Homework for
Ordinary Differential Equations

Nicholas Camacho
Department of Mathematics
University of Iowa
Fall 2017

Beware: Some solutions may be incorrect!

Exercise 1. Find the general solution $y(t)$ to the following second order scalar ODE.

$$\ddot{y} + \dot{y} - 6y = 2e^{2t}.$$

Solution:

The roots of the characteristic polynomial $r^2 + r - 6 = 0$ are $r = -3$ and $r = 2$. So our general solution is $y_h(t) = c_1 e^{-3t} + c_2 e^{2t}$.

Since our non-homogeneous term has the form ae^{ct}, then our particular solution will have the form $y_p(t) = t^s b e^{ct}$ for some s and b. If $s = 0$ then $y_p(t)$ will be linear dependent with a term in $y_h(t)$, and so $s = 1$. Now to find b, we solve the equation $\ddot{y_p} + \dot{y_p} - 6y_p - 2e^{2t} = 0$ and find that $b = 2/5$. So we get

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-3t} + c_2 e^{2t} + (2/5)te^{2t}.$$

Exercise 2. Consider the following scalar initial value problem (IVP).

$$\dot{x} = -tx^3, x(t_0) = x_0.$$

Find the solution $x(t, t_0, x_0)$ to this IVP explicitly in terms of t_0 and x_0. From the expression of $x(t, t_0, x_0)$ calculate directly $D_{x_0} x(t, t_0, x_0)$. In particular for $t_0 = 0$ and $x_0 = 2$ give the solution $x(t, 0, 2)$ and $D_{x_0} x(t, 0, 2)$.

Solution:

By a separation of variables, we get

$$\frac{dx}{dt} = -tx^3 \implies \int x^{-3} dx = -\int t dt \implies x(t) = \pm(t^2 + c)^{-1/2}.$$

for some arbitrary constant c. Considering the initial condition $x(t_0) = x_0$, we get

$$x(t) = \begin{cases} (t^2 + c)^{-1/2} & \text{if } x_0 > 0 \\ -(t^2 + c)^{-1/2} & \text{if } x_0 < 0. \end{cases}$$

Notice that $x_0 \neq 0$. Solving for c we get that $c = -t_0^2 + 1/x_0^2$. So

$$x(t, t_0, x_0) = \begin{cases} (t^2 - t_0^2 + x_0^{-2})^{-1/2} & \text{if } x_0 > 0 \\ -(t^2 - t_0^2 + x_0^{-2})^{-1/2} & \text{if } x_0 < 0. \end{cases}$$

Then

$$D_{x_0} x(t, t_0, x_0) = \begin{cases} x_0^{-3}(t^2 - t_0^2 - x_0^{-2})^{-3/2} & \text{if } x_0 > 0 \\ -x_0^{-3}(t^2 - t_0^2 - x_0^{-2})^{-3/2} & \text{if } x_0 < 0, \end{cases}$$

and

$$D_{x_0} x(t, 0, 2) = (1/8)(t^2 + 1/4)^{-3/2}.$$

Exercise 3. We consider systems of linear homogeneous autonomous ODEs $\dot{x} = Ax$ of dimension $n = 2$, i.e.,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

for the following 2 matrices:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Find the general solutions to these two systems of ODEs and draw approximately their phase portrait around the origin $x^* = (0,0) \in \mathbb{R}^2$.

Solution: The matrix A_1 has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with respective eignevectors $v_1 := [1\ 1]^T$ and $v_2 := [1\ -1]^T$. So the general solution to the system for A_1 is $x(t) = c_1 e^{3t} v_1 + c_2 e^{-t} v_2$. Since the eigenvalues are both real and are of opposite signs, the equilibrium solution is a saddle.

The matrix A_2 has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$ with respective eignevectors $v_1 := [1\ i]^T$ and $v_2 := [1\ -i]^T$. So the general solution to the system for A_2 is $x(t) = c_1 w_1 + c_2 w_2$ where $w_1 = \text{Re}(e^{(1+2i)t} v_1)$ and $w_1 = \text{Im}(e^{(1+2i)t} v_1)$. So,

$$\begin{aligned} e^{(1+2i)t} v_1 &= e^t(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^t \begin{bmatrix} \cos(2t) + i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} e^t\cos(2t) \\ -e^t\sin(2t) \end{bmatrix}}_{=w_1} + i \underbrace{\begin{bmatrix} e^t\sin(2t) \\ e^t\cos(2t) \end{bmatrix}}_{=w_2}. \end{aligned}$$

Since the eigenvalues are complex conjugates with $\text{Re}(1 \pm 2i) = 1 > 0$, the equilibrium solution is an unstable spiral. Choosing a position vector $u = [1\ 0]^T$ and considering Au, we get that the spiral is clockwise, since the velocity vector Au has negative second coordinate.

Exercise 4. Give the transition matrix $R(t, s)$ associated to $\dot{x} = Ax$ for the two matrices given in exercise 3. Remark: the transition matrix $R(t, s)$ associated to linear homogeneous autonomous ODEs $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$ satisfies $R(s, t) = R(t - s, 0)$. Note that this does not hold in general for linear homogeneous non-autonomous ODEs $\dot{x} = A(t)x$.

Solution: For A_1, we have the general solution $x(t) = c_1 e^{3t} v_1 + c_2 e^{-t} v_2$ for the system $\dot{x} = A_1 x$. So, if e_1 and e_2 denote the standard basis vectors in $\mathbb{R}^2$, then setting $x(t_0) = e_1$ gives $c_1 = 1/(2e^{3t_0})$ and $c_2 = e^{t_0}/2$. Setting $x(t_0) = e_2$, we get $c_1 = 1/(2e^{3t_0})$ and $c_2 = -e^{t_0}/2$. So

$$x(t, t_0, e_1) = \frac{1}{2} e^{3(t-t_0)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-t+t_0} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} e^{3(t-t_0)} + \frac{1}{2} e^{-t+t_0} \\ \\ \frac{1}{2} e^{3(t-t_0)} - \frac{1}{2} e^{-t+t_0} \end{bmatrix}$$

and

$$x(t, t_0, e_2) = \frac{1}{2} e^{3(t-t_0)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{-t+t_0} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} e^{3(t-t_0)} - \frac{1}{2} e^{-t+t_0} \\ \\ \frac{1}{2} e^{3(t-t_0)} + \frac{1}{2} e^{-t+t_0} \end{bmatrix}.$$

Then $R(t, t_0) = \big[x(t, t_0, e_1) \, \big| \, x(t, t_0, e_2) \big]$.

For A_2, we have the general solution $x(t) = c_1 w_1 + c_2 w_2$ for the system $\dot{x} = A_2 x$. Setting $x(0) = e_1$, we get $c_1 = 1$ and $c_2 = 0$; setting $x(0) = e_2$, we get that $c_1 = 0$ and $c_2 = 1$. So,

$$x(t, 0, e_1) = e^t \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}, \quad x(t, 0, e_2) = e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix},$$

and so

$$R(t, t_0) = R(t - t_0) = \begin{bmatrix} e^{t-t_0} \cos(2(t - t_0)) & e^{t-t_0} \sin(2(t - t_0)) \\ -e^{t-t_0} \sin(2(t - t_0)) & e^{t-t_0} \cos(2(t - t_0)) \end{bmatrix}.$$

Exercise 1.

(a) Give the transition matrix $R(t, t_0)$ for the system of linear homogeneous nonautonomous ODEs $\dot{x} = A(t)x$ given by

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution:

We have

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} tx_1 + x_2 \\ 0 \end{bmatrix},$$

which gives $x_2(t) = c$ for some constant c and $\dot{x_1} = tx_1 + c$. Using the integrating factor method to find $x_1(s)$, we have integrating factor $e^{\int -sdt} = e^{-s^2/2}$, and so

$$ce^{-s^2/2} = e^{-s^2/2}(\dot{x_1} - sx_1) = \frac{d}{ds}(x_1(s) \cdot e^{-s^2/2}).$$

Integrating on both sides,

$$c \int_{t_0}^{t} e^{-s^2/2} ds = (x_1(s) \cdot e^{-s^2/2})\big|_{s=t_0}^{s=t} = e^{-t^2/2} \cdot x_1(t) - e^{-t_0^2/2} \cdot x_1(t_0),$$

which gives

$$x_1(t) = e^{t^2/2} x_2(t_0) \int_{t_0}^{t} e^{-s^2/2} ds + e^{t^2/2 - t_0^2/2} x_1(t_0).$$

Finding $x(t, t_0, e_1)$, we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}.$$

Hence

$$x(t, t_0, e_1) = \begin{bmatrix} x_1(t, t_0, e_1) \\ x_2(t, t_0, e_1) \end{bmatrix} = \begin{bmatrix} e^{t^2/2 - t_0^2/2} \\ 0 \end{bmatrix}.$$

Similarly, we find that

$$x(t, t_0, e_2) = \begin{bmatrix} x_1(t, t_0, e_2) \\ x_2(t, t_0, e_2) \end{bmatrix} = \begin{bmatrix} e^{t^2} \int_{t_0}^{t} e^{-s^2/2} ds \\ 1 \end{bmatrix},$$

and then $R(t, t_0) = [x(t, t_0, e_1) | x(t, t_0, e_2)]$.

(b) For this system, verify that the Abel-Liouville-Jacobi-Ostroggradskii formula

$$\det(R(t, t_0)) = e^{\int_{t_0}^{t} Tr(A(s)) ds}$$

is satisfied.

Solution:

$$\det(R(t, t_0)) = e^{t^2/2 - t_0^2/2} = e^{\int_{t_0}^{t} sds} = e^{\int_{t_0}^{t} Tr(A(s)) ds}.$$

(c) For this system calculate the expression $e^{\int_{t_0}^t A(s)ds}$ using the definition of the exponential of a matrix and verify that it is not equal to the transition matrix $R(t, t_0)$.

Solution: We have

$$\int_{t_0}^t A(s)ds = \begin{bmatrix} \int_{t_0}^t sds & \int_{t_0}^t 1ds \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t^2/2 - t_0^2/2 & t - t_0 \\ 0 & 0 \end{bmatrix} = (t - t_0) \underbrace{\begin{bmatrix} \frac{t+t_0}{2} & 1 \\ 0 & 0 \end{bmatrix}}_{=:B}.$$

Notice that

$$B = \begin{bmatrix} \frac{t+t_0}{2} & 1 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -\frac{(t+t_0)}{2} \end{bmatrix}}_{=:P} \underbrace{\begin{bmatrix} \frac{(t+t_0)}{2} & 0 \\ 0 & 0 \end{bmatrix}}_{=:J} \underbrace{\begin{bmatrix} 1 & \frac{2}{t+t_0} \\ 0 & -\frac{2}{t+t_0} \end{bmatrix}}_{=P^{-1}},$$

and so

$$\begin{aligned} e^{\int_{t_0}^t A(s)ds} = e^{(t-t_0)B} = e^{(t-t_0)PJP^{-1}} &= \sum_{k\geq 0} \frac{(t-t_0)^k}{k!}(PJP^{-1})^k \\ &= P\left(\sum_{k\geq 0} \frac{(t-t_0)^k}{k!} J^k\right) P^{-1} \\ &= P \begin{bmatrix} e^{\frac{(t-t_0)(t+t_0)}{2}} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -\frac{(t+t_0)}{2} \end{bmatrix} \begin{bmatrix} e^{t^2/2 - t_0^2/2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{t+t_0} \\ 0 & -\frac{2}{t+t_0} \end{bmatrix} \\ &= \begin{bmatrix} e^{t^2/2 - t_0^2/2} & \frac{2e^{t^2/2 - t_0^2/2}}{t+t_0} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which is not equal to $R(t, t_0)$.

(d) Give the particular solution satisfying the following the initial conditions $x_1(2) = -1$ and $x_2(2) = 1$.

Solution:

$$\begin{bmatrix} x_1(t, 2, -1) \\ x_2(t, 2, 1) \end{bmatrix} = \begin{bmatrix} e^{t^2/2} \int_2^t e^{-s^2/2} ds - e^{(t^2/2)-2} \\ 1 \end{bmatrix}.$$

Exercise 2. Calculate the matrix exponential $e^{t-t_0}A$ for the following 2 matrices.

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Solution:
We have

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{=:P} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{=:J} \underbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{=P^{-1}}$$

So

$$\begin{aligned} e^{(t-t_0)A_1} = e^{(t-t_0)PJP^{-1}} &= P \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{3(t-t_0)} \end{bmatrix} P^{-1} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{3(t-t_0)} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{t-t_0} + e^{3(t-t_0)} & -e^{t-t_0} + e^{3(t-t_0)} \\ -e^{t-t_0} + e^{3(t-t_0)} & e^{t-t_0} + e^{3(t-t_0)} \end{bmatrix}. \end{aligned}$$

For A_2, notice that

$$A_2 = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = 2 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=:I} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:M}.$$

Then $M^{2\ell} = (-1)^\ell I$ and $M^{2\ell+1} = (-1)^\ell M$ for all $\ell \geq 0$, and so

$$\begin{aligned} e^{(t-t_0)M} &= \sum_{k\geq 0} \frac{(t-t_0)^k}{k!} M^k = \left(\sum_{\ell \geq 0} \frac{(-1)^\ell}{(2\ell)!} (t-t_0)^{2\ell} \right) I + \left(\sum_{\ell \geq 0} \frac{(-1)^\ell}{(2\ell+1)!} (t-t_0)^{2\ell+1} \right) M \\ &= I \cos(t-t_0) + M \sin(t-t_0) \\ &= \begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix}. \end{aligned}$$

We also have

$$e^{(t-t_0)2I} = \sum_{k\geq 0} \frac{(t-t_0)^k}{k!} (2I)^k = \begin{bmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{2(t-t_0)} \end{bmatrix}.$$

Putting it all together, we get

$$e^{(t-t_0)A_2} = e^{(t-t_0)2I} e^{(t-t_0)M} = \begin{bmatrix} e^{2(t-t_0)} \cos(t-t_0) & e^{2(t-t_0)} \sin(t-t_0) \\ -e^{2(t-t_0)} \sin(t-t_0) & e^{2(t-t_0)} \cos(t-t_0) \end{bmatrix}.$$

Exercise 3. Give the general solution to the following system of linear nonhomogeneous nonautonomous ODEs

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 2\sin(t)\end{bmatrix}.$$

Solution:

First we find a general solution to the homogeneous ODE $\dot{x} = A_1 x$. Letting $x_1(t_0) := x_{1,0}$, $x_2(t_0) := x_{2,0}$, and $x_0 = [x_{1,0}\ x_{2,0}]^T$, we have the following general solution to homogeneous ODE:

$$\begin{aligned} R(t,t_0)x_0 = e^{(t-t_0)A_1}x_0 &= \frac{1}{2}\begin{bmatrix} e^{t-t_0} + e^{3(t-t_0)} & -e^{t-t_0} + e^{3(t-t_0)} \\ -e^{t-t_0} + e^{3(t-t_0)} & e^{t-t_0} + e^{3(t-t_0)} \end{bmatrix}\begin{bmatrix}x_{1,0}\\ x_{2,0}\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} x_{1,0}(e^{t-t_0} + e^{3(t-t_0)}) + x_{2,0}(-e^{t-t_0} + e^{3(t-t_0)}) \\ x_{1,0}(-e^{t-t_0} + e^{3(t-t_0)}) + x_{2,0}(e^{t-t_0} + e^{3(t-t_0)}) \end{bmatrix}. \end{aligned}$$

A particular solution solution to $\dot{x} = A_1 x + b(t)$, where $b(t) = [0\ \ 2\sin(t)]^T$ is $\int_{t_0}^t R(t,s)b(s)ds$. So

$$R(t,s)b(s)ds = \begin{bmatrix} -\sin(s)e^{(t-s)} + \sin(s)e^{3(t-s)} \\ \sin(s)e^{(t-s)} + \sin(s)e^{3(t-s)} \end{bmatrix}.$$

and then

$$\begin{aligned} \int_{t_0}^t R(t,s)b(s)ds &= \begin{bmatrix} \displaystyle\int_{t_0}^t \Big(-\sin(s)e^{(t-s)} + \sin(s)e^{3(t-s)}\Big)ds \\ \displaystyle\int_{t_0}^t \Big(\sin(s)e^{(t-s)} + \sin(s)e^{3(t-s)}\Big)ds \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 2\sin t + 4\cos t - 5e^{t-t_0}(\sin t_0 + \cos t_0) + e^{3(t-t_0)}(3\sin t_0 + \cos t_0) \\ -8\sin t - 6\cos t + 5e^{t-t_0}(\sin t_0 + \cos t_0) + e^{3(t-t_0)}(3\sin t_0 + \cos t_0) \end{bmatrix} \end{aligned}$$

Then the general solution to the ODE $\dot{x} = A_1 x + b(x)$ is

$$x(t,t_0,x_0) = R(t,t_0)x_0 + \int_{t_0}^t R(t,s)b(s)ds.$$

Exercise 4. Consider the following scalar initial value problem

$$\dot{x} = -tx^3 =: f(t,x), x(0) = 2$$

with solution found in Exercise 2 of Homework 1. The matrix variational equation associated to this IVP is

$$\dot{R} = A(t)R$$

where $A(t) := D_x f(t, x(t,0,2))$ and $R(t) \in \mathbb{R}^{1\times 1}$. Solve this matrix variational equation explicitly for $R(t)$ (with $R(0) = 1$). Do we have $D_{x_0}x(t,0,2) = R(t)$ with $D_{x_0}x(t,0,2)$ as found in Exercise 2 of Homework 1?

Solution:

We have $D_x(f(t,x)) = -3tx^2$, and so

$$A(t) = D_x(f(t,x(t,0,2)) = -3t((t^2+1/4)^{-1/2})^2 = -3t(t^2+1/4)^{-1}.$$

So by a separation of variables, we get

$$\frac{dR}{dt} = \frac{-3tR}{t^2+1/4} \implies R(t) = \frac{c}{(t^2+1/4)^{3/2}}.$$

With the initial data $R(0) = 1$, we get $R(t) = (1/8)(t^2+1/4)^{-3/2}$, which in fact *does* equal $D_{x_0}x(t,0,2)$ from Exercise 2 of Homework 1.

1. A system of ODEs of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \nabla_{x_2} H(t, x_1, x_2) \\ -\nabla_{x_1} H(t, x_1, x_2) \end{bmatrix} \tag{1}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a called a **Hamiltonian system**. Assuming the Hamiltonian function $H(t, x_1, x_2)$ to be $\mathcal{C}^2$, i.e., twice continuously differentiable, show that the area of any set X_{t_0} of initial conditions in phase space $\mathbb{R}^2$ at some time t_0 is preserved under the evolution of the Hamiltonian system (1).

Proof. We have $\text{Vol}(X_t) = \int_{X_{t_0}} e^{\int_{t_0}^{t} Tr(D_x(f(s,x(s,t_0,x_0)))ds} dx_0$. Letting

$$f(t, x_1, x_2) = \begin{bmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2), \end{bmatrix} := \begin{bmatrix} \dfrac{\partial H}{\partial x_2}(t, x_1, x_2) \\ \\ -\dfrac{\partial H}{\partial x_1}(t, x_1, x_2) \end{bmatrix},$$

we get

$$D_x f(t, x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 H}{\partial x_1 x_2} & \dfrac{\partial^2 H}{\partial x_2^2} \\ \\ -\dfrac{\partial^2 H}{\partial x_1^2} & -\dfrac{\partial^2 H}{\partial x_2 x_1} \end{bmatrix}.$$

Since H is $\mathcal{C}^2$, mixed partials are the same, and so

$$Tr(D_x f(t, x)) = \frac{\partial^2 H}{\partial x_2 x_1} - \frac{\partial^2 H}{\partial x_1 x_2} = 0.$$

Hence

$$\text{Vol}(X_t) = \int_{X_{t_0}} e^0 dx_0 = \text{Vol}(X_{t_0}).$$

□

2. Consider the **planar pendulum equations** (q being the angle and p the angular momentum/velocity)

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} p \\ -\sin(q) \end{bmatrix}. \tag{2}$$

These differential equations form an example of a Hamiltonian system (1) with Hamiltonian $H(q,p) = p^2/2 - \cos(q)$. Initial conditions (q_0, p_0) at t_0 are assumed to be given. For a fixed stepsize h of integration consider the following 4 numerical schemes where (q_n, p_n) approximates the exact solution at $t_n = t_0 + nh$ for $n = 0, 1, 2, \ldots, N$

- the **explicit Euler method**

$$q_{n+1} = q_n + hp_n, \qquad p_{n+1} = p_n + h(-\sin(q_n));$$

- the **partitioned Euler method I**

$$q_{n+1} = q_n + hp_{n+1}, \qquad p_{n+1} = p_n + h(-\sin(q_n));$$

- the **partitioned Euler method II**

$$q_{n+1} = q_n + hp_n, \qquad p_{n+1} = p_n + h(-\sin(q_{n+1}));$$

- the **implicit Euler method**

$$q_{n+1} = q_n + hp_{n+1}, \qquad p_{n+1} = p_n + h(-\sin(q_{n+1})).$$

(a) Each of these schemes defines a mapping

$$\Phi_h : (q_n, p_n) \mapsto (q_{n+1}, p_{n+1})$$

depending on h. Which ones of these schemes satisfy

$$\det\left(D\Phi_h(q_n, p_n)\right) \stackrel{?}{=} 1$$

for all h (and for which the mapping Φ_h is defined)? For schemes satisfying this equality, the area in phase space $\mathbb{R}^2$ of any set X_{t_0} of initial conditions is preserved by the numerical flow, analogously to the exact flow of (2). Hint: use implicit differentiation when needed.

(b) For the planar pendulum equations (2), consider the initial conditions $q_0 = \pi/8$, $p_0 = 0$ at $t_0 = 0$. Write a simple code (in Matlab preferably) for the explicit Euler method and the partitioned Euler method I for $n = 0, 1, \ldots, N$ with $N = 1000$ using a fixed stepsize $h = 0.1$. For these 2 methods plot the quantities $H(q_n, p_n) = p_n^2/2 - \cos(q_n)$ with respect to t_n for $n = 0, 1, \ldots, N$. Compare the 2 plots, what do you observe? (We are not asking for an explanation).

Solution:

- the **explicit Euler method**

$$\det(D\Phi_h(q_n,p_n) = \begin{vmatrix} \frac{\partial q_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{vmatrix} = \begin{vmatrix} 1 & h \\ -h\cos q_n & 1 \end{vmatrix} = 1 + h^2 \cos q_n$$

- the **partitioned Euler method I**

$$\det(D\Phi_h(q_n,p_n) = \begin{vmatrix} \frac{\partial q_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{vmatrix} = \begin{vmatrix} 1 + h\frac{\partial p_{n+1}}{\partial q_n} & h\frac{\partial p_{n+1}}{\partial q_n} \\ -h\cos q_n & 1 \end{vmatrix} = \begin{vmatrix} 1 - h^2\cos q_n & h \\ -h\cos q_n & 1 \end{vmatrix} = 1$$

- the **partitioned Euler method II**

$$\det(D\Phi_h(q_n,p_n) = \begin{vmatrix} \frac{\partial q_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{vmatrix} = \begin{vmatrix} 1 & h \\ -h\frac{\partial \sin(q_n + hp_n)}{\partial q_n} & 1 - h\frac{\partial \sin(q_n + hp_n)}{\partial p_n} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & h \\ -h\cos(q_n + hp_n) & 1 - h^2\cos(q_n + hp_n) \end{vmatrix}$$

$$= 1$$

- the **implicit Euler method**

  We have the equations

$$\begin{aligned} \frac{\partial q_{n+1}}{\partial q_n} &= 1 + h\frac{\partial p_{n+1}}{\partial q_n} &&\implies 1 = -h\frac{\partial p_{n+1}}{\partial q_n} + \frac{\partial q_{n+1}}{\partial q_n} \\ \frac{\partial q_{n+1}}{\partial p_n} &= h\frac{\partial p_{n+1}}{\partial p_n} &&\implies 0 = -h\frac{\partial p_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} = -h\frac{\partial \sin q_{n+1}}{\partial q_n} &= -\cos q_{n+1}\frac{\partial q_{n+1}}{\partial q_n} &&\implies 0 = h\cos q_{n+1}\frac{\partial q_{n+1}}{\partial q_n} + \frac{\partial p_{n+1}}{\partial q_n} \\ \frac{\partial p_{n+1}}{\partial p_n} &= 1 - h\cos q_{n+1}\frac{\partial q_{n+1}}{\partial p_n} &&\implies 1 = \frac{\partial p_{n+1}}{\partial p_n} + h\cos q_{n+1}\frac{\partial q_{n+1}}{\partial p_n} \end{aligned}$$

  And so we get a linear system

$$\begin{bmatrix} 1 & 0 & -h & 0 \\ 0 & 1 & 0 & -h \\ h\cos q_{n+1} & 0 & 1 & 0 \\ 0 & h\cos q_{n+1} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial q_{n+1}}{\partial q_n} \\ \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} \\ \frac{\partial p_{n+1}}{\partial p_n} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving, we get

$$\begin{bmatrix} \dfrac{\partial q_{n+1}}{\partial q_n} \\ \dfrac{\partial q_{n+1}}{\partial p_n} \\ \dfrac{\partial p_{n+1}}{\partial q_n} \\ \dfrac{\partial p_{n+1}}{\partial p_n} \end{bmatrix} = \frac{1}{1+h^2\cos q_{n+1}} \begin{bmatrix} 1 \\ h \\ -h\cos q_{n+1} \\ 1 \end{bmatrix},$$

and so

$$\det(D\Phi_h(q_n,p_n)) = \begin{vmatrix} \dfrac{\partial q_{n+1}}{\partial q_n} & \dfrac{\partial q_{n+1}}{\partial p_n} \\ \dfrac{\partial p_{n+1}}{\partial q_n} & \dfrac{\partial p_{n+1}}{\partial p_n} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{1+h^2\cos q_{n+1}} & \dfrac{h}{1+h^2\cos q_{n+1}} \\ \dfrac{-h\cos q_{n+1}}{1+h^2\cos q_{n+1}} & \dfrac{1}{1+h^2\cos q_{n+1}} \end{vmatrix}$$

$$= \frac{1}{1+h^2\cos q_{n+1}}$$

For part (b), I do not understand what exactly I am looking for. In the explicit Euler case, the graph "starts off" as a "parabolic curve" for smaller n, and then as n gets larger, the graph starts to oscillate at more and more frequency as n increases. In the partition Euler method I, the graph is highly oscillatory with the same frequency throughout.

3. We consider systems of linear homogeneous autonomous ODEs $\dot{x} = Ax$ of dimension $n = 2$ for various matrices A

$$A_1 := \begin{bmatrix} 2 & 4 \\ -4 & -2 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 2 & -4 \\ -4 & -2 \end{bmatrix}, \quad A_3 := \begin{bmatrix} -2 & 4 \\ -4 & -2 \end{bmatrix}.$$

For each matrix A, determine if the solutions to $\dot{x} = Ax$ are Lyapunov-stable, asymptotically Lyapunov-stable, or Lyapunov-unstable.
Solution: We look at the determinant and trace:

$$\begin{aligned} \delta(A_1) = 12, \tau(A_1) = 0 &\implies \text{solutions to } \dot{x} = A_1x \text{ are Lyapunov-stable} \\ \delta(A_2) = -20, \tau(A_1) = 0 &\implies \text{solutions to } \dot{x} = A_2x \text{ are Lyapunov-unstable} \\ \delta(A_3) = 20, \tau(A_1) = -4 &\implies \text{solutions to } \dot{x} = A_3x \text{ are asymptotically Lyapunov-stable} \end{aligned}$$

4. Consider the system of linear homogeneous nonautonomous ODEs $\dot{x} = A(t)x$ with matrix $A(t)$ given by $A(t) := T(t)MT(t)^T$ where the matrices $T(t)$ and M are given respectively by

$$T(t) := \begin{bmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{bmatrix}, \qquad M := \begin{bmatrix} -2 & 6 \\ 0 & -2 \end{bmatrix}.$$

(a) Show that the eigenvalues $\lambda_i(A(t))$ $(i = 1, 2)$ of $A(t)$ satisfy $Re(\lambda_i(A(t)) \leq -2$. Hint: $T(t)$ is an orthogonal matrix, i.e., $T(t)^T T(t) = I_2 = T(t)T(t)^T$.

Solution:
Similar matrices have the same eigenvalues, and so $\lambda_i(A(t)) = \lambda_i(M)$. Hence $Re(\lambda_i(A(t))) = Re(\lambda_i(M)) = Re(\{-2\}) = -2$.

(b) Consider the change of coordinates $y := T(t)^T x$ and give the system of differential equations satisfied by $y = (y_1, y_2)$.

Solution:

$$\dot{y} = (\dot{T}(t)^T T(t) + M)y = \underbrace{\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}}_{=:B} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(c) Find the general solution for $x(t)$. In particular give the solution $x(t, 0, (1, -1))$, i.e., the one which satisfies the initial conditions $x(0) = (1, -1)$. Is this solution Lyapunov-stable? Hint: find the general solution for $y(t)$ first.

Solution: The eigenvalues of B are $\lambda_1 = -5$ and $\lambda_2 = 1$, with respective eigenvectors $[-1\ 1]^T$ and $[1\ 1]^T$. So a general solution to the system $\dot{y} = By$ is

$$y(t) = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 e^{-5t} + c_2 e^t \\ c_1 e^{-5t} + c_2 e^t \end{bmatrix}$$

So

$$\begin{aligned} x(t) = T(t)y(t) &= \begin{bmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{bmatrix} \begin{bmatrix} -c_1 e^{-5t} + c_2 e^t \\ c_1 e^{-5t} + c_2 e^t \end{bmatrix} \\ &= \begin{bmatrix} \cos 3t(-c_1 e^{-5t} + c_2 e^t) + \sin 3t(c_1 e^{-5t} + c_2 e^t) \\ -\sin 3t(-c_1 e^{-5t} + c_2 e^t) + \cos 3t(c_1 e^{-5t} + c_2 e^t) \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{-5t}(\sin 3t - \cos 3t) + c_2 e^t(\sin 3t + \cos 3t) \\ c_1 e^{-5t}(\sin 3t + \cos 3t) + c_2 e^t(-\sin 3t + \cos 3t) \end{bmatrix} \end{aligned}$$

Then

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x(0) = \begin{bmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} \implies c_1 = -1, c_2 = 0.$$

So

$$x(t) = \begin{bmatrix} -e^{-5t}(\sin 3t - \cos 3t) \\ -e^{-5t}(\sin 3t + \cos 3t) \end{bmatrix}$$

(d) Are the solutions $x(t)$ to the system $\dot{x} = A(t)x$ Lyapunov-stable or Lyapunov-unstable? You can use the following result: since $x(t) = T(t)y(t)$ and $T(t)$ is orthogonal, the Lyapunov-stability of $x(t)$ is the same as the Lyapunov-stability of $y(t)$.

Solution: Using the hint, we consider the determinant and trace of B: $\delta(B) = -5, \tau(B) = -4$. This means that the solutions are unstable.

(e) Calculate $\mu_2(A(t))$, the logarithmic norm of $A(t)$ corresponding to the Euclidean norm.

Solution:

We have

$$\begin{aligned}\frac{1}{2}(A(t)+A(t)^T) &= \frac{1}{2}(TMT^T+(TMT^T)^T)\\ &= \frac{1}{2}(TMT^T+TM^TT^T)\\ &= \frac{1}{2}(T(M+M^T)T^T)\\ &= \frac{1}{2}\left(T\begin{bmatrix}-4 & 6\\ 6 & -4\end{bmatrix}T^T\right)\\ &= TBT^T\end{aligned}$$

Since similar matrices have the same eigenvalues, the eigenvalues of $(1/2)(A(t)+A(t)^T)$ are -5 and 1. So,

$$\mu_2(A(t)) = \max_{i=1,2}\lambda_i(T(t)BT(t)^T) = \max\{-5,1\} = 1.$$

5. Consider the following system of linear nonhomogeneous nonautonomous ODEs

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-4x_1+3\cos(2t)x_2+e^{7t}\\ t^2x_1-(t^2+1)x_2+2t^6\end{bmatrix}.$$

Are solutions to this system Lyapunov-stable or even asymptotically Lyapunov-stable? Hint: do not attempt to solve this system of ODEs.

Solution:

Let

$$A = \begin{bmatrix}-4x_1+3\cos(2t)x_2+e^{7t}\\ t^2x_1-(t^2+1)x_2+2t^6\end{bmatrix}.$$

We have

$$\begin{aligned}\mu_\infty(A) &= \max\{-4+3\cos 2t, -(t^2+1)+t^2\}\\ &= \max\{-4+3\cos 2t, -1\}\\ &\le -1,\end{aligned}$$

and so the solutions to the system are asymptotically stable.

6. Consider the following system of nonlinear ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 - \sin(3t)x_2 - x_1^3 \\ \sin(3t)x_1 - 3x_2 - 3x_2^5 \end{bmatrix}.$$

Are solutions to this system Lyapunov-stable or even asymptotically Lyapunov-stable? Hint: do not attempt to solve this system of ODEs.

Solution:

Let

$$f(t,x) = \begin{bmatrix} -2x_1 - \sin(3t)x_2 - x_1^3 \\ \sin(3t)x_1 - 3x_2 - 3x_2^5 \end{bmatrix}.$$

Then

$$D_x f(t,x) = \begin{bmatrix} -2 - 3x_1^2 & -3\sin 3t \\ \sin 3t & -3 - 15x_2^4 \end{bmatrix},$$

and so

$$\frac{1}{2}(D_x f(t,x) + D_x f(t,x)^T) = \begin{bmatrix} -2 - 3x_1^2 & 0 \\ 0 & -3 - 15x_2^4 \end{bmatrix},$$

For any x_1, x_2, the eigenvalues of $D_x f(t,x)$ remain negative. So

$$\mu_2(D_x f(t,x)) < 0$$

which means the solutions to the system are asymptotically stable.

Exercise 1. Consider the following autonomous nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1\left(1 - \sqrt{x_1^2 + x_2^2}\right) \\ -x_1 + x_2\left(1 - \sqrt{x_1^2 + x_2^2}\right) \end{bmatrix}. \tag{1}$$

To find the corresponding flow map $\varphi_{2\pi}(x_0)$ with $x_0 = (x_{0_1}, x_{0_2})$ proceed here as follows:

(a) Cartesian coordinates (x_1, x_2) and polar coordinates (r, θ) are related by $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$. When these coordinates vary with t, prove that the following relations are satisfied

$$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}, \qquad x_1\dot{x}_2 - \dot{x}_1 x_2 = r^2\dot{\theta} \tag{2}$$

(of course these relations are totally independent of (1)).

Solution:

$$\begin{aligned}
x_1\dot{x_1} + x_2\dot{x_2} &= x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} \\
&= r\cos\theta\frac{d(r\cos\theta)}{dt} + r\sin\theta\frac{d(r\sin\theta)}{dt} \\
&= r\cos\theta\left(r\frac{d\cos\theta}{dt} + \cos\theta\frac{dr}{dt}\right) + r\sin\theta\left(r\frac{d\sin\theta}{dt} + \sin\theta\frac{dr}{dt}\right) \\
&= r^2\cos\theta\frac{d\cos\theta}{dt} + r\cos^2\theta\frac{dr}{dt} + r^2\sin\theta\frac{d\sin\theta}{dt} + r\sin^2\theta\frac{dr}{dt} \\
&= r^2\left(\cos\theta\left(-\sin\theta\frac{d\theta}{dt}\right) + \sin\theta\left(\cos\theta\frac{d\theta}{dt}\right)\right) + r\frac{dr}{dt}\left(\cos^2\theta + \sin^2\theta\right) \\
&= r\frac{dr}{dt} \\
&= r\dot{r}
\end{aligned}$$

$$\begin{aligned}
x_1\dot{x_2} - \dot{x_1}x_2 &= x_1\frac{dx_2}{d-}\frac{dx_1}{dt}x_2 \\
&= r\cos\theta\left(r\frac{d\sin\theta}{dt} + \sin\theta\frac{dr}{dt}\right) - \left(r\frac{d\cos\theta}{dt} + \cos\theta\frac{dr}{dt}\right)r\sin\theta \\
&= r\cos\theta\left(r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt}\right) - \left(-r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt}\right)r\sin\theta \\
&= r^2\cos^2\theta\frac{d\theta}{dt} + r\cos\theta\sin\theta\frac{dr}{dt} + r^2\sin^2\theta\frac{d\theta}{dt} - r\cos\theta\sin\theta\frac{dr}{dt} \\
&= r^2\frac{d\theta}{dt}(\cos^2\theta + \sin^2\theta) \\
&= r^2\dot{\theta}
\end{aligned}$$

(b) Convert the system of differential equations (1) given in Cartesian coordinates to polar coordinates using the relations (2).

Solution:

$$\begin{aligned} r\dot{r} = x_1\dot{x_1} + x_2\dot{x_2} &= x_1x_2 + x_1^2 - x_1^2\sqrt{x_1^2+x_2^2} - x_1x_2 + x_2^2 - x_2^2\sqrt{x_1^2+x_2^2} \\ &= r^2\cos^2\theta - r^3\cos^2\theta - r^2\sin^2\theta + r^2\sin^2\theta \\ &= r^2 - r^3 \\ r^2\dot{\theta} = x_1\dot{x_2} - \dot{x_1}x_2 &= -x_1^2 + x_1x_2 - x_1x_2\sqrt{x_1^2+x_2^2} - x_1^2 - x_1x_2 + x_1x_2\sqrt{x_1^2+x_2^2} \\ &= -(x_1^2+x_2^2) \\ &= -(r^2\cos^\theta + r^2\sin^2\theta) \\ &= -r^2 \end{aligned}$$

So the converted system is

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} r - r^2 \\ -1 \end{bmatrix}$$

(c) Solve the system of differential equations that you have just obtained in polar coordinates with initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$. Express the solution for any time t as a flow map $\phi_t(r_0, \theta_0)$ in polar coordinates.

Solution:

By a separation of variables, we obtain

$$r(t) = \frac{c_1e^t}{c_1e^t + 1} \quad \text{and} \quad \theta(t) = -t + c_2$$

for arbitrary c_1 and c_2. When $r(0) = r_0$ and $\theta(0) = \theta_0$, we get

$$r(t, 0, r_0) = \frac{r_0e^t}{r_0e^t + 1 - r_0} \quad \text{and} \quad \theta(t, 0, \theta_0) = -t + \theta_0.$$

So our flow map is

$$\phi_t(r_0, \theta_0) = \begin{bmatrix} r(t, 0, r_0) \\ \theta(t, 0, \theta_0) \end{bmatrix}.$$

(d) Convert back the flow map $\phi_{2\pi}(r_0, \theta_0)$ in polar coordinates to the corresponding flow map $\varphi_{2\pi}(x_{0_1}, x_{0_2})$ in Cartesian coordinates.

Solution:

At 2π, we have

$$\phi_{2\pi}(r_0, \theta_0) = \begin{bmatrix} \dfrac{r_0e^{2\pi}}{r_0e^{2\pi} + 1 - r_0} \\ \\ -2\pi + \theta_0 \end{bmatrix}.$$

Converting to cartesian coordinates, we use the relations

$$r_0 = \sqrt{x_{0_1}^2 + x_{0_2}^2}, \quad x_{0_1} = r_0 \cos(\theta_0), \quad \text{and} \quad x_{0_2} = r_0 \sin(\theta_0).$$

So

$$\begin{aligned} x_1(2\pi, 0, x_{0_1}) = r(2\pi, 0, r_0)\cos\theta(2\pi, 0, \theta_0) &= \frac{r_0 e^{2\pi}}{r_0 e^{2\pi} + 1 - r_0}\cos(-2\pi + \theta_0) \\ &= \frac{r_0 e^{2\pi}}{r_0 e^{2\pi} + 1 - r_0}\cos(\theta_0) \\ &= \frac{x_{0_1} e^{2\pi}}{\sqrt{x_{0_1}^2 + x_{0_2}^2}(e^2\pi - 1) + 1}, \end{aligned}$$

$$\begin{aligned} x_2(2\pi, 0, x_{0_2}) = r(2\pi, 0, r_0)\sin\theta(2\pi, 0, \theta_0) &= \frac{r_0 e^{2\pi}}{r_0 e^{2\pi} + 1 - r_0}\sin(-2\pi + \theta_0) \\ &= \frac{r_0 e^{2\pi}}{r_0 e^{2\pi} + 1 - r_0}\sin(\theta_0) \\ &= \frac{x_{0_2} e^{2\pi}}{\sqrt{x_{0_1}^2 + x_{0_2}^2}(e^2\pi - 1) + 1}. \end{aligned}$$

So $\varphi_{2\pi}(x_{0_1}, x_{0_2}) = \begin{bmatrix} x_1(2\pi, 0, x_{0_1}) \\ x_2(2\pi, 0, x_{0_2}) \end{bmatrix}$.

(e) Show that the first component of $\varphi_{2\pi}(0, x_{0_2})$ vanishes.

Solution:

$$x_1(2\pi, 0, 0) = \frac{(0)e^{2\pi}}{\sqrt{(0)^2 + x_{0_2}^2}(e^2\pi - 1) + 1} = 0.$$

Exercise 2. Find the fixed points of the following autonomous nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - x_2^2 \\ x_1 - x_1^2 x_2 \end{bmatrix}.$$

For each fixed point determine if they are hyperbolic or nonhyperbolic. If a fixed point is hyperbolic determine if it is a source, a sink, or a saddle.

Solution:

The fixed points are $(0,0), (1,1)$, and $(-1,-1)$. Letting $f = (\dot{x_1}, \dot{x_2})$, we have

$$Df = \begin{bmatrix} x_2 & x_1 - 2x_2 \\ 1 - 2x_1 x_2 & -x_1^2 \end{bmatrix}.$$

So

$$\begin{aligned} Df(0,0) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ has repeated eigenvalue } 0 \implies \text{nonhyperbolic} \\ Df(1,1) &= \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \text{ has eigenvalues } \pm\sqrt{2} \implies \text{hyperbolic saddle} \\ Df(-1,-1) &= \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \text{ has eigenvalues } -1 \pm i \implies \text{hyperbolic sink} \end{aligned}$$

Exercise 3. The origin $x^* = (0, 0)$ is a fixed point of the following autonomous nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1^3 - 6x_1x_2^2 \\ 8x_1^2x_2 - 2x_2^3 \end{bmatrix}.$$

(a) Is the origin $x^* = (0, 0)$ a hyperbolic or nonhyperbolic fixed point?

Solution:

Letting $f = (\dot{x_1}, \dot{x_2})$, we have

$$Df = \begin{bmatrix} -6x_1^2 - 6x_2^2 & -12x_1x_2 \\ 16x_1x_2 & 8x_1^2 - 6x_2^2 \end{bmatrix}$$

So $Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has repeated eigenvalue $0 \implies$ nonhyperbolic.

(b) Is the function $V(x_1, x_2) := \frac{1}{2}(x_1^2 + x_2^2)$ a strict Lyapunov function for this system of ODEs at $x^* = (0, 0)$?

Solution:

The fixed point $(0, 0)$ is a local minimizer for $V(x)$. Also,

$$(\mathcal{L}_f V)(x) = Dvf(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -2x_1^3 - 6x_1x_2^2 \\ 8x_1^2x_2 - 2x_2^3 \end{bmatrix} = -2(x_1^4 + x_2^4 - x_1^2x_2^2)$$

Notice that

$$x_1^4 + x_2^4 - x_1^2x_2^2 = x_1^4 - 2x_1^2x_2^2 + x_2^4 + x_1^2x_2^2 = (x_1^2 - x_2^2)^2 + x_1^2x_2^2 > 0,$$

and hence $(\mathcal{L}_f V)(x) < 0$ whenever $(x_1, x_2) \neq (0, 0)$, giving that $V(x)$ is indeed a strict Lyapunov function for this system at $(0, 0)$.

(c) Is the origin $x^* = (0, 0)$ a sink? (In other words is the constant solution $x(t) \equiv x^* \equiv (0, 0)$ asymptotically Lyapunov-stable?)

Solution:

Since $V(x)$ is a strict Lyapunov function for this system at $(0, 0)$, the origin is asymptotially Lyapunov-stable, and hence a sink.

Exercise 4. We consider a scalar function H of $n = 2m$ variables with $H \in \mathcal{C}^2$.

(a) We consider the Hamiltonian system $\dot{x} = J\nabla H(x)$ with

$$J = \begin{bmatrix} O & I_m \\ -I_m & O \end{bmatrix}$$

and Hamiltonian $H(x)$. For the same scalar function H we consider the gradient system $\dot{y} = -\nabla H(y)$. Show that solutions $y(t)$ of the gradient system $\dot{y} = -\nabla H(y)$ are orthogonal to solutions $x(t)$ of the Hamiltonian system $\dot{x} = J\nabla H(x)$, i.e., show that $\dot{y}(t)^T \dot{x}(t) = 0$ at any intersection point $w = x(t) = y(t)$.

Solution:

$$\begin{aligned}
\dot{y}(t)^T \dot{x}(t) &= -\nabla H(y)^T J \nabla H(x) \\
&= \begin{bmatrix} -\nabla_{x_1} H(y) \cdots - \nabla_{x_n} H(y) \end{bmatrix} \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \begin{bmatrix} \nabla_{x_1} H(x) \\ \vdots \\ \nabla_{x_n} H(x) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{x_{m+1}} H(y) \big| \cdots \big| \nabla_{x_n} H(y) \big| - \nabla_{x_1} H(y) \big| \cdots \big| - \nabla_{x_m} H(y) \end{bmatrix} \begin{bmatrix} \nabla_{x_1} H(x) \\ \vdots \\ \nabla_{x_n} H(x) \end{bmatrix} \\
&= \sum_{i=1}^{n/2} \nabla_{x_{m+i}} H(y) \nabla_{x_i} H(x) - \sum_{j=1}^{n/2} \nabla_{x_{m+j}} H(y) \nabla_{x_j} H(x).
\end{aligned}$$

So if $x(t) = y(t)$, then $\dot{y}(t)^T \dot{x}(t) = 0$.

(b) For the case $m = 1$ ($n = 2$) show that a hyperbolic fixed point of a Hamiltonian system $\dot{x} = J\nabla H(x)$ cannot be a source or a sink, but must be a saddle.

Solution: We have $\dot{x} = JH(x) = \begin{bmatrix} -\nabla_{x_2} H(x) \\ \nabla_{x_1} H(x) \end{bmatrix} =: f(x)$. Let x^* be a hyperbolic fixed point of this system. Then

$$Df(x^*) = \begin{bmatrix} \dfrac{\partial^2 H}{\partial x_1 \partial x_2}(x^*) & \dfrac{\partial^2 H}{\partial x_2^2}(x^*) \\ -\dfrac{\partial^2 H}{\partial x_1^2}(x^*) & -\dfrac{\partial^2 H}{\partial x_2 \partial x_1}(x^*) \end{bmatrix}.$$

Since the trace of $Df(x^*)$ is 0, it has characteristic polynomial of the form $\lambda^2 + \alpha$, $\alpha \in \mathbb{R}$. The solutions of the characteristic polynomial are

$$\lambda_\pm = \begin{cases} \pm i\sqrt{\alpha} & \text{if } \alpha \geq 0 \\ \pm\sqrt{-\alpha} & \text{if } \alpha < 0 \end{cases}$$

But x^* is a hyperbolic fixed point, i.e., $Re(\lambda_\pm(Df(x^*)) \neq 0$. Hence $\lambda_\pm = \pm\sqrt{-\alpha}$, which means the fixed point must be a saddle.

Exercise 5. Consider the following autonomous nonlinear system of ODEs $\dot{x} = f(x)$ given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_2 \end{bmatrix}.$$

(a) Show that the origin $x^* = (0,0)$ is the unique fixed point (hence an isolated fixed point). Is $x^* = (0,0)$ an hyperbolic or a nonhyperbolic fixed point?

Solution:

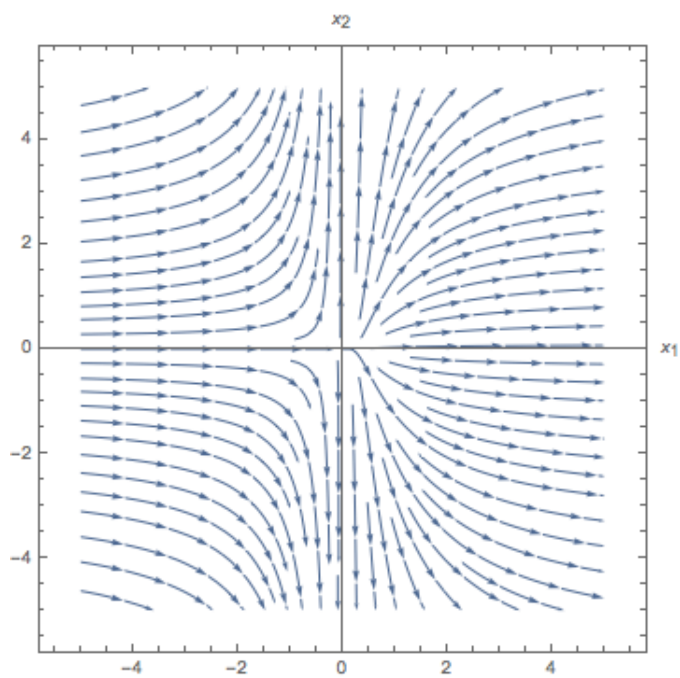
Setting $x_1^2 = x_2 = 0$, we easily see that the only fixed point is $(0,0)$. Moreover, letting $f = (\dot{x_1}, \dot{x_2})$

$$Df(0,0) = \begin{bmatrix} 2x_1 & 0 \\ 0 & 1 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which has an eigenvalue 0, giving that $(0,0)$ is a nonhyperbolic fixed point.

(b) Draw approximately the phase portrait.

Solution:



(c) Is $x^* = (0,0)$ a Lyapunov-stable, an asymptotically Lyapunov-stable, or a Lyapunov-unstable solution?

Solution:

Since 1 is an eigenvalue of $Df(0,0)$ and has positive real part, then $(0,0)$ is an Lyapunov-unstable solution.

(d) Give the linearized system

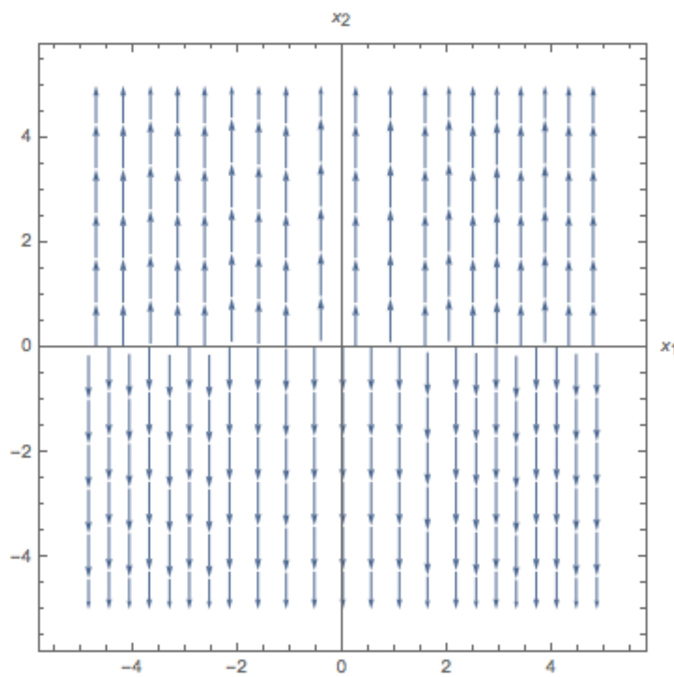
$$\dot{y} = Df(x^*)(y - x^*)$$

at $x^* = (0,0)$ and solve it. Show that $x^* = (0,0)$ is not an isolated fixed point of the linearized system. Draw approximately the phase portrait of the linearized system.

Solution:

$$\dot{y} = Df(x^*)(y - x^*) = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}.$$

For any y_1^*, $(y_1^*, 0)$ will be a fixed point of the system, so $(0,0)$ is not an isolated fixed point.



(e) For the linearized system is the constant solution $x^* = (0,0)$ Lyapunov-stable, asymptotically Lyapunov-stable, or Lyapunov-unstable?

Solution:

Letting $g = (\dot{y_1}, \dot{y_2})$, we have $Dg(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, which has 0 as an eigenvalue, and so $(0,0)$ is a Lyapunov-unstable solution.

(f) Would you say that the linearized system captures the dynamics of the original nonlinear system of ODEs well? (state your opinion, you will not lose any point on this question).

Solution: No.

(g) Is the Hartman-Grobman Theorem applicable here?

Solution: No; the fixed point in nonhyperbolic.

Exercise 6. Consider the following autonomous nonlinear system of ODEs $\dot{x} = f(x)$ given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 - x_1(x_1^2 + x_2^2) \\ x_1 - x_2(x_1^2 + x_2^2) \end{bmatrix},$$

and which has a fixed point at the origin $x^* = (0, 0)$.

(a) Is $x^* = (0, 0)$ an hyperbolic or a nonhyperbolic fixed point?

Solution:

Letting $f = (\dot{x_1}, \dot{x_2})$, we have

$$Df(0,0) = \begin{bmatrix} -3x_1^2 - x_2^2 & -1 - 2x_1x_2 \\ \\ 1 - 2x_1x_2 & -x_1^2 - 3x_2^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which has eigenvalues $\pm i$. Hence $(0, 0)$ is a nonhyperbolic fixed point.

(b) Convert the nonlinear system of ODEs given in Cartesian coordinates (x_1, x_2) to polar coordinates (r, θ) $(x_1 = r\cos(\theta), x_2 = r\sin(\theta))$.

Solution:

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -r^3 \\ 1 \end{bmatrix}.$$

(c) Solve the system of ODEs that you have just obtained in polar coordinates with initial conditions $r(t_0) = r_0$ and $\theta(t_0) = \theta_0$.

Solution:

Since the origin is a fixed point of our system, if $r_0 = 0$, then our system requires that we have the solution $r(t, t_0, 0) = 0$. Suppose now $r_0 \neq 0$. With the conditions $r(t_0) = r_0$ and $\theta(t_0) = 0$, we obtain

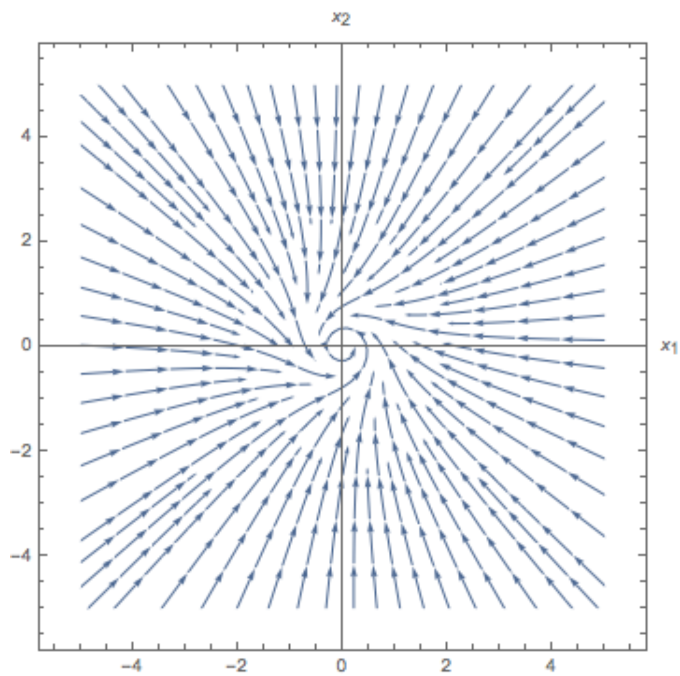
$$r(t, t_0, r_0) = \frac{1}{\sqrt{2t - 2t_0 + r_0^{-2}}} \quad \text{and} \quad \theta(t, t_0, \theta_0) = t + \theta_0 - t_0$$

(d) Express the solution back to Cartesian coordinates and draw approximately the phase portrait.

Solution:

$$x_1(t,t_0,x_{0_1}) = r(t,t_0,r_0)\cos\theta(t,t_0,\theta_0) = \frac{\cos(t+\theta_0-t_0)}{\sqrt{2t-2t_0+r_0^{-2}}} = \frac{\cos(t-t_0+\arctan(x_{0_2}/x_{0_1}))}{\sqrt{2t-2t_0+(x_{0_1}^2+x_{0_2}^2)^{-1}}}$$

$$x_2(t,t_0,x_{0_2}) = r(t,t_0,r_0)\sin\theta(t,t_0,\theta_0) = \frac{\sin(t+\theta_0-t_0)}{\sqrt{2t-2t_0+r_0^{-2}}} = \frac{\sin(t-t_0+\arctan(x_{0_2}/x_{0_1}))}{\sqrt{2t-2t_0+(x_{0_1}^2+x_{0_2}^2)^{-1}}}$$



(e) Is the constant solution $x^* = (0,0)$ Lyapunov-stable, asymptotically Lyapunov-stable, or Lyapunov-unstable? Is $x^* = (0,0)$ a sink? Hint: for a nonhyperbolic fixed point try to find a (strict) Lyapunov function, e.g., try $V(x) := \frac{1}{2}(x_1^2 + x_2^2)$.

Solution:

Using the hint, $V(x)$ is a locally minimized by $(0,0)$. Moreover, letting $f = (\dot{x_1}, \dot{x_2})$, we have

$$(\mathcal{L}_f V)(x) = DV(x)f(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -x_2 - x_1(x_1^2+x_2^2) \\ x_1 - x_2(x_1^2+x_2^2) \end{bmatrix} = -(x_1^4 + x_1^4) < 0$$

whenever $(x_1, x_2) \neq (0,0)$. So $V(x)$ is a strict Lyapunov function for the system at $(0,0)$, giving that the constant solution $(0,0)$ is an asymptotically Lyapunov-stable solution, hence a sink.

(f) Give the linearized system

$$\dot{y} = Df(x^*)(y - x^*)$$

at $x^* = (0,0)$ and solve it. Draw approximately the phase portrait of the linearized system.

Solution:

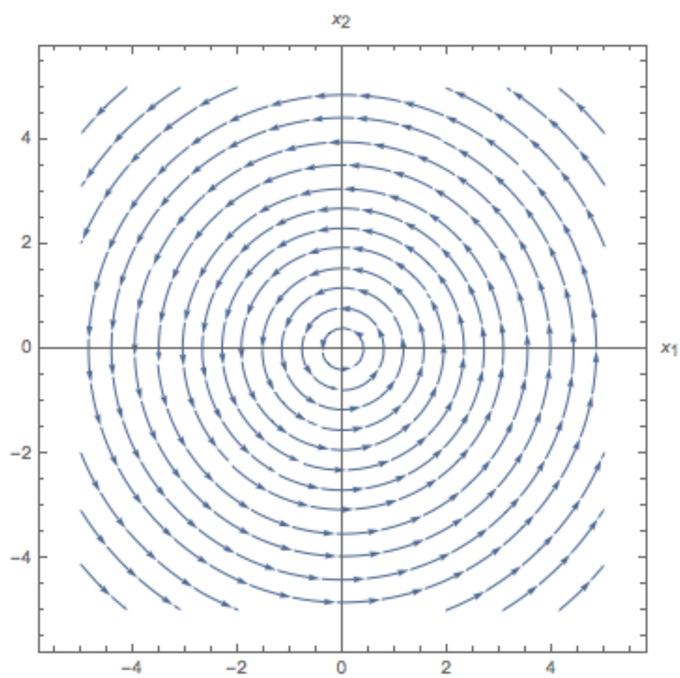
$$\dot{y} = Df(x^*)(y - x^*) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

We have $A^{2\ell} = (-1)^\ell I$ and $A^{2\ell+1} = (-1)^\ell A$ for all $\ell \geq 0$. So,

$$\begin{aligned} e^{(t-t_0)A} = \sum_{k\geq 0} \frac{(t-t_0)^k}{k!} A^k &= \sum_{\ell \geq 0} \frac{(t-t_0)^{2\ell}}{(2\ell)!} A^{2\ell} + \sum_{\ell\geq 0} \frac{(t-t_0)^{2\ell+1}}{(2\ell+1)!} A^{2\ell+1} \\ &= I\cos(t-t_0) + A\sin(t-t_0) \\ &= \begin{bmatrix} \cos(t-t_0) & -\sin(t-t_0) \\ \sin(t-t_0) & \cos(t-t_0) \end{bmatrix} \end{aligned}$$

So the solution to the linear system is

$$\begin{aligned} y(t,t_0,y_0) &= e^{(t-t_0)A} y_0 \\ &= \begin{bmatrix} \cos(t-t_0) & -\sin(t-t_0) \\ \sin(t-t_0) & \cos(t-t_0) \end{bmatrix} \begin{bmatrix} y_{0_1} \\ y_{0_2} \end{bmatrix} \\ &= \begin{bmatrix} y_{0_1}\cos(t-t_0) - y_{0_2}\sin(t-t_0) \\ y_{0_1}\sin(t-t_0) + y_{0_2}\cos(t-t_0) \end{bmatrix} \end{aligned}$$



(g) For the linearized system is the constant solution $x^* = (0,0)$ Lyapunov-stable, asymptotically Lyapunov-stable, or Lyapunov-unstable?

Solution:

Letting $V(y) = \frac{1}{2}(y_1^2 + y_2^2)$, we have that the origin is a local minimizer for V and if $g = (\dot{y_1}, \dot{y_2})$, then we have

$$DV(y)g(y) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix} = -y_1 y_2 + y_1 y_2 = 0,$$

which means that V is a weak Lyapunov function for this system, giving that $(0,0)$ is a Lyapunov-stable solution.

(h) Would you say that the linearized system captures the dynamics of the original nonlinear system of ODEs well? (state your opinion, you will not lose any point on this question).

Solution: No.

(i) Is the Hartman-Grobman Theorem applicable here?

Solution: No; the fixed point in nonhyperbolic.

1. Consider the following autonomous nonlinear system of ODEs $\dot{x} = f(x)$ given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + \frac{x_2}{\ln(\sqrt{x_1^2+x_2^2})} \\ -x_2 - \frac{x_1}{\ln(\sqrt{x_1^2+x_2^2})} \end{bmatrix}$$

for $x \neq (0,0)$. At $x^* := (0,0)$ we define

$$f(x^*) := \lim_{x \to x^*} f(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

since $\lim_{r \to 0^+} 1/\ln(r) = 0$. Thus $x^* = (0,0)$ is a fixed point.

(a) Based on the definition of a Jacobian matrix, show that

$$Df(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Is $x^* = (0,0)$ an hyperbolic or a nonhyperbolic fixed point? If it is hyperbolic determine if it is a source, a sink, or a saddle.

Solution:

Let $f = (\dot{x_1}, \dot{x_2})$. Then

$$Df = \begin{bmatrix} -1 - \frac{x_1 x_2}{(x_1^2+x_2^2)\ln\sqrt{x_1^2+x_2^2}} & \frac{1}{\ln\sqrt{x_1^2+x_2^2}} - \frac{x_2^2}{(x_1^2+x_2^2)\ln\sqrt{x_1^2+x_2^2}} \\ -\frac{1}{\ln\sqrt{x_1^2+x_2^2}} - \frac{x_1^2}{(x_1^2+x_2^2)\ln\sqrt{x_1^2+x_2^2}} & -1 - \frac{x_1 x_2}{(x_1^2+x_2^2)\ln\sqrt{x_1^2+x_2^2}} \end{bmatrix}$$

Taking limits, we get $Df(x^*) = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

(b) Is the constant solution $x^* = (0,0)$ Lyapunov-stable, asymptotically Lyapunov-stable, or Lyapunov-unstable?

Solution:

The eigenvalues of $DF(x^*)$ are both -1, so the constant solution is a hyperbolic sink.

(c) Convert the nonlinear system of ODEs given in Cartesian coordinates (x_1, x_2) to polar coordinates (r, θ) $(x_1 = r\cos(\theta), x_2 = r\sin(\theta))$.

Solution:

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -r \\ -\frac{1}{\ln r} \end{bmatrix}$$

(d) Assuming $r_0 = \sqrt{x_{1_0}^2 + x_{2_0}^2} < 1$, solve the system of ODEs that you have just obtained in polar coordinates with initial conditions $r(t_0) = r_0$ and $\theta(t_0) = \theta_0$. Show that $\lim_{t \to +\infty} r(t) = 0$, $\dot{\theta}(t) > 0$ for $0 < r_0 < 1$ and $t \geq t_0$, and $\lim_{t \to +\infty} \theta(t) = +\infty$.

Solution:

$$r(t, t_0, r_0) = r_0 e^{t_0 - t} \quad \text{and} \quad \theta(t, t_0, \theta_0) = \ln|(\ln r_0) + t_0 - t| + \theta_0 - \ln|\ln r_0|.$$

Then

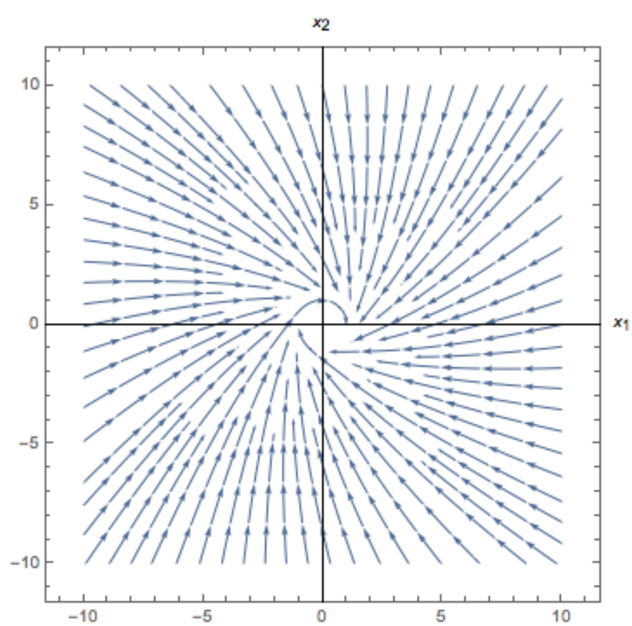
$$\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} r_0 e^{t_0-t} = 0$$
$$\dot{\theta}(t) = -\frac{1}{\ln r} = -\frac{1}{\ln(r_0 e^{t_0-t})} > 0 \text{ if } 0 < r_0 < 1 \text{ and } t \geq t_0$$
$$\lim_{t\to+\infty} \theta(t) = \lim_{t\to+\infty} \ln|(\ln r_0) + t_0 - t| + \theta_0 - \ln|\ln r_0| = +\infty$$
$$\text{since } |(\ln r_0) + t_0 - t| \to \infty \text{ as } t \to \infty.$$

(e) Express the solution back to Cartesian coordinates and draw approximately the phase portrait.

Solution:

Why? This is just gross:

$$\begin{aligned} x_1(t, t_0, x_{1_0}) &= r(t, t_0, r_0)\cos(\theta(t, t_0, \theta_0)) \\ &= r_0 e^{t_0-t}\cos(\ln|(\ln r_0) + t_0 - t| + \theta_0 - \ln|\ln r_0|) \\ &= \sqrt{x_{1_0}^2 + x_{2_0}^2}\, e^{t_0-t}\cos\left(\ln\left|(\ln\sqrt{x_{1_0}^2 + x_{2_0}^2}) + t_0 - t\right|\right. \\ &\qquad\left. + \arctan\left(\frac{x_{2_0}}{x_{1_0}}\right) - \ln\left|\ln\sqrt{x_{1_0}^2 + x_{2_0}^2}\right|\right) \\ x_2(t, t_0, x_{2_0}) &= r(t, t_0, r_0)\sin(\theta(t, t_0, \theta_0)) \\ &= r_0 e^{t_0-t}\sin(\ln|(\ln r_0) + t_0 - t| + \theta_0 - \ln|\ln r_0|) \\ &= \sqrt{x_{1_0}^2 + x_{2_0}^2}\, e^{t_0-t}\sin\left(\ln\left|(\ln\sqrt{x_{1_0}^2 + x_{2_0}^2}) + t_0 - t\right|\right. \\ &\qquad\left. + \arctan\left(\frac{x_{2_0}}{x_{1_0}}\right) - \ln\left|\ln\sqrt{x_{1_0}^2 + x_{2_0}^2}\right|\right) \end{aligned}$$



(f) Give the linearized system

$$\dot{y} = Df(x^*)(y - x^*)$$

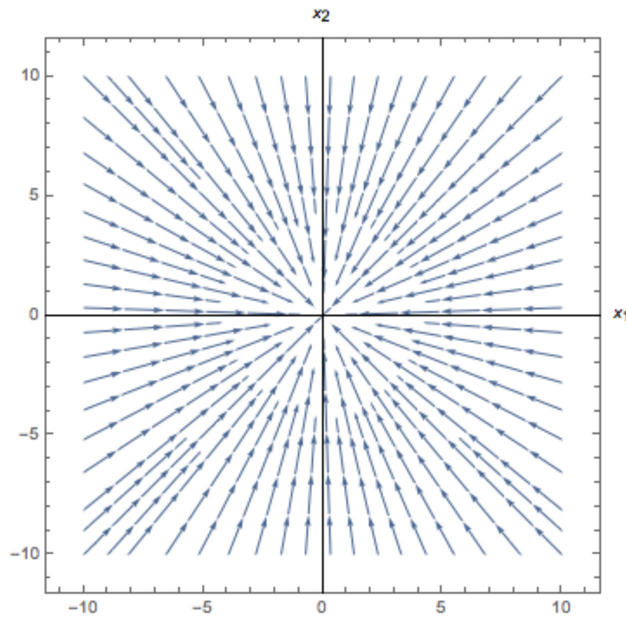
at $x^* = (0,0)$ and solve it. Draw approximately the phase portrait of the linearized system.

Solution:

$$\dot{y} = Df(x^*)(y - x^*) = \begin{bmatrix} -y_1 \\ -y_2 \end{bmatrix}.$$

Then

$$y_1(t, t_0, y_{0_1}) = y_{1_0} e^{t_0 - t} \quad \text{and} \quad y_2(t, t_0, y_{2_0}) = y_{2_0} e^{t_0 - t}.$$



(g) For the linearized system is the constant solution $x^* = (0,0)$ Lyapunov-stable, asymptotically Lyapunov-stable, or Lyapunov-unstable?

Solution:
Let $g = (\dot{y_1}, \dot{y_2})$. Then

$$Dg(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which has a repeated eigenvalue -1, so the constant solution is a hyperbolic sink, giving that the constant solution is asymptotically Lyapunov-stable.

(h) Would you say that the linearized system captures the dynamics of the original nonlinear system of ODEs well? What is captured and what is not captured here? (state your opinion, you will not lose any point on this last question).

Solution: Sorta; the phase portraits are similar a wee bit.

(i) Is the Hartman-Grobman Theorem applicable here?

Solution: Yup! The fixed point is hyperbolic.

2. We consider the autonomous nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_2^3 + 2x_1x_2^2 - x_2\cos(x_1x_2) \\ -6x_1x_2^2 + 2x_1^2x_2 - x_1\cos(x_1x_2) \end{bmatrix}.$$

Is it a gradient system? If yes, find $U(x)$ such that $\dot{x} = -\nabla U(x)$. Does this system possess a periodic solution?

Solution:

Yes is it a gradient system since $\dfrac{\partial f_1}{\partial x_2} = \dfrac{\partial f_2}{\partial x_1}$ where $(f_1, f_2) = (\dot{x_1}, \dot{x_2})$. Then $U(x) = 2x_1x_2^3 - x_1^2x_2^2 + \sin(x_1x_2) + C$ is such that $\dot{x} = -\nabla U(x)$. This system does not have a periodic solution since it is a gradient system. (Theorem 2.54).

3. The following nonlinear system of ODEs describes two species F and R, F predating (unrelated to dating!) R,

$$\begin{bmatrix} \dot{F} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} F(R-r) \\ R(f-F) \end{bmatrix} \tag{1}$$

with parameters $r > 0, f > 0$. It is a **Lotka-Volterra predator-prey model** (a very simplistic model). For example in a Canadian forest, F may represent a population of foxes and R a population of rabbits (foxes eat rabbits, but not the opposite!). For $F > 0$ and $R > 0$ show that the quantity

$$I(F, R) := F - f\ln(F) + R - r\ln(R) \tag{2}$$

is a **first integral** of the system (1), i.e., it stays constant along solutions. Hence, the closed curves

$$\{F > 0, R > 0 \mid I(F, R) = Const\}$$

represent periodic orbits (except when $(F, R) = (f, r)$ which is a fixed point).

Solution:

We have

$$\begin{aligned} DI \begin{bmatrix} \dot{F} \\ \dot{R} \end{bmatrix} &= \begin{bmatrix} 1 - f/F & 1 - r/R \end{bmatrix} \begin{bmatrix} F(r-R) \\ R(f-F) \end{bmatrix} \\ &= F(R-r) - \frac{fF(R-r)}{F} + R(f-F) - \frac{rR(f-F)}{R} \\ &= FR - Fr - fR + fr + Rf - RF - rf + rF \\ &= 0, \end{aligned}$$

and so I is a first integral.

4. Consider the autonomous nonlinear system of ODEs given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 - 2x_1^3 \\ x_1 + x_2 - 2x_2^3 \end{bmatrix}.$$

(a) Convert this nonlinear system of ODEs given in Cartesian coordinates (x_1, x_2) to polar coordinates (r, θ).

Solution:

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} r - 2r^3 + r^3 \sin^2(2\theta) \\ 1 + r^2 \sin(2\theta)(\cos^2\theta - \sin^2\theta) \end{bmatrix}$$

(b) Determine a circle of radius r_1 centered at the origin $(0,0)$ such that $\dot{r} \geq 0$ for $r \leq r_1$. Determine also a circle of radius r_2 centered at the origin $(0,0)$ such that $\dot{r} \leq 0$ for $r \geq r_2$. Note: $2\sin(\theta)\cos(\theta) = \sin(2\theta)$.

Solution:

We use that the $0 \leq \sin^2(2\theta) \leq 1$:

$$r - 2r^3 \leq \dot{r} \leq r - 2r^3 + r^3$$
$$r(1 - 2r^2) \leq \dot{r} \leq r(1 - r^2)$$

So if $r \leq \frac{1}{\sqrt{2}} =: r_1$ then $\dot{r} \geq 0$ and if $r \geq 1 =: r_2$, then $\dot{r} \leq 0$.

(c) Does the above system of ODEs have a periodic orbit?

Solution:

First notice that

$$K = \left\{ x \in \mathbb{R}^2 \ \mid r_1^2 = \frac{1}{2} \leq x_1^2 + x_2^2 \leq 1 = r_2^2 \right\}$$

is a closed annulus and hence compact. Moreover, $(0,0)$ is the only fixed point of our system which does not lie in the annulus. Hence by the Poincare-Bendixon Theorem, the system has a periodic orbit.

1. For **partitioned** systems of ODEs of the form $\dot{q} = f(q,p)$, $\dot{p} = g(q,p)$, one can consider the following 2 numerical schemes where (q_n, p_n) approximates the exact solution at $t_n = t_0 + nh$ for $n = 0, 1, 2, 3, \ldots, N$:

   - the **explicit Euler method**

   $$q_{n+1} = q_n + hf(q_n, p_n), \qquad p_{n+1} = p_n + hg(q_n, p_n);$$

   - the **partitioned Euler method II**

   $$q_{n+1} = q_n + hf(q_{n+1}, p_n), \qquad p_{n+1} = p_n + hg(q_{n+1}, p_n).$$

   For the Lotka-Volterra predator-prey model

   $$\begin{bmatrix} \dot{F} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} F(R-3) \\ R(2-F) \end{bmatrix}$$

   consider the initial conditions $F(0) = 1.8$, $R(0) = 3.4$. Write simple codes (in MATLAB preferably) for the explicit Euler method and the partitioned Euler method II for $n = 0, 1, \ldots, N$ with $N = 500$ using a fixed stepsize $h = 0.05$. Plot the numerical solution obtained in phase space (F, R) for each method. For each method plot also the quantities $I(F_n, R_n)$ for

   $$I(F, R) := F - 2\ln(F) + R - 3\ln(R)$$

   with respect to t_n for $n = 0, 1, \ldots, N$. Compare the plots for the 2 methods, what do you observe? (We are not asking for a mathematical explanation). The exact solution is known to be periodic, see exercise 3 of homework 5.

   Solution:

   The explicit Euler method:

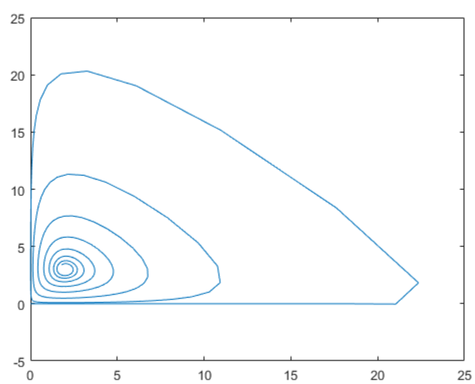


Figure 1: Numerical Solution

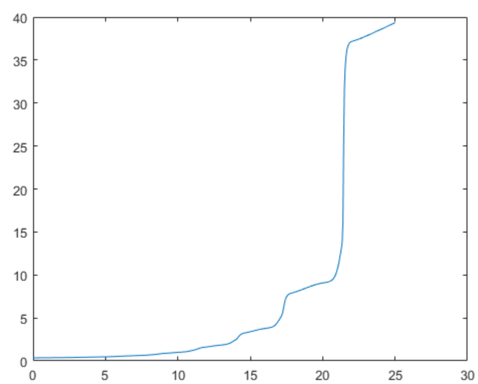


Figure 2: $I(F_n, R_n)$

The partitioned Euler method II:

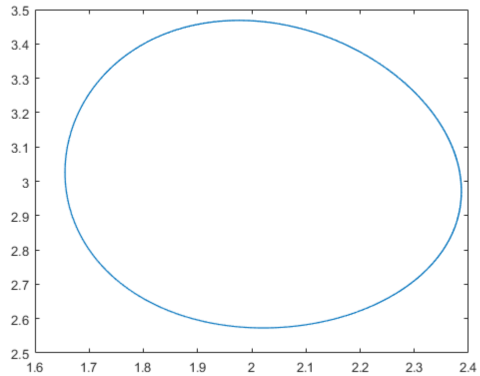


Figure 3: Numerical Solution

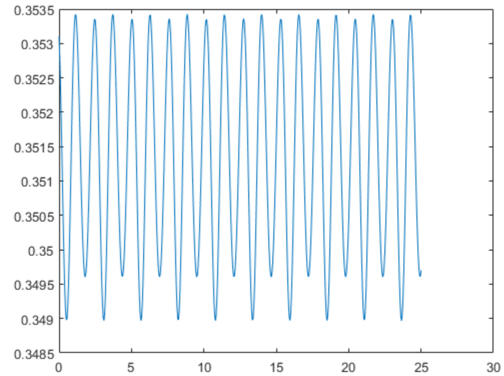


Figure 4: $I(F_n, R_n)$

In the explicit Euler method, the numerical solution is not periodic, and the first integral diverges. In the partitioned Euler method II, the numerical solution is periodic and the first integral (as a function) is also periodic and doesn't diverge like in the first method.

2. Apply Liénard's Theorem to show that for any $\mu > 0$ the second-order nonlinear ODE

$$\ddot{y} + \mu(y^2 - 1)\dot{y} + \tanh(y) = 0$$

has a periodic orbit which is asymptotically orbitally stable. Note: by definition

$$\tanh(y) := \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Solution:

Set $g(y) = \mu(y^2 - 1)$ and $h(y) = \tanh(y)$. We need to check a bunch of conditions on these functions to apply Liénard's Theorem. If all the conditions are met, then the theorem says that the system has a periodic orbit which is asymptotically orbitally stable.

(a) Both g and h are $\mathcal{C}^1$

(b) *h is odd*: $\tanh(-y) = \dfrac{e^{-y} - e^y}{e^{-y} + e^y} = -\left(\dfrac{e^y - e^{-y}}{e^y + e^{-y}}\right) = -\tanh(y)$.

(c) $h(y) > 0$ *if* $y > 0$: $e^y + e^{-y}$ is always positive, and since $y > 0$ and e^y is strictly increasing, $y > -y \implies e^y > e^{-y} \implies e^y - e^{-y} > 0$.

(d) *g is even*: $g(-y) = \mu((-y)^2 - 1) = \mu(y^2 - 1) = g(y)$.

(e) $G(y) := \int_0^y g(u)du = \mu \int_0^y (u^2 - 1)du$ satisfies

i. $G(a) = 0$ *for some* $a > 0$: Let $a = \sqrt{3}$. Then

$$G(\sqrt{3}) = \mu \int_0^{\sqrt{3}} (u^2 - 1)du = \mu \left(\frac{(\sqrt{3})^3}{3} - \sqrt{3} \right) = 0.$$

ii. $G(y) < 0$ *for* $0 < y < \sqrt{3}$:

$$G(y) = \mu\left(\frac{y^3}{3} - y\right) = \underbrace{\mu}_{>0} \left(\underbrace{\frac{y}{3}}_{>0} \underbrace{(y - \sqrt{3})}_{<0} \underbrace{(y + \sqrt{3})}_{>0} \right) < 0.$$

iii. $G(y) > 0$ *for* $\sqrt{3} < y$:

$$G(y) = \underbrace{\mu}_{>0} \left(\underbrace{\frac{y}{3}}_{>0} \underbrace{(y - \sqrt{3})}_{>0} \underbrace{(y + \sqrt{3})}_{>0} \right) > 0.$$

iv. $G'(y) = g(y) \geq 0$ *for* $\sqrt{3} < y$: By the Fundamental Theorem of Calculus, $G'(y) = g(y)$. Moreover, $g(y) = \mu(y^2 - 1) > 2\mu > 0$.

v. $\lim_{y \to +\infty} G(y) = +\infty$:

$$\lim_{y \to +\infty} \mu \left(\frac{y^3}{3} - y\right) = \mu \lim_{y \to +\infty} \left(\frac{y^3}{3} - y\right) = +\infty.$$

3. Consider the autonomous nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -7x_1 - 3x_2 + 5x_1^2 + 2x_2^2 \end{bmatrix}.$$

Does this system have a periodic orbit or not? Hint: Apply Dulac's criterion with $B(x_1, x_2) := 3e^{-4x_1}$.

Solution:

Let $f = (\dot{x_1}, \dot{x_2})$.

$$\begin{aligned} \nabla \cdot (B(x)f(x)) &= \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} 3x_2 e^{-4x_1} \\ 3x_2 e^{-4x_1}(-7x_1 - 3x_2 + 5x_1^2 + 2x_2^2) \end{bmatrix} \\ &= \frac{\partial(3x_2 e^{-4x_1})}{\partial x_1} + \frac{\partial(3e^{-4x_1}(-7x_1 - 3x_2 + 5x_1^2 + 2x_2^2))}{\partial x_2} \\ &= -12x_2 e^{-4x_1} + 3e^{-4x_1}(-3 + 4x_2) \\ &= \frac{-9}{e^{4x_1}} \\ &\neq 0 \end{aligned}$$

So in fact this system does not have a periodic orbit on all of $\mathbb{R}^2$.

4. Consider the autonomous nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 - 2x_1(x_1^2 + x_2^2) \\ x_1 + 2x_2 - 2x_2(x_1^2 + x_2^2) \end{bmatrix}.$$

Show that there exists a periodic orbit which is asymptotically orbitally stable.

Solution:

Converting to polar coordinates, we have

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 2r(1 - r^2) \\ 1 \end{bmatrix}.$$

So we see that if $r < 1$, then $\dot{r} > 0$ and if $r > 1$ then $\dot{r} < 0$. Hence at $r = 1$, we have a periodic solution:

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} 1 \\ t + \theta_0 \end{bmatrix}.$$

Converting back to cartesian coordinates, this solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(t + \theta_0) \\ \sin(t + \theta_0) \end{bmatrix},$$

and so we have a period T of 2π. Letting $f = (r, \theta)$ from our periodic solution, set

$$A(t) := Df|_{r=1} = \begin{bmatrix} 2 - 6r^2 & 0 \\ 0 & 0 \end{bmatrix}\Bigg|_{r=1} \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

We want to find a solution to $\dot{S}(t) = A(t)S(t)$. Since $A(t)$ is autonomous and is diagonal, we get that

$$S(t) = \begin{bmatrix} e^{-4t} & - \\ 0 & 1 \end{bmatrix}.$$

Since out period is 2π, we get

$$S(2\pi) = \begin{bmatrix} e^{-8\pi} & - \\ 0 & 1 \end{bmatrix}.$$

So $\lambda_i(S(2\pi, 0)) = \{e^{-8\pi}, 1\}$. By Lemma 2.67, the eigenvalues of $S(2\pi, 0)$ are the same as the monodromy matrix $R(t) = D_{x_0}(x)$ where x is our solution in cartesian coordinates. Now since $|e^{-8\pi}| < 1$, then Theorem 2.66 says that our periodic solution is asymptotically orbitally stable.

1. Find $y \in \mathcal{C}^1([1,2],\mathbb{R})$ satisfying $y(1) = 0$, $y(2) = 1$ such that the first variation of

$$\mathcal{A}[y] := \int_1^2 \frac{1}{x}\sqrt{1+y'^2}dx.$$

satisfies $\delta\mathcal{A}[y] \equiv 0$, i.e., $\delta\mathcal{A}[y][h] = 0 \; \forall h \in \mathcal{C}_0^1([1,2],\mathbb{R})$.

Solution:

Setting $L = \frac{1}{x}\sqrt{1+y'^2}$ and using the Euler equation,

$$0 = \frac{d}{dx}(L_{y'}) - L_y = \frac{d}{dx}(L_{y'}) - 0,$$

we get $L_{y'} \equiv C$ for some constant C. So

$$C = L_{y'} = \frac{y'}{x\sqrt{1+y'^2}} \implies y' = \pm\frac{Cx}{\sqrt{1-C^2x^2}}.$$

Hence

$$y(x) = -\frac{\sqrt{1-C^2x^2}}{C} + \tilde{C}$$

for some other constant $\tilde{C}$. Using the initial conditions,

$$0 = y(1) = -\frac{\sqrt{1-C^2}}{C} + \tilde{C} \implies \tilde{C} = \frac{\sqrt{1-C^2}}{C}$$

$$1 = y(2) = -\frac{\sqrt{1-4C^2}}{C} + \frac{\sqrt{1-C^2}}{C} \implies C = \frac{1}{\sqrt{5}}.$$

So

$$y(x) = \sqrt{5-x^2} + 2.$$

2. Is there $y \in \mathcal{C}^1([a,b],\mathbb{R})$ such that $y(a) = A$, $y(b) = B$ (a, b, A, B are fixed quantities) minimizing

$$\mathcal{A}_1[y] := \int_a^b y^2 dx?$$

If yes under which conditions on a, b, A, B? Answer the same questions for

$$\mathcal{A}_2[y] := \int_a^b yy' dx.$$

Solution:

First let $L = y^2$. Then again using the Euler equation,

$$0 = \frac{d}{dx}(L_{y'}) - L_y = -2y,$$

which implies $y \equiv 0$. So a and b can be any real numbers and $A = B = 0$.

Next, let $L = yy'$. Then

$$0 = \frac{d}{dx}(L_{y'}) - L_y = \frac{d}{dx}y - y' = y' - y' \implies y' = y',$$

so any $y \in \mathcal{C}^1([a,b],\mathbb{R})$ works.

3. Find $y \in \mathcal{C}^1([-1,1],\mathbb{R})$ satisfying $y(-1) = y(1) = A \geq 1$ such that the first variation of

$$\mathcal{A}[y] := \int_{-1}^{1} \sqrt{y\left(1 + y'^2\right)}dx$$

satisfies $\delta\mathcal{A}[y] \equiv 0$, i.e., $\delta\mathcal{A}[y][h] = 0 \ \forall h \in \mathcal{C}_0^1([-1,1],\mathbb{R})$.

Solution:

We use Beltrami's identity. Let $L = \sqrt{y\left(1+y'^2\right)}$. So

$$E_{\text{gen}} = L_{y'} - L = -\frac{y}{\sqrt{y(1+y'^2)}},$$

must satisfy

$$\frac{d}{dx}\left(E_{\text{gen}}\right) = -L_x = 0,$$

so $E_{\text{gen}} \equiv C$ for some constant C. Solving for y', we get

$$y' = \pm\frac{\sqrt{y - C^2}}{C}.$$

Solving this differential equation, we get

$$y(x) = C^2 + \frac{x^2}{4C^2} + C^2\tilde{C}^2 + \tilde{C}x,$$

for some constant $\tilde{C}$. The initial condition $y(1) = y(-1)$ gives $\tilde{C} = 0$. So $y(x) = C^2 + \frac{x^2}{4C^2}$. Then the initial condition $y(1) = A$ gives

$$C = \pm\sqrt{\frac{A \pm \sqrt{A^2 - 1}}{2}}.$$

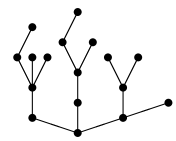
4. Consider the brachistochrone problem of finding $y \in \mathcal{C}^1([0,b],\mathbb{R})$ satisfying $y(0) = 0$ and $y(b) = B$ (technically $y(0) = \varepsilon > 0$ and we look at the limit $\varepsilon \to 0^+$) and $\delta\mathcal{A}[y] \equiv 0$ where
$$\mathcal{A}[y] := \int_0^b \sqrt{\frac{1+y'^2}{y}}dx$$
Show that the cycloids given parametrically by
$$x = X(\theta) = \frac{C}{2}(\theta - \sin(\theta)), \qquad y = Y(\theta) = \frac{C}{2}(1-\cos(\theta))$$
for $\theta \in]0, 2\pi[$ satisfy
$$y(1+y'^2) = C$$
by showing that
$$y'(x) = \frac{\frac{d}{d\theta}Y(\theta)}{\frac{d}{d\theta}X(\theta)} = \cot\left(\frac{\theta}{2}\right).$$
Note that for $x = X(\theta)$ and $y = Y(\theta) = y(X(\theta))$, the chain rule gives
$$\frac{d}{d\theta}Y(\theta) = y'(X(\theta))\frac{d}{d\theta}X(\theta) = y'(x)\frac{d}{d\theta}X(\theta).$$

Solution:

We have
$$y'(x) = \frac{\frac{d}{d\theta}Y(\theta)}{\frac{d}{d\theta}X(\theta)} = \frac{\frac{C}{2}(1-\cos\theta)}{\frac{C}{2}(\sin\theta)} = \frac{1-\cos\theta}{\sin\theta} = \cot\frac{\theta}{2},$$
where the last equality is a trig identity. Now,
$$\cot^2\frac{\theta}{2} = \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{\frac{1+\cos\theta}{2}}{\frac{1-\cos\theta}{2}} = \frac{1+\cos\theta}{1-\cos\theta},$$
and so
$$\begin{aligned}
y(1-y'^2) &= \frac{C}{2}(1-\cos\theta)\left(1+\cot^2\frac{\theta}{2}\right) \\
&= \frac{C}{2}\left(1-\cos\theta+\cot^2\frac{\theta}{2}-\cos\theta\cot^2\frac{\theta}{2}\right) \\
&= \frac{C}{2}\left(1-\cos\theta+\frac{1+\cos\theta}{1-\cos\theta}-\frac{\cos\theta+\cos^2\theta}{1-\cos\theta}\right) \\
&= \frac{C}{2}\left(\frac{(1-\cos\theta)^2+1+\cos\theta-\cos\theta-\cos^2\theta}{1-\cos\theta}\right) \\
&= \frac{C}{2}\left(\frac{1-2\cos\theta+\cos^2\theta+1+\cos\theta-\cos\theta-\cos^2\theta}{1-\cos\theta}\right) \\
&= \frac{C}{2}\left(\frac{2-2\cos\theta}{1-\cos\theta}\right) \\
&= C\left(\frac{1-\cos\theta}{1-\cos\theta}\right) \\
&= C.
\end{aligned}$$

Justify your answers and show your work.

1. Associated to the autonomous system of ODEs $\dot{x} = f(x)$ we consider the following rooted tree T



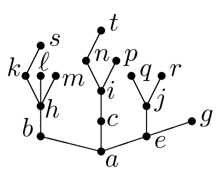
In which derivative $x^{(q)}$ does the corresponding elementary differential $F(T)(x)$ appear? Give $F(T)(x)$ in vector notation and also in componentwise notation.

Solution:

Since $T \in T_{17}$, the corresponding elementary differential appears in the 17^{th} derivative, $x^{(17)}$. In vector notation, we have

$$F(T)(x) = f_{xxx}\Big(f_x\big(f_{xxx}(f_x f, f, f)\big),\ f_x\big(f_{xx}(f_x f, f)\big),\ f_{xx}\big(f_{xx}(f,f), f\big)\Big).$$

In componentwise notation,



$$\begin{aligned}
&\sum_{b=1}^{d}\sum_{c=1}^{d}\sum_{e=1}^{d} \frac{\partial^3 f_a}{\partial x_b \partial x_c \partial x_e}(x) \cdot \sum_{h=1}^{d} \frac{\partial f_b}{\partial x_n}(x) \\
&\cdot \sum_{k=1}^{d}\sum_{\ell=1}^{d}\sum_{m=1}^{d} \frac{\partial^3 f_n}{\partial x_k \partial x_\ell \partial x_m}(x) f_\ell(x) f_m(x) \cdot \sum_{s=1}^{d} \frac{\partial f_k}{\partial x_s}(x) f_s(x) \\
&\cdot \sum_{i=1}^{d} \frac{\partial f_c}{\partial x_i}(x) \cdot \sum_{n=1}^{d}\sum_{p=1}^{d} \frac{\partial^2 f_i}{\partial x_n \partial x_p}(x) f_p(x) \cdot \sum_{t=1}^{d} \frac{\partial f_n}{\partial x_t}(x) f_t(x) \\
&\cdot \sum_{j=1}^{d}\sum_{g=1}^{d} \frac{\partial^2 f_e}{\partial x_j \partial x_g}(x) f_g(x) \cdot \sum_{q=1}^{d}\sum_{r=1}^{d} \frac{\partial^2 f_j}{\partial x_q \partial x_r}(x) f_q(x) f_r(x).
\end{aligned}$$

2. We consider the following initial value problem

$$\dot{x} = -100x, \qquad x(0) = 2.$$

(a) Find the exact solution to this initial value problem.

Solution:

$$x(t) = 2e^{-100t}$$

(b) By hand and thanks to an explicit formula give the result after 20 steps of the explicit Euler method with a constant stepsize $h = 0.05$ on the interval $[0, 1]$ and give the numerical approximation and its error obtained at $t_{end} = 1$. Is this numerical approximation satisfactory? Same questions for $h = 0.005$ by applying this time 200 steps of the explicit Euler method.

Solution:

We have $\dot{x}_{n+1} = x_n + h(-100x_n) = x_n(1 - 100h)$, and so

$$\begin{aligned} x_1 &= x_0(1 - 100h) \\ x_2 &= x_1(1 - 100h) \\ &= x_0(1 - 100h)^2 \\ &\vdots \\ x_n &= x_{n-1}(1 - 100h) \\ &= x_0(1 - 100h)^n. \end{aligned}$$

So for $n = 20, h = 0.05$,

$$x_{20} = x_0(1 - 100h)^{20} = 2(1 - 100(0.05))^{20} \approx 2.199023256 \times 10^{12};$$

and for $n = 200, h = 0.005$

$$x_{200} = x_0(1 - 100(0.005))^{200} = 2(1 - 100(0.005))^{200} \approx 1.2460306 \times 10^{-60}.$$

In our solution from part (a), we have at $t_{end} = 1$,

$$x(1) = 2e^{-100(1)} \approx 7.44015195 \times 10^{-44}.$$

(c) Find also the largest possible constant stepsize h such that the numerical approximations x_n obtained by the explicit Euler method satisfy $|x_n| \leq |x(0)|$.

Solution:

$$\begin{aligned} |x_n| = |x_0(1 - 100h)^n| &\leq |x(0)| \\ 2|(1 - 100h)^n| &\leq 2 \\ |(1 - 100h)|^n &\leq 1 \\ |(1 - 100h)| &\leq 1 \\ -1 \leq 1 - 100h &\leq 1 \\ -2 \leq -100h &\leq 0 \\ 0 \leq h &\leq \frac{2}{100}, \end{aligned}$$

so h can be at most 0.02.

3. Consider the following nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \cos(tx_2) \\ \cos(x_1)x_2 - tx_1 \end{bmatrix}.$$

Give the value $x_1 = (x_{11}, x_{12})$ after one step with stepsize $h = 0.1$ of the Taylor series method of order 3 starting from $x_0 = [0, 1]$ at $t_0 = 0$.

Solution:

We have

$$\begin{aligned}
\dot{x}_1 &= \cos(tx_2) \\
\dot{x}_2 &= \cos(x_1)x_2 - tx_1 \\
\\
\ddot{x}_1 &= -(t\dot{x}_2 + x_2)\sin(tx_2) \\
\ddot{x}_2 &= -\dot{x}_1\sin(x_1)x_2 + \cos(x_1)\dot{x}_2 - t\dot{x}_1 - x_1 \\
\\
\dddot{x}_1 &= -\Big(\big(t\ddot{x}_2 + \dot{x}_2 + \dot{x}_2\big)\sin(tx_2) + (t\dot{x}_2 + x_2)^2\cos(tx_2)\Big) \\
\dddot{x}_2 &= -\Big(\dot{x}_1\sin(x_1)\dot{x}_2 + \big(\ddot{x}_1\sin(x_1) + \dot{x}_1^2\cos(x_1)\big)x_2\Big) \\
&\quad + \Big(- \dot{x}_1\dot{x}_2\sin(x_1) + \cos(x_1)\ddot{x}_2\Big) - \Big(t\ddot{x}_1 + \dot{x}_1\Big) - \dot{x}_1
\end{aligned}$$

So,

$$\begin{aligned}
\dot{x}_1(0) &= \cos(0) = 1 \\
\dot{x}_2(0) &= \cos(x_1(0))x_2(0) - 0 = 1 - 0 = 1 \\
\\
\ddot{x}_1(0) &= -(0 + x_2(0))\sin(0) = 0 \\
\ddot{x}_2(0) &= -\dot{x}_1(0)\sin(x_1(0))x_2(0) + \cos(x_1(0))\dot{x}_2(0) - 0 - x_1(0) \\
&= 1 \\
\\
\dddot{x}_1(0) &= -\Big(\big(0 + \dot{x}_2(0) + \dot{x}_2(0)\big)\sin(0) + (0 + x_2(0))^2\cos(0)\Big) \\
&= -1 \\
\dddot{x}_2(0) &= -\Big(\dot{x}_1(0)\sin(x_1(0))\dot{x}_2(0) + \big(\ddot{x}_1(0)\sin(x_1(0)) + \dot{x}_1(0)^2\cos(x_1(0))\big)x_2(0)\Big) \\
&\quad + \Big(- \dot{x}_1(0)\dot{x}_2(0)\sin(x_1(0)) + \cos(x_1(0))\ddot{x}_2(0)\Big) - \Big(0 + \dot{x}_1(0)\Big) - \dot{x}_1(0) \\
&= -2.
\end{aligned}$$

So,

$$\begin{aligned}
x(t_0 + h) = x(0) + \frac{\dot{x}(0)}{1!}h^1 + \frac{\ddot{x}(0)}{2!}h^2 + \frac{\dddot{x}(0)}{3!}h^3 &= x_0 + \dot{x}(0)(.1) + \frac{\ddot{x}(0)}{2}(.01) + \frac{\dddot{x}(0)}{6}(.001) \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} .1 \\ .1 \end{bmatrix} + \begin{bmatrix} 0 \\ .005 \end{bmatrix} + \begin{bmatrix} -\frac{1}{6}(.001) \\ -\frac{1}{3}(.001) \end{bmatrix} \\
&= \begin{bmatrix} 0.0998\bar{3} \\ 1.104\bar{6} \end{bmatrix}.
\end{aligned}$$

4. Give the expression after one step with stepsize h of the Taylor series method of order p applied to the scalar initial value problem

$$\dot{x} = \lambda x, \qquad x(t_0) = x_0$$

where λ is a constant. Express the exact solution and the Taylor series method of order p at $t_1 := t_0 + h$ in term of the quantity $z := h\lambda$.

Solution:

The exact solution is $x(t) = x_0 e^{\lambda(t-t_0)}$. Then

$$x(t_1) = x_0 e^{\lambda(t_1 - t_0)} = x_0 e^{\lambda(t_0+h-t_0)} = x_0 e^z.$$

For the Taylor series method of order p, we obtain

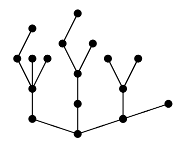
$$\begin{aligned}
x_{n+1} &= x_n + h_n \dot{x}_n + \frac{h_n^2}{2!}\ddot{x}_n + \frac{h_n^3}{3!}\dddot{x}_n + \cdots + \frac{h_n^p}{p!}x_n^{(p)} \\
&= x_n + h_n \dot{x}_n + \frac{h_n^2}{2!}\lambda \dot{x}_n + \frac{h_n^3}{3!}\lambda^2 \dot{x}_n + \cdots + \frac{h_n^p}{p!}\lambda^{p-1}\dot{x}_n \\
&= x_n + h_n \lambda x_n + \frac{h_n^2}{2!}\lambda^2 x_n + \frac{h_n^3}{3!}\lambda^3 x_n + \cdots + \frac{h_n^p}{p!}\lambda^p x_n \\
&= x_n \left(1 + h_n \lambda + \frac{h_n^2}{2!}\lambda^2 + \frac{h_n^3}{3!}\lambda^3 + \cdots + \frac{h_n^p}{p!}\lambda^p\right)
\end{aligned}$$

So at $t_1 = t_0 + h$, we have $n = 0$, which gives

$$\begin{aligned}
x_1 &= x_0 \left(1 + h\lambda + \frac{h^2}{2!}\lambda^2 + \frac{h^3}{3!}\lambda^3 + \cdots + \frac{h^p}{p!}\lambda^p\right) \\
&= x_0 \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^p}{p!}\right)
\end{aligned}$$

Justify your answers and show your work.

1. We consider the following rooted tree T



For this tree give

(a) the value of $\rho(T)$;

Solution: $\rho(T) = 17$

(b) the value of $\gamma(T)$;

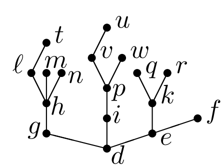
Solution: Let

$T_1 =$ $T_2 =$ $T_3 =$

Then $\gamma(T) = \rho(T)\gamma(T_1)\gamma(T_2)\gamma(T_3) = 17 \cdot 60 \cdot 40 \cdot 15 = 612,000$.

(c) for a Runge-Kutta method the sum expression $\sum_j b_j \Phi_j(T)$ in terms of Runge-Kutta coefficients.

Solution:



$$\begin{aligned}
\sum_{j=1}^{s} b_j \Phi_j(T) &= \sum_{j=1}^{s} b_j \sum_{g=1}^{s} a_{dg} \sum_{h=1}^{s} a_{gh} \sum_{\ell=1}^{s} a_{h\ell} \sum_{m=1}^{s} a_{hm} \sum_{n=1}^{s} a_{hn} \sum_{t=1}^{s} a_{\ell t} \\
&\quad \sum_{i=1}^{s} a_{di} \sum_{p=1}^{s} a_{ip} \sum_{v=1}^{s} a_{pv} \sum_{w=1}^{s} a_{pw} \sum_{u=1}^{s} a_{vu} \\
&\quad \sum_{e=1}^{s} a_{de} \sum_{k=1}^{s} a_{ek} \sum_{q=1}^{s} a_{kq} \sum_{r=1}^{s} a_{kr} \sum_{f=1}^{s} a_{ef} \\
&= \sum_{j=1}^{s} b_j \sum_{g=1}^{s} a_{dg} \sum_{h=1}^{s} a_{gh} \sum_{\ell=1}^{s} a_{h\ell} c_h c_h c_\ell \sum_{i=1}^{s} a_{di} \sum_{p=1}^{s} a_{ip} \\
&\quad \sum_{v=1}^{s} a_{pv} c_p c_v \sum_{e=1}^{s} a_{de} \sum_{k=1}^{s} a_{ek} c_k c_k c_e \\
&= \sum_{j=1}^{s}\sum_{g=1}^{s}\sum_{h=1}^{s}\sum_{\ell=1}^{s}\sum_{i=1}^{s}\sum_{p=1}^{s}\sum_{v=1}^{s}\sum_{e=1}^{s}\sum_{k=1}^{s} b_j a_{dg} a_{gh} a_{h\ell} c_h^2 c_\ell a_{di} a_{ip} a_{pv} c_p c_v a_{de} a_{ek} c_k^2 c_e \\
&= \sum_{j,g,h,\ell,i,p,v,e,k=1}^{s} b_j a_{dg} a_{gh} a_{h\ell} c_h^2 c_\ell a_{di} a_{ip} a_{pv} c_p c_v a_{de} a_{ek} c_k^2 c_e
\end{aligned}$$

2. Consider the following 4-stage explicit Runge-Kutta method given by its tableau of coefficients

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/3 & 1/3 & 0 & & \\ 2/3 & 1/3 & 0 & 1/3 & \\ \hline & 0 & -2 & 3/2 & 3/2 \end{array}$$

Give explicitly one step of this method when applied to $\dot{x} = f(t, x)$. What is the local order of this method?

Solution:

The first step is given by x_1, where

$$\begin{aligned} x_1 &= x_0 + h\left(0 \cdot f(t_0, X_1) - 2f(t_0 + h/2, X_2) + \frac{3}{2}f(t_0 + h/3, X_3) + \frac{3}{2}f(t_0 + 2h/3, X_4)\right) \\ &= x_0 + h\left(-2f(t_0 + h/2, X_2) + \frac{3}{2}f(t_0 + h/3, X_3) + \frac{3}{2}f(t_0 + 2h/3, X_4)\right) \end{aligned}$$

and

$$\begin{aligned} X_1 &= x_0 \\ X_2 &= x_0 + h/2f(t_0, X_1) = x_0 + h/2f(t_0, x_0) \\ X_3 &= x_0 + h\left(1/3f(t_0, X_1) + 0 \cdot f(t_0 + h/2, X_2)\right) \\ &= x_0 + h/3f(t_0, x_0) \\ X_4 &= x_0 + h\left(1/3f(t_0, X_1) + 0 \cdot f(t_0 + h/2, X_2) + 1/3f(t_0 + h/3, X_3)\right) \\ &= x_0 + h/3f(t_0, X_1) + h/3f(t_0 + h/3, X_3) \\ &= x_0 + h/3f(t_0, x_0) + h/3f(t_0 + h/3, x_0 + h/3f(t_0, x_0)). \end{aligned}$$

Now, since

$$\sum_{j=1}^{1} a_{2j} = 1/2 = c_2, \quad \sum_{j=1}^{2} a_{3j} = 1/3 + 0 = c_3, \quad \sum_{j=1}^{3} a_{4j} = 1/3 + 0 + 1/3 = 2/3 = c_4,$$

then we can apply the conditions given on page 71 to determine the local order of the given explicit RK method.

- 1 condition for order $p = 1$ is satisfied:

$$\sum_{j=1}^{4} b_j = 0 - 2 + 3/2 + 3/2 = 1.$$

- 1 additional condition for order $p = 2$ is satisfied:

$$\sum_{j=1}^{4} b_j c_j = 0 \cdot 0 + (-2)(1/2) + (3/2)(1/3) + (3/2)(2/3) = -1 + 1/2 + 1 = 1/2.$$

- 2 additional conditions for order $p = 3$ is satisfied:

$$\sum_{j=1}^{4} b_j c_j^2 = 0 \cdot 0^2 + (-2)(1/2)^2 + (3/2)(1/3)^2 + (3/2)(2/3)^2 = 1/3$$

$$\sum_{j=1}^{4}\sum_{k=1}^{4} b_j a_{jk} c_k = 3/2 \cdot 0 \cdot 1/2 + 3/2(0 \cdot 1/2 + (1/3)(1/3)) = 1/6$$

- however, one of the 4 additional conditions for order $p = 4$ is not satisfied:

$$\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4} b_j a_{jk} a_{kl} c_l = (3/2)(1/3)(0)(1/2) = 0 \neq 1/24.$$

Hence the local order of this method is 3.

3. Find all explicit Runge-Kutta methods of local order 3 satisfying $\sum_{j=1}^{i-1} a_{ij} = c_i$ for $i = 1, 2, 3$ with coefficients $c_1 = 0, c_2 = 1/2, c_3 = 1$, i.e.,

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & a_{21} & & \\ 1 & a_{31} & a_{32} & \\ \hline & b_1 & b_2 & b_3 \end{array}$$

Is there any method of local order 4 in this form?

Solution:

First, note that there is no method of local order 4 in this form since it is only a 3-stage method.

Since we are given that $\sum_{j=1}^{i-1} a_{ij} = c_i$, we can consider the conditions given on page 71 to derive the unknowns. We see immediately that $a_{21} = 1/2$. The conditions

$$1/2 = \sum_{j=1}^{3} b_j c_j = b_2/2 + b_3 \quad \text{and} \quad 1/3 = \sum_{j=1}^{3} b_j c_j^2 = b_2/4 + b_3$$

imply $b_2 = 2/3$ and $b_3 = 1/6$. Then the condition $b_1 + b_2 + b_3 = 1$ implies $b_1 = 1/6$. The condition

$$1/6 \sum_{j=1}^{3} \sum_{k=1}^{3} b_j a_{jk} c_k = (a_{32}/6)(1/2)$$

implies $a_{32} = 2$. Then $1 = c_3 = a_{31} + a_{32} = a_{31} + 2$ implies $a_{31} = -1$. Altogether, we have the tableau

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

4. Show that the local order p of an s-stage explicit Runge-Kutta method cannot be greater than the number s of internal stages, i.e., $p \leq s$. Hint: Apply the Runge-Kutta method with stepsize h to $\dot{x} = x$, $x(0) = 1$, and show that the numerical solution $x_1(h)$ is a polynomial of degree s in h.

Solution:

Using the hint, with $f(t, x) = x$, notice that $f(t_0, X_1) = x(0) = 1$. So,

$$
\begin{aligned}
X_1 &= 1 \\
X_2 &= 1 + ha_{21} \\
X_3 &= 1 + h(a_{31} + a_{32}(1 + ha_{21})) \\
X_4 &= 1 + h\Big(a_{41} + a_{42}(1 + ha_{21}) + a_{43}(1 + h(a_{31} + a_{32}(1 + ha_{21})))\Big) \\
&\vdots \\
X_s &= 1 + h\Big(a_{s1} + a_{s2}(1 + ha_{21}) + \cdots + a_{s,s-1}(1 + h(a_{s-1,1} + a_{s-1,2} + \cdots + a_{s-1,s-2}X_{s-1})\Big)
\end{aligned}
$$

By induction, we can see that the highest power of h in X_s is $s - 1$. Since

$$x_1(h) = 1 + h\Big(b_1 + b_2X_2 + \cdots + b_sX_s\Big),$$

then the $x_1(h)$ is a polynomial of degree at most s in the variable h. Let $x(t)$ be the exact solution. Then the definition of "local order p", (that $x(t_0+h)-x_1 = O(h^{p+1})$), together with the fact that $x_1(h)$ has order at most s, imply that $p \leq s$, as desired.

5. What is the (linear) stability function $R(z)$ of an s-stage explicit Runge-Kutta method of local order $p = s$? In particular give the (linear) stability function of Runge's (RK2) method of local order 2 and of Kutta's (RK4) method of local order 4. The (linear) stability domain of explicit Runge-Kutta methods of local order $p = s$ is drawn in the figure below for $s = 1, 2, 3, 4$

Solution:

For an s-stage explicit Runge-Kutta method of local order $p = s$, the (linear) stability function $R(z)$ is

$$R(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^s}{s!}.$$

For RK2 and RK4 we have, respectively, the stability functions

$$R(z) = 1 + z + \frac{z^2}{2!} \quad \text{and} \quad R(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}.$$

Justify your answers and show your work. Throughout, we let $f(x,\mu) = \dot{x}$.

1. In each of the following two one-dimensional examples, find all the fixed points, determine the Lyapunov stability of hyperbolic fixed points, sketch a bifurcation diagram, and find the bifurcation values of the parameter $\mu \in \mathbb{R}$:

   (a) $\dot{x} = \mu - x^2 + 4x^4$;

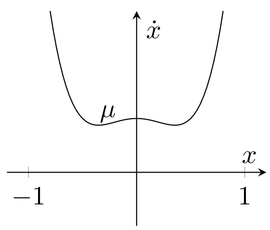
   Solution:

   The system has fixed points $x^*_{1_1|1_2|2_1|2_2} = \pm\frac{\sqrt{1\pm\sqrt{1-16\mu}}}{2\sqrt{2}}$, and we have

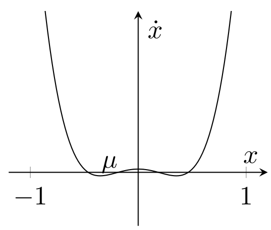
$$\frac{\partial f}{\partial x}(x^*) = \pm\frac{\sqrt{1\pm\sqrt{1-16\mu}}\,\sqrt{1-16\mu}}{\sqrt{2}}.$$

So x^* is non-hyperbolic if and only if $\mu = 0$ or $\mu = \frac{1}{16}$. For any other values of μ, the fixed points of the system (if it has any) will be hyperbolic.

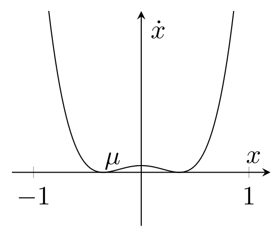
$\boldsymbol{\mu > \frac{1}{16}}$, 0 fixed points



$\boldsymbol{0 < \mu < \frac{1}{16}}$, 4 fixed points



$\boldsymbol{\mu = \frac{1}{16}}$, 2 fixed points

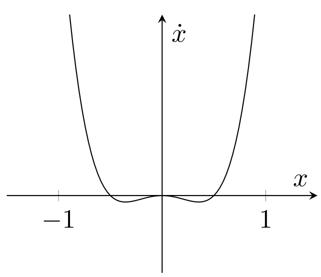


When $0 < \mu < \frac{1}{16}$, we have (from left to right):

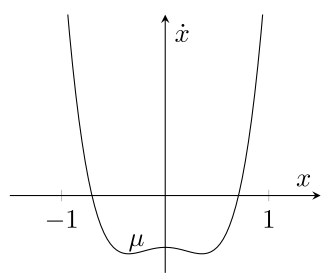
$$x^*_{2_1} \rightsquigarrow \text{sink}; \quad x^*_{2_2} \rightsquigarrow \text{source}; \quad x^*_{1_2} \rightsquigarrow \text{sink}; \quad x^*_{1_1} \rightsquigarrow \text{source}.$$

where sinks are Lyapunov stable and sources are Lyapunov unstable.

$\boldsymbol{\mu = 0}$, 3 fixed points

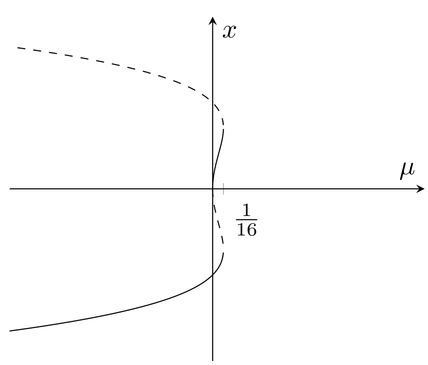


$\boldsymbol{\mu < 0}$, 2 fixed points



When $\mu = 0$, we have a two hyperbolic fixed points: $x^* = 1/2$ source; $x^* = -1/2$ sink. When $\mu < 0$, we have two fixed points:$x^* = \sqrt{-\mu}/2$ source; $x^* = -\sqrt{-\mu}/2$ sink.

So we get the following bifurcation diagram, with bifurcation values $\mu = 0$ and $\mu = 1/16$.



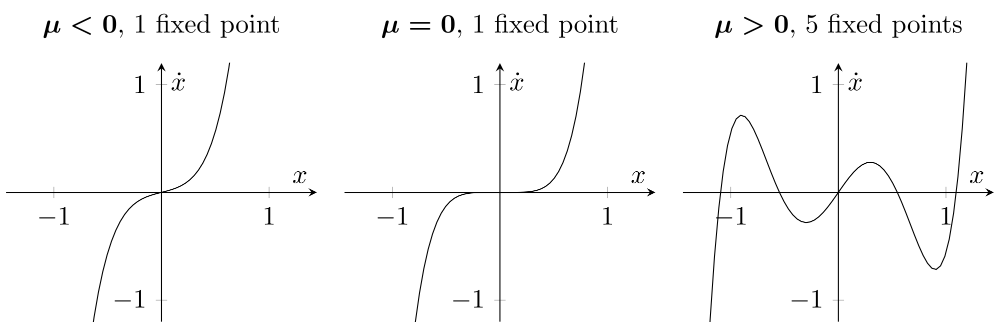
(b) $\dot{x} = x(\mu - x^2)(\mu - 4x^2)$.

Solution:

The system has fixed points $x_1^* = 0, x_{2_1|2_2}^* = \pm\sqrt{\mu}$, and $x_{3_1|3_2}^* = \pm\frac{\sqrt{\mu}}{2}$. Also,

$$\frac{\partial f}{\partial x}(0) = \mu^2, \quad \frac{\partial f}{\partial x}(\pm\sqrt{\mu}) = 6\mu^2 \quad \frac{\partial f}{\partial x}\left(\frac{\pm\sqrt{\mu}}{2}\right) = -39\mu^2.$$

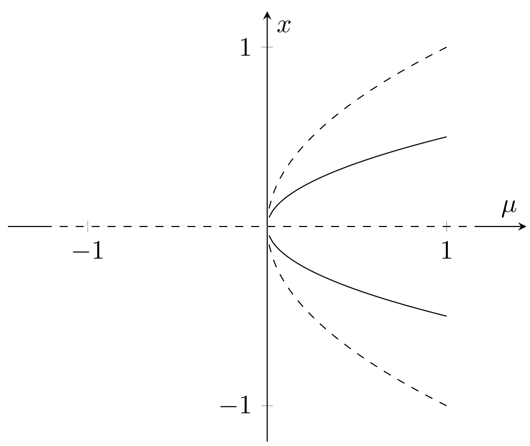
So our fixed points are non-hyperbolic if and only if $\mu = 0$.



If $\mu < 0$, then the 1 fixed point $x_1^* = 0$ is a source. If $\mu > 0$, then we have

$$x_{2_1|2_2}^* = \pm\sqrt{\mu} \rightsquigarrow \text{sources} \quad x_1^* = 0 \rightsquigarrow \text{source} \quad \text{and} \quad x_{3_1|3_2}^* = \pm\frac{\sqrt{\mu}}{2} \rightsquigarrow \text{sinks}.$$

So we get the following bifurcation diagram with bifurcation value $\mu = 0$.



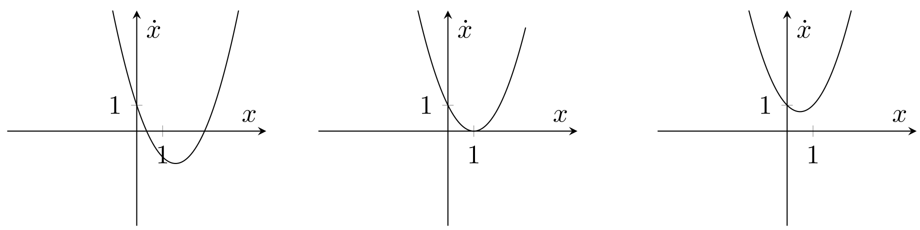
2. In each of the following one-dimensional examples, show that bifurcations occur at critical values of the parameter $\mu \in \mathbb{R}$ (to be determined), sketch a bifurcation diagram, and classify the bifurcations as being of the type *fold*, *transcritical*, *supercritical/subcritical pitchfork*, or *unknown* (if not seen in class):

   (a) $\dot{x} = 1 + \mu x + x^2$;

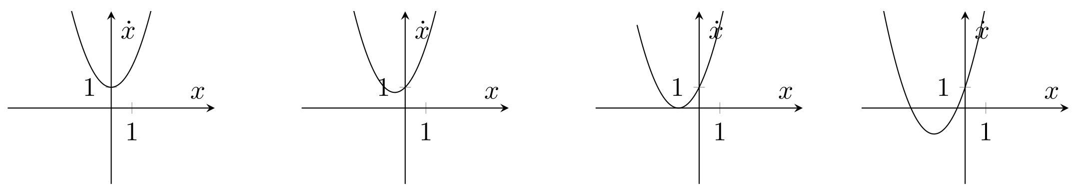
   Solution:

   The system has fixed points $x^*_{1|2} = \frac{-\mu \pm \sqrt{\mu^2-4}}{2}$. Then

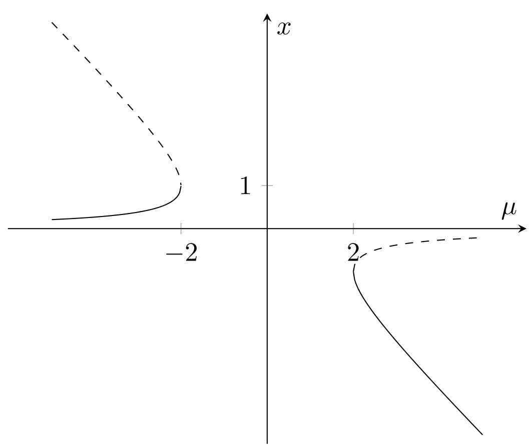
   $\boldsymbol{\mu < -2}$, 2 fixed points $\boldsymbol{\mu = -2}$, 1 fixed point $\boldsymbol{-2 < \mu < 0}$, 0 fixed points



   $\boldsymbol{\mu = 0}$, 0 fixed points $\boldsymbol{0 < \mu < 2}$, 0 fixed points $\boldsymbol{\mu = 2}$, 0 fixed points $\boldsymbol{\mu > 2}$, 2 fixed points



   And so we have the following bifurcation diagram, with bifurcation values $\mu = -2$ and $\mu = 2$, both fold bifurcations. These bifurcation values are precisely those for which our number of fixed points change, hence



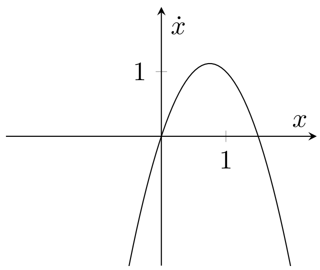
   The equations $f(x, \mu) = 0$ and $f_x(x, \mu) = 0$ together imply that we have critical values $(x, \mu) = (\pm 1, \mp 2)$, which agrees with our bifurcation values for μ.

(b) $\dot{x} = x - \mu x(1 - x)$;

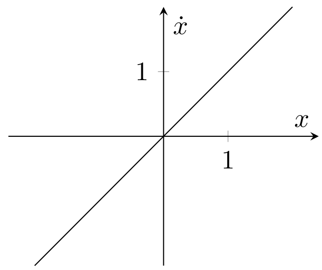
Solution:

When $\mu \neq 0$, the system has fixed points $x_1^* = 0$ and $x_2^* = \frac{\mu - 1}{\mu}$. When $\mu = 0$, the system has a single fixed point, $x_1^* = 0$. Then

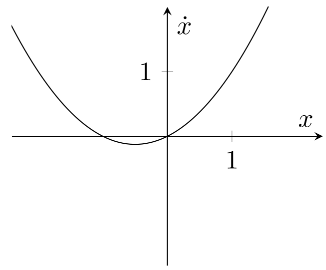
$\mu < 0$, 2 fixed points



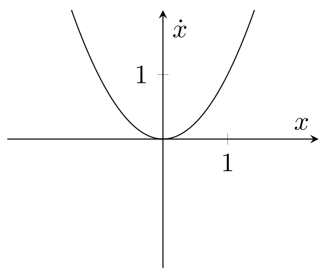
$\mu = 0$, 1 fixed point



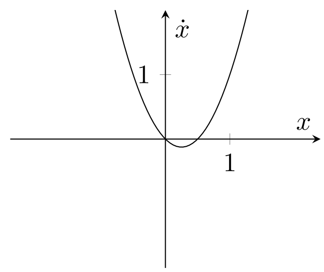
$0 < \mu < 1$, 2 fixed points



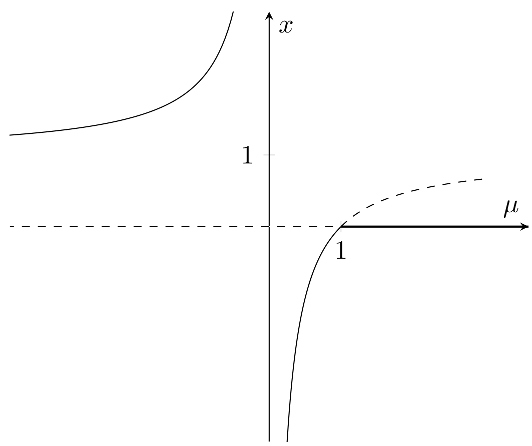
$\mu = 1$, 1 fixed points



$\mu > 1$, 2 fixed points



And so we have the following bifurcation diagram, with transcritical bifurcation value at $\mu = 1$.

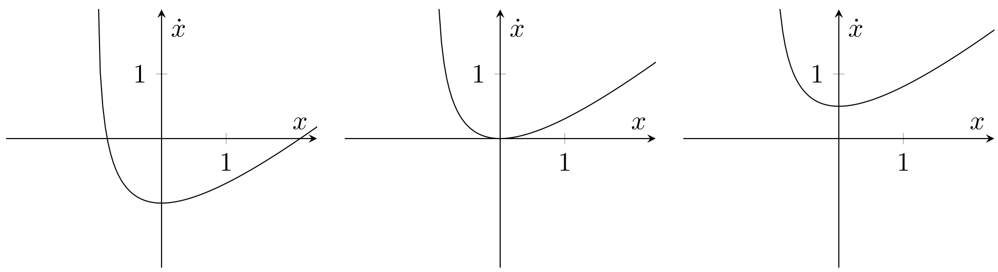


The equations $f(x, \mu) = 0$ and $f_x(x, \mu) = 0$ together imply that we have a critical value $(x, \mu) = (0, 1)$, which agrees with our bifurcation value for μ.

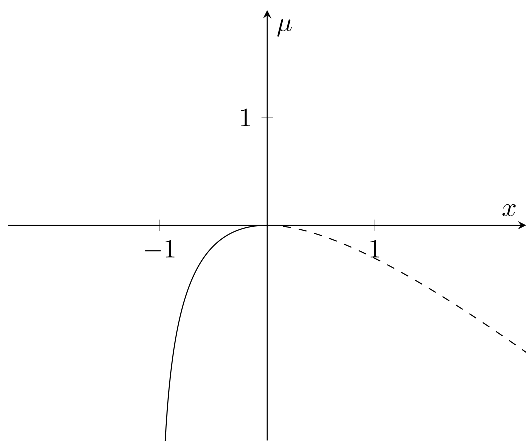
(c) $\dot{x} = \mu + x - \ln(1+x)$.

Solution:
We have

$\boldsymbol{\mu < 0}$, 2 fixed points $\boldsymbol{\mu = 0}$, 1 fixed point $\boldsymbol{\mu > 0}$, 0 fixed points



So when $\mu < 0$ the fixed point for when $x < 0$ is a sink, and the fixed point for when $x > 0$ is a source. Notice that when $\dot{x} = 0$, we have $\mu = -x + \ln(1+x)$. So we have the following bifurcation diagram, with a fold bifurcation at $\mu = 0$.



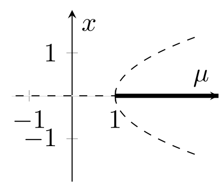
Again, The equations $f(x,\mu) = 0$ and $f_x(x,\mu) = 0$ together imply that we have a critical value $(x,\mu) = (0,0)$, which agrees with our bifurcation value for μ.

Justify your answers and show your work.

1. In the following two 1-dimensional examples, show that bifurcations occur at critical values of the parameter $\mu \in \mathbb{R}$ (to be determined), sketch a bifurcation diagram, and classify the bifurcations as being of the type *fold*, *transcritical*, or *supercritical/subcritical pitchfork* this time by also checking the conditions given in Theorems 6.9, 6.10, or 6.12 of the class notes:

   (a) $\dot{x} = x - \frac{\mu x}{1+x^2}$

   Solution:
   We have the following bifurcation diagram, with a subcritical pitchfork bifurcation at $\mu = 1$:



The equations $f(x, \mu) = 0$ and $f_x(x, \mu) = 0$ together imply that we have a critical value $(x^*, \mu^*) = (0, 1)$. which agrees with our bifurcation value for μ. Now,

$$f_\mu(x, \mu) = -\frac{x}{x^2+1}, \qquad f_{xx}(x, \mu) = -\frac{2\mu x(x^2-3)}{(x^2+1)^3},$$
$$f_{\mu x}(x, \mu) = \frac{x^2-1}{(x^2+1)^2}, \qquad f_{xxx}(x, \mu) = \frac{6\mu(x^4-6x^2+1)}{(x^2+1)^4}.$$

Therefore,

$$f_\mu(x^*, \mu^*) = 0, \qquad f_{xx}(x^*, \mu^*) = 0$$
$$f_{\mu x}(x^*, \mu^*) = -1 \neq 0, \qquad f_{xxx}(x^*, \mu^*) = 6 \neq 0.$$

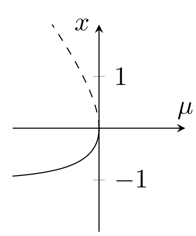
Moreover,

$$f_{\mu x}(x^*, \mu^*) f_{xxx}(x^*, \mu^*) = -6 < 0 \quad \text{and} \quad f_{xxx}(x^*, \mu^*) = 6 > 0.$$

These satisfy the conditions of Theorem 6.12, so we do indeed have a subcritical pitchfork bifurcation at $\mu^* = 1$, with the "inner middle" fixed point a sink, and the "outer" pair of fixed points each sources.

   (b) $\dot{x} = \mu + x - \ln(1 + x)$

   Solution:
   We have the following bifurcation diagram, with a fold bifurcation at $\mu = 0$:



The equations $f(x, \mu) = 0$ and $f_x(x, \mu) = 0$ together imply that we have a critical value $(x^*, \mu^*) = (0, 0)$. which agrees with our bifurcation value for μ. Now, $f_\mu(x, \mu) = 1$ and $f_{xx}(x, \mu) = \frac{1}{(1+x)^2}$, and so

$$f_\mu(x^*, \mu^*) = 1 \neq 0, \qquad f_{xx} f(x^*, \mu*) = 1 \neq 0,$$
$$f_\mu(x^*, \mu^*) f_{xx} f(x^*, \mu*) = 1 > 0, \qquad f_{xx}(x^*, \mu^*) = 1 > 0.$$

These satisfy the conditions of Theorem 6.9, so we do indeed have a fold bifurcation at $\mu^* = 0$, with the "upper" fixed point a source, and the "lower" fixed point a sink.

2. Show that the system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ (2\mu - 4x_1^2)x_2 - x_1 \end{bmatrix}$$

(equivalent to the second order scalar ODE $\ddot{x}_1 + (4x_1^2 - 2\mu)\dot{x}_1 + x_1 = 0$) has a *supercritical Hopf* bifurcation at $\mu^* = 0$ by checking the conditions of Theorem 6.13 of the class notes.

Solution:

Let $f(x, \mu) = (f_1(x, \mu), f_2(x, \mu)) = (\dot{x}_1, \dot{x}_2)$ and let $x^* = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Then

$$f(x^*, \mu^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad D_x f(x^*, \mu^*) = \begin{bmatrix} 0 & 1 \\ -8x_1x_2 - 1 & 2\mu - 4x_1^2 \end{bmatrix}\Bigg|_{\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], 0\right)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

so the first two conditions of the Theorem are satisfied, with $\omega = -1$. Now,

$$\begin{aligned} a &= \partial^2_{\mu x_1} f_1(x^*, \mu^*) + \partial^2_{\mu x_2} f_2(x^*, \mu^*) \\ &= 0 + 2 \neq 0 \\ b &= 0 + 0 + 0 - 8 + \frac{1}{-1}\Big((0)(0+0) - (-8x_1^*)(-1x_2^* + 0) - (0)(-8x_2^*) - (0)(0)\Big) \\ &= -8 \neq 0, \end{aligned}$$

and so the last two conditions of the Theorem are satisfied.

3. Consider the Lorenz equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma \cdot (y - x) \\ rx - y - xz \\ -bz + xy \end{bmatrix}$$

with parameters $\sigma = 10$, $b = 8/3$, r, and initial conditions

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \\ r - 1 \end{bmatrix}.$$

Using the numerical integration method given by the method of Kutta $RK4$ (see pp. 67-68 of the class notes) with a constant stepsize $h = 0.004$, plot in different plots the numerical approximations to the solution $(x(t), y(t), z(t))$ in phase space $\mathbb{R}^3$ (using for example the MATLAB command `plot3`) for $t \in [0, 120]$ and the following 8 different values of the parameter r: $r = 1,\ 14,\ 24,\ 24.2,\ 28,\ 100,\ 102,\ 400$.
Solution:

Code:

```
t(1)=0;
h=0.004;
r=100;
N=120/h;
s=10;
b=8/3;

x(1)=-8;
y(1)=8;
z(1)=r-1;
for n=1:N
        X1=x(n);
        Y1=y(n);
        Z1=z(n);

        X2=x(n)+(h/2)*(s*(Y1-X1));
        Y2=y(n)+(h/2)*(r*X1-Y1-X1*Z1);
        Z2=z(n)+(h/2)*(-b*Z1+X1*Y1);

        X3=x(n)+(h/2)*(s*(Y2-X2));
        Y3=y(n)+(h/2)*(r*X2-Y2-X2*Z2);
        Z3=z(n)+(h/2)*(-b*Z2+X2*Y2);

        X4=x(n)+h*(s*(Y3-X3));
        Y4=y(n)+h*(r*X3-Y3-X3*Z3);
        Z4=z(n)+h*(-b*Z3+X3*Y3);

        x(n+1)=x(n)+(h/6)*(s*(Y1-X1)+2*s*(Y2-X2)+2*s*(Y3-X3)+s*(Y3-X3));
        y(n+1)=y(n)+(h/6)*((r*X1-Y1-X1*Z1)+2*(r*X2-Y2-X2*Z2)+2*(r*X3-Y3-X3*Z3)+(r*X4-Y4-X4*Z4));
        z(n+1)=z(n)+(h/6)*((-b*Z1+X1*Y1)+2*(-b*Z2+X2*Y2)+2*(-b*Z3+X3*Y3)+(-b*Z4+X4*Y4));
end
plot3(x,y,z)
```

Plots:

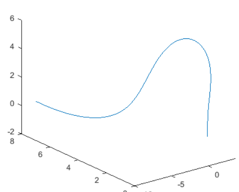


Figure 1: $r = 1$

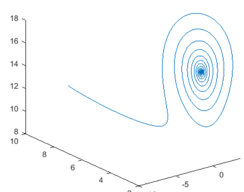


Figure 2: $r = 14$

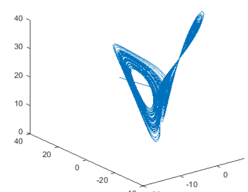


Figure 3: $r = 24$

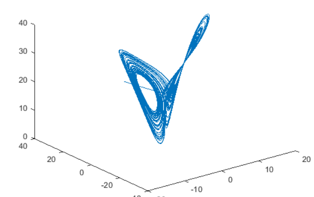


Figure 4: $r = 24.2$

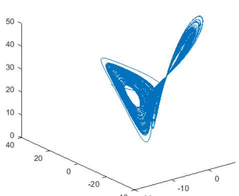


Figure 5: $r = 28$

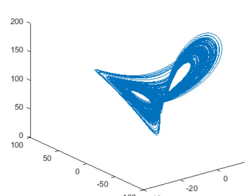


Figure 6: $r = 100$

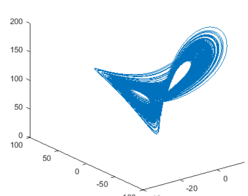


Figure 7: $r = 102$

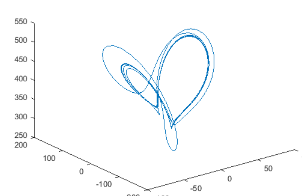


Figure 8: $r = 400$