aligned}$$

   Since the final line is a *false* statement, our final answer is *no solution*, or $\varnothing$, the empty set. In other words, *no* values for x will satisfy the original inequality.

1.3 Combining Two Inequalities: Compound Inequalities

Compound inequalities are those which have *two* inequality symbols in one sentence at the same time. First, we look at problems where we are given two linear inequalities with the words "or" or "and" in between them. **"Or" means union. "And" means intersection.**

These are the two examples from the lecture notes:

Examples.

1. $2x + 7 \leq 1$ or $3x - 2 < 4(x - 1)$.
   Solution: Solving each linear inequality, we get
   $$x \leq -3 \text{ or } x > 2,$$
   which, in interval notation is
   $$(-\infty, -3] \cup (2, \infty).$$
   Because these intervals do not overlap, we cannot simply their union any further.

2. $2(x - 3) + 5 < 9$ and $3(1 - x) - 2 \leq 7$.
   Solution: Solving each linear inequality, we get
   $$x < 5 \text{ and } x \geq -2,$$
   which, in interval notation is
   $$(-\infty, 5) \cap [-2, \infty).$$
   Graphing these two intervals on a number line (do it!), we see that their intersection is
   $$(-\infty, 5) \cap [-2, \infty) = [-2, 5).$$

Now we look at compound inequalities that are written without an "or" or "and", but instead put all together in one line. All the rules are the same as before, except now we do all the operations to the "left, middle, and right", instead of just the "left and right" sides of an inequality. As always, the goal is to get the x by itself, except now we want to get the x by itself in the *middle*.

Examples.

1. $-6 \leq 4x - 2 < 4$.
   Solution:
   $$\begin{aligned} -6 &\leq 4x - 2 < 4 \\ -4 &\leq 4x < 6 && \text{(add 2 to the left, middle, \& right)} \\ -1 &\leq x < \frac{3}{2} && \text{(divide the left, middle, \& right by 4)} \end{aligned}$$
   Now we want to convert the last line to an interval.

   Remember this: When both inequality symbols are "pointing" left, and x is in the middle, the last line can be read "x is in between -1 and $\frac{3}{2}$". So our final answer in interval notation is
   $$\left[-1, \frac{3}{2}\right),$$
   because this interval contains all the points "in between -1 and $\frac{3}{2}$".

2. $-5 < 2x + 3 \leq 9$.
   Solution:

$$\begin{aligned} &-5 < 2x + 3 \leq 9 \\ &-8 < 2x \leq 6 \\ &-4 < x \leq 3. \end{aligned}$$

   So our final answer in interval notation is

$$(-4, 3] \, .$$

1.4 Quadratic & Rational Inequalities

What if an inequality has highest power on x equal to 2? What if an inequality involves a rational expression, i.e. has a fraction in it? Then we have to apply the "test point method":

Step 1 Get everything to the left hand side, leaving a 0 on the right. Simplify.

Step 2 Factor the numerator (and denominator, if there is one) as much as possible.

Step 3 Set each factor equal to zero, and solve for x in each equation. Plot these x-values on a number line.

Step 4 Pick test points in each interval determined by the x-values in Step 3. For all the test points that satisfy the criterion of the original inequality, keep the corresponding interval.

Step 5 To obtain the final answer in interval notation, take the union of all "kept" intervals, i.e. take the union of all the intervals whose corresponding test points satisfy the original inequality.

Examples.

1. Instead of re-typing an example from the book, I refer you to Example 10 in section 1.6 on page 152. Also look at the examples from the lecture notes on section 1.6.