Homework for Algebraic Topology

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Beware: Some solutions may be incorrect!

Algebraic Topology
Homework 1
Name: 16977

1 Adjudication of adjunctions

1.1 In the beginning

Definition 1. An object $a \in Ob(\mathcal{C})$ is *initial* if for each $b \in Ob(\mathcal{C})$ there is a unique morphism $a \to b$. An object $z \in Ob(\mathcal{C})$ is *final* if it is initial in $\mathcal{C}^{op}$.

Since final objects are dual by a different exercise, we'll focus on initial objects.

1. Prove that initial objects are uniquely determined up to isomorphism when they exist.

   Proof. Let $a, a' \in Ob(\mathcal{C})$ be initial. Then there exists unique morphisms $a \xrightarrow{f} a'$ and $a' \xrightarrow{g} a$. However, the identity morphism $a \xrightarrow{1_a} a$ is the unique morphism from a to itself, and so $g \circ f = 1_a$. Similarly, $f \circ g = 1_{a'}$, and so $a \cong a'$. □

2. Show that 0 is both initial and final in Mod_R.

   Proof. For any $m \in Ob(Mod_R)$ any morphism $f \in \mathrm{Hom}_{Mod_R}(0, m)$ is a group homomorphism, and so the only map we have is $f : 0 \mapsto 0$. Similarly, $0 \mapsto 0$ is the only map in $\mathrm{Hom}_{Mod_R}(m, 0)$. □

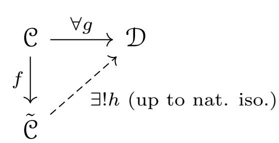
3. If $\mathcal{C}$ is a category then prove that there is a category $\tilde{\mathcal{C}}$ uniquely determined up to equivalence of categories by the properties:

   (a) $\tilde{\mathcal{C}}$ contains an initial object

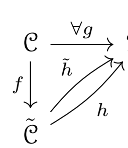
   (b) There is a canonical functor $\mathcal{C} \to \tilde{\mathcal{C}}$ which is an initial object in the category of functors from $\mathcal{C}$ to categories $\mathcal{D}$ containing initial objects.

   Proof. First, existence of $\tilde{\mathcal{C}}$: Define a category $\tilde{\mathcal{C}}$ which has the same objects and morphisms as $\mathcal{C}$ and add an object X such that $\mathrm{Hom}_{\mathcal{C}}(X, X) = \{1_{\mathcal{C}}\}$ and for all $Y \in Ob(\tilde{\mathcal{C}})$ different from X, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set with cardinality equal to 1. By construction, $\mathcal{C}$ is a category with an initial object X.

   Let Cat_{in} be the category of (small) categories containing initial objects, where we require that morphisms send initial objects to initial objects. Let $f : \mathcal{C} \to \tilde{\mathcal{C}}$ be the inclusion functor. We want to show that f is an initial object in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$[1].



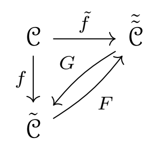
   So, let $g : \mathcal{C} \to \mathcal{D}$ be any object in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$ with $D \in Ob(\mathcal{D})$ initial. Then define $h : \tilde{C} \to \mathcal{D}$ by the rules: $h = g$ for all objects and morphisms in $\mathcal{C}$, $h(X) = D$, and let $h(\alpha : X \to (-)) = h(\alpha) : D \to h(-)$ be the unique map in $\mathcal{D}$ from D to $(-)$. Then $h \circ f = g$ by construction.



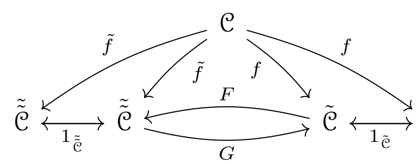
[1] Here, we only require that a morphism from an initial object in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$ to another object in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$ is unique *up to natural isomorphism.*

If $\tilde{h} : \tilde{\mathcal{C}} \to \mathcal{D}$ is another morphism in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$ so that $\tilde{h} \circ f = g$, then $h = \tilde{h}$ on $\mathcal{C}$. Suppose $\tilde{h}(X) = \tilde{D}$, where $\tilde{D}$ is an initial object in $\mathcal{D}$. Let $\alpha : D \xrightarrow{\sim} \tilde{D}$ be an isomorphism in $\mathcal{D}$ (since initial objects are unique up to isomorphism). Then define a natural transformation $\eta : h \implies \tilde{h}$ whose components $\eta_Y : h(Y) \to \tilde{h}(Y)$ are the identity for all $Y \in Ob(\mathcal{C})$. For $h(X) = D$, let $\eta_X = \alpha$. This makes η a natural isomorphism, as the reader can check. Hence $h \cong \tilde{h}$.

Now, uniqueness of $\tilde{\mathcal{C}}$. Suppose there exists $\tilde{\tilde{\mathcal{C}}}$ in $Ob(\mathrm{Cat}_{in})$ and a functor $\tilde{f} : \mathcal{C} \to \tilde{\tilde{\mathcal{C}}}$ which is an initial object in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$. Since both $f : \mathcal{C} \to \tilde{\mathcal{C}}$ and $\tilde{f} : \mathcal{C} \to \tilde{\tilde{\mathcal{C}}}$ are initial in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$, there exists unique functors (again, up to natural isomorphism) $F : \tilde{\mathcal{C}} \rightleftarrows \tilde{\tilde{\mathcal{C}}} : G$ so that $F \circ f = \tilde{f}$ and $G \circ \tilde{f} = f$.



Consider the functor $GF : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$. We have $G \circ F \circ f = G \circ \tilde{f} = f$. But since f is initial in $\mathcal{C} \downarrow \mathrm{Cat}_{in}$ then $1_{\tilde{\mathcal{C}}}$ is the unique morphism (up to natural isomorphism) for which $1_{\tilde{\mathcal{C}}} \circ f = f$. Hence $GF \cong 1_{\tilde{\mathcal{C}}}$.



Similarly, the functor FG is such that $F \circ G \circ \tilde{f} = F \circ f = \tilde{f}$, and so $FG \cong 1_{\tilde{\tilde{\mathcal{C}}}}$. So $\tilde{\mathcal{C}}$ and $\tilde{\tilde{\mathcal{C}}}$ are equivalent categories, completing the proof.

□

1.2 Fermenation above the firmament

Let's formalize the last part of the last problem a bit more.

Definition 2. If $\mathcal{C}$ is a category and $x \in Ob(\mathcal{C})$ then the *overcategory* $\mathcal{C}\_x$ whose objects are arrows

$$Ob(\mathcal{C} \downarrow x) = \{y \to x : y \in Ob(\mathcal{C})\}$$

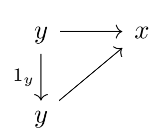
Similarly, the undercategory satisfies

$$Ob(x \downarrow \mathcal{C}) = \{x \to y : y \in Ob(\mathcal{C})\}$$

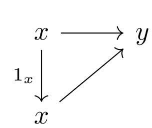
In both cases, morphisms are commutative diagrams. If $f : \mathcal{C} \to \mathcal{D}$ is a functor then there are categories $x \downarrow f$ consisting of objects $y \to f(x)$ and $f \downarrow x$ consisting of objects $f(x) \to y$. In these cases a map is one of the form $f(a : x \to x')$.

1. Check that overcategories and undercategories are categories.

   Solution: For any object $(y \to x) \in Ob(\mathcal{C} \downarrow x)$, the identity morphism $(1_y : y \to y)$ give an an identity morphism in $\mathrm{Hom}_{\mathcal{C}\downarrow x}(y \to x, y \to x)$:



Similarly, for any object $(x \to y) \in Ob(x \downarrow \mathcal{C})$, the identity morphism $(1_x : x \to x)$ gives an an identity morphism in $\mathrm{Hom}_{x\downarrow\mathcal{C}}(x \to y, x \to y)$:



Let $\alpha : y \to x, \beta : w \to x, \gamma : z \to x, \delta : v \to x \in Ob(\mathcal{C} \downarrow x)$. Given morphisms $f : y \to w, g : w \to z, h : z \to v$, we have composition defined by

$$\begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle f}\downarrow & \nearrow_{\beta} \\ w & \end{array} \circ \begin{array}{cc} w & \xrightarrow{\beta} x \\ {\scriptstyle g}\downarrow & \nearrow_{\gamma} \\ z & \end{array} = \begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle f}\downarrow & \\ w & \nearrow_{\gamma} \\ {\scriptstyle g}\downarrow & \\ z & \end{array} = \begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle g\circ f}\downarrow & \nearrow_{\gamma} \\ z & \end{array}$$

Which is well defined since the commutativity of the left-hand side diagrams and associativity in $\mathcal{C}$ gives $\gamma \circ (g \circ f) = (\gamma \circ g) \circ f = \beta \circ f = \alpha$. We then get associative composition in $Ob(\mathcal{C} \downarrow x)$:

$$\begin{aligned} \left(\begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle f}\downarrow & \nearrow_{\beta} \\ w & \end{array} \circ \begin{array}{cc} w & \xrightarrow{\beta} x \\ {\scriptstyle g}\downarrow & \nearrow_{\gamma} \\ z & \end{array} \right) \circ \begin{array}{cc} z & \xrightarrow{\gamma} x \\ {\scriptstyle h}\downarrow & \nearrow_{\delta} \\ v & \end{array} &= \begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle g\circ f}\downarrow & \nearrow_{\gamma} \\ z & \end{array} \circ \begin{array}{cc} z & \xrightarrow{\gamma} x \\ {\scriptstyle h}\downarrow & \nearrow_{\delta} \\ v & \end{array} \\ &= \begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle h\circ g\circ f}\downarrow & \nearrow_{\delta} \\ v & \end{array} \\ &= \begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle f}\downarrow & \nearrow_{\beta} \\ w & \end{array} \circ \begin{array}{cc} w & \xrightarrow{\beta} x \\ {\scriptstyle h\circ g}\downarrow & \nearrow_{\delta} \\ v & \end{array} \\ &= \begin{array}{cc} y & \xrightarrow{\alpha} x \\ {\scriptstyle f}\downarrow & \nearrow_{\beta} \\ w & \end{array} \circ \left(\begin{array}{cc} w & \xrightarrow{\beta} x \\ {\scriptstyle g}\downarrow & \nearrow_{\gamma} \\ z & \end{array} \circ \begin{array}{cc} z & \xrightarrow{\gamma} x \\ {\scriptstyle h}\downarrow & \nearrow_{\delta} \\ v & \end{array} \right) \end{aligned}$$

Showing that composition is associative in $x \downarrow \mathcal{C}$ is very similar.

2. Suppose F is left adjoint to G. Prove that this is equivalent to the statement that the unit and counit maps

$$1 \to GF \quad \text{and} \quad FG \to 1$$

are non-degenerate, i.e. the compositions

$$G \to GFG \to G \quad \text{and} \quad F \to FGF \to F$$

are both identity natural transformations.

Proof. ($\Rightarrow$) Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be adjoint functors, i.e., there is a natural isomorphism

$$\Psi_{x,y} : \operatorname{Hom}_{\mathcal{D}}(F(x), y) \implies \operatorname{Hom}_{\mathcal{C}}(x, G(y))$$

for each $x \in Ob(\mathcal{C})$ and each $y \in Ob(\mathcal{D})$. Let

$$\eta : 1_{\mathcal{C}} \implies GF \quad \text{and} \quad \epsilon : FG \implies 1_{\mathcal{D}}$$

be the unit and counit maps of the adjunction, i.e.,

$$\eta_x = \Psi_{x,F(x)}(1_{F(x)}) \quad \text{and} \quad \epsilon_y = \Psi^{-1}_{G(y),y}(1_{G(y)})$$

for each $x \in Ob(\mathcal{C})$ and each $y \in Ob(\mathcal{D})$. Then by whiskering η with G and also G with η, we get natural transformations

$$\eta G : (1_{\mathcal{C}} \circ G) \implies (GF \circ G) \quad \text{and} \quad G\epsilon : (G \circ FG) \implies (G \circ 1_{\mathcal{D}}).$$

i.e.,

$$\eta G : G \implies GFG \quad \text{and} \quad G\epsilon : GFG \implies G.$$

For $y \in Ob(\mathcal{D})$, $G\epsilon \circ \eta G$ gives a composition in $\mathcal{C}$:

$$G(y) \xrightarrow[\eta_{G(y)}]{} GFG(y) \xrightarrow[G(\epsilon_y)]{} G(y), \qquad G(y) \xrightarrow{G(\epsilon_y) \circ \eta_{G(y)}} G(y)$$

We want to show $G(\epsilon_y) \circ \eta_{G(y)} = 1_{G(y)}$. The naturality of Ψ gives a commutative diagram for $f : c' \to c \in Hom_{\mathcal{C}^{op}}(c', c)$ and $g : d \to d' \in \operatorname{Hom}_{\mathcal{D}}(d, d')$:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{D}}(F(c), d) & \xrightarrow{\Psi_{c,d}} & \operatorname{Hom}_{\mathcal{C}}(c, G(d)) \\
\Big\downarrow {\scriptstyle \operatorname{Hom}_{\mathcal{D}}(F(f), g)} & & \Big\downarrow {\scriptstyle \operatorname{Hom}_{\mathcal{C}}(f, G(g))} \\
\operatorname{Hom}_{\mathcal{D}}(F(c'), d') & \xrightarrow{\Psi_{c',d'}} & \operatorname{Hom}_{\mathcal{C}}(c', G(d'))
\end{array}$$

where the maps $\operatorname{Hom}_{\mathcal{D}}(F(f), g)$ and $\operatorname{Hom}_{\mathcal{C}}(f, G(g))$ are respectively defined

$$(h : F(c) \to d) \longmapsto g \circ h \circ F(f) : F(c') \to F(c) \to d \to d'$$
$$(k : c \to G(d)) \longmapsto G(g) \circ k \circ f : c' \to c \to G(d) \to G(d')$$

So in particular, if we make the substitutions

$$f = 1_{G(y)} : G(y) \longrightarrow G(y) \quad \text{and} \quad g = \epsilon_y : FG(y) \longrightarrow y$$

we get the commuting diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{D}}(FG(y), FG(y)) & \xrightarrow{\Psi_{G(y),FG(y)}} & \operatorname{Hom}_{\mathcal{C}}(G(y), GFG(y)) \\
\Big\downarrow {\scriptstyle \operatorname{Hom}_{\mathcal{D}}(F(1_{G(y)}), \epsilon_y)} & & \Big\downarrow {\scriptstyle \operatorname{Hom}_{\mathcal{C}}(1_{G(y)}, G(\epsilon_y))} \\
\operatorname{Hom}_{\mathcal{D}}(FG(y), y) & \xrightarrow{\Psi_{G(y),y}} & \operatorname{Hom}_{\mathcal{C}}(G(y), G(y))
\end{array}$$

So, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(1_{G(y)}, G(\epsilon_y))\left(\Psi_{G(y),FG(y)}(1_{FG(y)})\right) &= \Psi_{G(y),y}\left(\mathrm{Hom}_{\mathcal{C}}(F(1_{G(y)}), \epsilon_y)(1_{FG(y)})\right) \\
G(\epsilon_y) \circ \Psi_{G(y),FG(y)}(1_{FG(y)}) \circ 1_{G(y)} &= \Psi_{G(y),y}\left(\epsilon_y \circ 1_{FG(y)} \circ F(1_{G(y)})\right) \\
G(\epsilon_y) \circ \eta_{G(y)} &= \Psi_{G(y),y}(\epsilon_y) \\
&= \Psi_{G(y),y}\left(\Psi^{-1}_{G(y),y}(1_G(y))\right) \\
&= 1_{G(y)},
\end{aligned}$$

as desired. Let's do it all again the other way around. This time, we whisker together F with η and also ϵ with F to obtain natural transformations

$$F\eta : F \implies FGF \quad \text{and} \quad \epsilon F : FGF \implies F.$$

For $x \in Ob(\mathcal{C})$, $\epsilon F.F\eta$ gives a composition in $\mathcal{D}$

$$F(x) \xrightarrow[F(\eta_x)]{} FGF(x) \xrightarrow[\epsilon_{F(x)}]{} F(x), \qquad F(x) \xrightarrow{\epsilon_{F(x)} \circ F(\eta_x)} F(x)$$

We want to show $\epsilon_{F(x)} \circ F(\eta_x) = 1_{F(x)}$. As before, we make substitutions

$$f = \eta_x : x \to GF(x) \quad \text{and} \quad g = 1_{F(x)} : F(x) \to F(x)$$

to get a commuting diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FGF(x), F(x)) & \xleftarrow{\Psi^{-1}_{GF(x),F(x)}} & \mathrm{Hom}_{\mathcal{C}}(GF(x), GF(x)) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{D}}(F(\eta_x), 1_{F(x)})} & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(\eta_x, G(1_{F(x)})} \\
\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) & \xleftarrow{\Psi^{-1}_{x,F(x)}} & \mathrm{Hom}_{\mathcal{C}}(x, GF(x))
\end{array}$$

So, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(F(\eta_x), 1_{F(x)})\left(\Psi^{-1}_{GF(x),F(x)}(1_{GF(x)})\right) &= \Psi^{-1}_{x,F(x)}\left(\mathrm{Hom}_{\mathcal{C}}(\eta_x, G(1_{F(x)}))(1_{GF(x)})\right) \\
1_{F(x)} \circ \Psi^{-1}_{GF(x),F(x)}(1_{GF(x)}) \circ F(\eta_x) &= \Psi^{-1}_{x,F(x)}\left(G(1_{F(x)}) \circ 1_{GF(x)} \circ \eta_x\right) \\
\epsilon_{F(x)} \circ F(\eta_x) &= \Psi^{-1}_{x,F(x)}(\eta_x) \\
&= \Psi^{-1}_{x,F(x)}\left(\Psi_{x,F(x)}(1_{F(x)})\right) \\
&= 1_{F(x)}
\end{aligned}$$

($\Leftarrow$) Now we assume we have unit and counit maps and the equations

$$G(\epsilon_y) \circ \eta_{G(y)} = 1_{G(y)} \quad \text{and} \quad \epsilon_{F(x)} \circ F(\eta_x) = 1_{F(x)}.$$

Then we define functors

$$\Psi_{x,y} : \mathrm{Hom}_{\mathcal{D}}(F(x), y) \Longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(x, G(y)) : \Psi^{-1}_{x,y}$$

in the obvious ways:

$$(F(x) \xrightarrow{h} y) \overset{\Psi_{x,y}}{\longmapsto} (G(h) \circ \eta_x), \qquad \begin{array}{ccc} x & \xrightarrow{\eta_x} & GF(x) \\ & {\scriptstyle G(h)\circ\eta_x} \searrow & \Big\downarrow {\scriptstyle G(h)} \\ & & G(y) \end{array}$$

and

$$(x \xrightarrow{k} G(y)) \xmapsto{\Psi^{-1}_{x,y}} (\epsilon_y \circ F(k)), \qquad \begin{array}{ccc} F(x) & \xrightarrow{F(k)} & FG(y) \\ & \searrow^{\epsilon_y \circ F(k)} & \downarrow \epsilon_y \\ & & y \end{array}$$

These maps are inverses of each other:

$$\begin{aligned} \Psi^{-1}_{x,y}(\Psi_{x,y}(h)) = \Psi^{-1}_{x,y}(G(h) \circ \eta_x) = \epsilon_y \circ F(G(h) \circ \eta_x) &= \epsilon_y \circ F(G(h)) \circ F(\eta_x) \\ &= h \circ \epsilon_{F(x)} \circ F(\eta_x) \qquad (*) \\ &= h, \\ \Psi_{x,y}(\Psi^{-1}_{x,y}(k)) = \Psi_{x,y}(\epsilon_y \circ F(k)) = G(\epsilon_y \circ F(k)) \circ \eta_x &= G(\epsilon_y) \circ GF(k) \circ \eta_x \\ &= G(\epsilon_y) \circ \eta_{G(y)} \circ k \qquad (**) \\ &= k \end{aligned}$$

where $(*)$ and $(**)$ come from the commutative diagrams

$$\begin{array}{ccc} FGF(x) & \xrightarrow{FG(h)} & FG(y) \\ \epsilon_{F(x)} \downarrow & & \downarrow \epsilon_y \\ F(x) & \xrightarrow{h} & y \end{array} \quad \text{and} \quad \begin{array}{ccc} x & \xrightarrow{k} & G(y) \\ \eta_x \downarrow & & \downarrow \eta_{G(y)} \\ GF(x) & \xrightarrow{GF(k)} & GFG(y) \end{array}.$$

We also need to check that $\Psi_{x,y}$ is natural, i.e., for $(f : x' \to x) \in \mathcal{C}^{op}$ and $(g : y \to y') \in \mathcal{D}$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(x), y) & \xrightarrow{\Psi_{x,y}} & \mathrm{Hom}_{\mathcal{C}}(x, G(y)) \\ \mathrm{Hom}_{\mathcal{D}}(F(f), g) \downarrow & & \downarrow \mathrm{Hom}_{\mathcal{C}}(f, G(g)) \\ \mathrm{Hom}_{\mathcal{D}}(F(x'), y') & \xrightarrow{\Psi_{x',y'}} & \mathrm{Hom}_{\mathcal{C}}(x', G(y')) \end{array}$$

So let $h \in \mathrm{Hom}_{\mathcal{D}}(F(x), y)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(f, G(g))(\Psi_{x,y}(h)) &= \mathrm{Hom}_{\mathcal{C}}(f, G(g))(G(h) \circ \eta_x) \\ &= G(g) \circ G(h) \circ \eta_x \circ f \end{aligned}$$

and on the other hand

$$\begin{aligned} \Psi_{x',y'}(\mathrm{Hom}_{\mathcal{D}}(F(f), g))(h)) &= \Psi_{x',y'}(g \circ h \circ F(f)) \\ &= G(g \circ h \circ F(f)) \circ \eta_{x'} \\ &= G(g) \circ G(h) \circ GF(f)) \circ \eta_{x'} \\ &= G(g) \circ G(h) \circ \eta_x \circ f, \end{aligned}$$

where the last equality comes from the commutative diagram

$$\begin{array}{ccc} x' & \xrightarrow{f} & x \\ \eta_{x'} \downarrow & & \downarrow \eta_x \\ GF(x') & \xrightarrow{GF(f)} & GF(x) \end{array}.$$

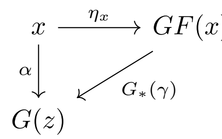
□

3. Suppose $F : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$. Observe that g determines a functor

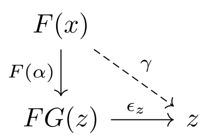
$$G_* : F \downarrow x \to x \downarrow F$$

defined so that $G_*(F(x) \to y) = GF(x) \to G(y)$, which makes sense because $GF(x) \in Ob(\mathcal{C})$. The unit $\eta_x : x \to GF(x)$ is called *universal* if it is initial. In long form: for each map $x \to G(z)$ there exists a unique map $F(x) \to z$ so that $G_*(F(x) \to z)$ commutes. Show that if F is left adjoint to G then $\eta_x : x \to GF(x)$ is universal. BONUS: prove the converse.

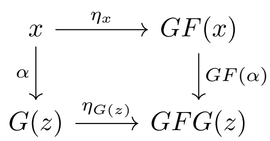
Proof. We first show the existence of a map $\gamma : F(x) \to z$ such that the diagram commutes:



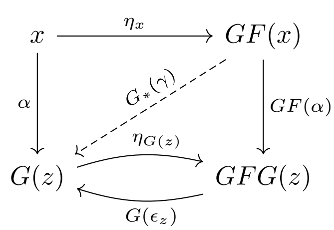
Applying the functor F to the morphism $\alpha : x \to G(z)$, we get a map $F(\alpha) : F(x) \to FG(z)$. But we also have the map $\epsilon_z : FG(z) \to z$, and so it seems a good choice to define $\gamma = \epsilon_z \circ F(\alpha)$.



Then $G_*(\gamma) = G(\epsilon_z) \circ GF(\alpha)$. Since η is a natural transformation between $1_{\mathcal{C}}$ and GF, we get the commutative diagram



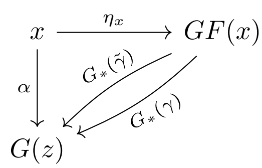
Considering $G(\epsilon_z)$, we get



Since F is left adjoint to G, we saw in the previous problem that $G(\epsilon_z) \circ \eta_{G(z)} = 1_{G(z)}$. So

$$G_*(\gamma) \circ \eta_x = G(\epsilon_z) \circ GF(\alpha) \circ \eta_x = G(\epsilon_z) \circ \eta_{G(z)} \circ \alpha = \alpha,$$

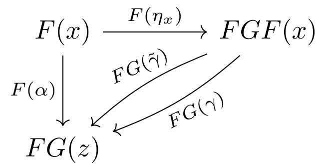
as desired. Now, uniqueness: Suppose there were another map $\tilde{\gamma} : F(x) \to z$ making the diagram commute



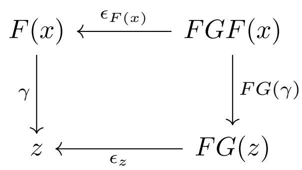
Applying F to the equality $G_*(\gamma) \circ \eta_x = \alpha = G_*(\tilde{\gamma}) \circ \eta_x$, we get

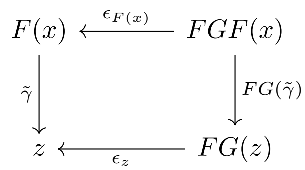
$$FG(\gamma) \circ F(\eta_x) = F(\alpha) = FG(\tilde{\gamma}) \circ F(\eta_x),$$

making the diagram commute:

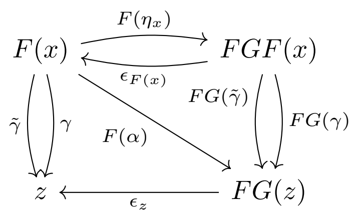


Since ϵ is a natural transformation from FG to $1_{\mathcal{D}}$, we get the commutative diagrams





Putting the last three diagrams together, we have



Since F is left adjoint to G, we proved in the last problem that $\epsilon_{F(x)} \circ F(\eta_x) = 1_{F(x)}$. So

$$\begin{aligned}\gamma = \gamma \circ \epsilon_{F(x)} \circ F(\eta_x) = \epsilon_z \circ FG(\gamma) \circ F(\eta_x) &= \epsilon_z \circ FG(\tilde{\gamma}) \circ F(\eta_x) \\ &= \tilde{\gamma} \circ \epsilon_{F(x)} \circ F(\eta_x) \\ &= \tilde{\gamma}\end{aligned}$$

□

1.3 Adjunctions

1. There is a forgetful functor from R-modules to sets. What are the left and right adjoints of this functor? Do they always exist? Prove or disprove.

2. If $f_1 : \mathcal{A} \to \mathcal{B}$ and $f_2 : \mathcal{B} \to \mathcal{C}$ have right adjoints $g_1 : \mathcal{B} \to \mathcal{A}$ and $g_2 : \mathcal{C} \to \mathcal{B}$ respectively. Then prove that $f_2 \circ f_1$ has right adjoint $g_1 \circ g_2$.

Proof. The hypotheses give natural isomorphisms

$$\begin{aligned} \Psi_{x,y} &: \mathrm{Hom}_{\mathcal{B}}(f_1(x), y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(x, g_1(y)) \quad \forall x \in Ob\mathcal{A}, \forall y \in Ob\mathcal{B} \\ \Phi_{w,z} &: \mathrm{Hom}_{\mathcal{C}}(f_2(w), z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(w, g_2(z)) \quad \forall w \in Ob\mathcal{B}, \forall z \in Ob\mathcal{C} \end{aligned}$$

For $x \in Ob\mathcal{A}$ and $z \in Ob\mathcal{C}$, set $g_2(z) = y$ in $\Psi_{x,y}$ and $f_1(x) = w$ in $\Phi_{w,z}$. Then

$$\begin{aligned} \Psi_{x,g_2(z)} &: \mathrm{Hom}_{\mathcal{B}}(f_1(x), g_2(z)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(x, g_1 g_2(z)) \\ \Phi_{f_1(x),z} &: \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(x), z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(f_1(x), g_2(z)) \end{aligned}$$

So then the map $\Xi_{x,z} := \Psi_{x,g_2(z)} \circ \Phi_{f_1(x),z}$ is a bijection. To show that the isomorphism is natural, we check for $\alpha : x' \to x$ in $\mathcal{C}^{op}$ and $\beta : z \to z'$ in $\mathcal{A}$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(x), z) & \xrightarrow{\Xi_{x,z}} & \mathrm{Hom}_{\mathcal{A}}(x, g_1 g_2(z)) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f_1 f_2(\alpha), \beta)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(\alpha, g_1 g_2(\beta))} \\ \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(x'), z') & \xrightarrow{\Xi_{x',z'}} & \mathrm{Hom}_{\mathcal{A}}(x', g_1 g_2(z')) \end{array}$$

This will essentially follow since Ξ is defined in terms of other natural isomorphisms:

$$\begin{array}{ccccc} & & \overset{\Xi_{x,z}}{\frown} & & \\ \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(x), z) & \xrightarrow{\Phi_{f_1(x),z}} & \mathrm{Hom}_{\mathcal{B}}(f_1(x), g_2(z)) & \xrightarrow{\Psi_{x,g_2(z)}} & \mathrm{Hom}_{\mathcal{A}}(x, g_1 g_2(z)) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(\alpha), \beta)} \downarrow & & {\scriptstyle \mathrm{Hom}_{\mathcal{B}}(f_1(\alpha), g_2(\beta))} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(\alpha, g_1 g_2(\beta))} \\ \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(x'), z') & \xrightarrow{\Phi_{f_1(x'),z'}} & \mathrm{Hom}_{\mathcal{B}}(f_1(x'), g_2(z')) & \xrightarrow{\Psi_{x',g_2(z')}} & \mathrm{Hom}_{\mathcal{A}}(x', g_1 g_2(z')) \\ & & \underset{\Xi_{x',z'}}{\smile} & & \end{array}$$

Let $\gamma \in \mathrm{Hom}_{\mathcal{C}}(f_2 f_1(x), z)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(\alpha, g_1 g_2(\beta)) \left(\Xi_{x,z}(\gamma)\right) &= \mathrm{Hom}_{\mathcal{A}}(\alpha, g_1 g_2(\beta)) \left(\Psi_{x,g_2(z)}(\Phi_{f_1(x),z}(\gamma))\right) \\ &= \Psi_{x',g_2(z')} \left(\mathrm{Hom}_{\mathcal{B}}(f_1(\alpha), g_2(\beta))(\Phi_{f_1(x),z}(\gamma))\right) \\ &= \Psi_{x',g_2(z')} \left(\Phi_{f_1(x'),z'}(\mathrm{Hom}_{\mathcal{C}}(f_1, f_2(\alpha), \beta)(\gamma))\right) \\ &= \Xi_{x',z'} \left(\mathrm{Hom}_{\mathcal{C}}(f_1, f_2(\alpha), \beta)(\gamma)\right), \end{aligned}$$

showing naturality. To show that $\Xi^{-1} = \Phi^{-1} \circ \Psi^{-1}$ is natural is very similar. So $f_2 \circ f_1$ has right adjoint $g_1 \circ g_2$. $\square$

3. In the category of, say, sets prove:

$$Hom(\coprod_i C_i, A) \cong \prod_i Hom(C_i, A) \quad \text{and} \quad Hom(A, \prod_i C_i) \cong \prod_i Hom(A, C_i)$$

Find criteria which allows you to conclude that this holds in a general setting.

Proof. The coproduct $\coprod_i C_i$ comes equipped with maps $\{\alpha_j : C_j \to \coprod_i C_i\}$. Define a map x

$$\Phi : Hom(\coprod_i C_i, A) \to \prod_i Hom(C_i, A) \quad \text{by} \quad f \mapsto \{f_i := f \circ \alpha_i\}_i.$$

If $\{g_i\}_i \in \prod_i Hom(C_i, A)$, then there exists a unique map $g : \coprod_i C_i \to A$ such that $g_i = g \circ \alpha_i$ for all i. Define a map

$$\Psi : \prod_i Hom(C_i, A) \to Hom(\coprod_i C_i, A) \quad \text{by} \quad \{g_i\}_i \mapsto g.$$

Then

$$\Psi\Phi(f) = \Psi(\{f \circ \alpha\}) = \tilde{f},$$

where $\tilde{f}$ is the unique map so that $f_i = \tilde{f} \circ \alpha$ for all i. Since also $f_i = f \circ \alpha$ for all i, then $f = \tilde{f}$, giving that $\Psi\Phi(f) = f$. Conversely,

$$\Phi\Psi(\{g_i\}_i) = \Phi(g) = \{\tilde{g_i} = g \circ \alpha_i\}_i.$$

But g is defined so that $g_i = g \circ \alpha_i$, and so $\tilde{g_i} = g_i$ for all i, i.e., $\Phi\Psi(\{g_i\}_i) = \{g_i\}_i$. Therefore we've shown the first isomorphism. The second isomorphism is very similar, and is left as an exercise for the reader. □

4. Let R be a commutative ring, $S \subset R$ a multiplicative set. There is a localization $\ell : R \to S^{-1}R$. Prove that for M an R-module $\ell_*(M) = S^{-1}R \otimes_R M$ is a $S^{-1}R$-module find the right adjoint of ℓ_*.

5. Define the category of abstract simplicial complexes. Characterize pushouts in this category.

2 In Soviet Russia: if you know Yoneda then *a yes knows you*

2.1 The theme

Definition 3. Let $\mathcal{C}$ be a category. The category of *presheaves on* $\mathcal{C}$ is the category of contravariant functors $PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$ from $\mathcal{C}$ to sets.

1. For each object $a \in \mathcal{C}$ prove that there is a functor $R_a \in Ob(PSh(\mathcal{C}))$ which is determined by the assignment

$$R_a(b) = Hom_{\mathcal{C}}(b, a)$$

Proof. Let $a \in Ob(\mathcal{C})$. We've defined how the map R_a acts on objects of $\mathcal{C}^{op}$, so we need to define how R_a acts on morphisms and show that these definitions make R_a into a functor, i.e., an object in $PSh(\mathcal{C})$. Given a morphism $f : x \to y$ in $\mathcal{C}$, we want to define a morphism in Set:

$$R_a(f) : \mathrm{Hom}_{\mathcal{C}}(y, a) \to \mathrm{Hom}_{\mathcal{C}}(x, a).$$

Define $R_a(f) = f_a^*$, where f_a^* is pre-composition with f, i.e. if $\alpha \in \mathrm{Hom}_{\mathcal{C}}(y, a)$, then $f_a^*(\alpha) = \alpha \circ f$.

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & \underset{f^*(\alpha)=\alpha\circ f}{\searrow} & \downarrow{\scriptstyle\alpha} \\ & & a \end{array}$$

Now, if $g : y \to z$ is another morphism in $\mathcal{C}$, then we have $(g \circ f)_a^* : \mathrm{Hom}_{\mathcal{C}}(z, a) \to \mathrm{Hom}_{\mathcal{C}}(x, a)$, and so if $\beta \in \mathrm{Hom}_{\mathcal{C}}(z, a)$, then

$$(g \circ f)_a^*(\beta) = \beta \circ g \circ f = f_a^*(\beta \circ g) = f_a^*(g_a^*(\beta)) = (f_a^* \circ g_a^*)(\beta).$$

Moreover, if $1_x : x \to x$ is the identity morphism for $x \in Ob(\mathcal{C})$, then for $\gamma \in \mathrm{Hom}_{\mathcal{C}}(x, a)$

$$(1_x)_a^*(\gamma) = \gamma \circ 1_x = \gamma = 1_{\mathrm{Hom}_{\mathcal{C}}(x,a)}(\gamma) = 1_{(1_x)_a^*}(\gamma).$$

Hence R_a is a contravariant functor. ∎

2. Show that this extends to a functor: $\psi : \mathcal{C} \to PSh(\mathcal{C})$

Proof. Define the map $\psi : \mathcal{C} \to PSh(\mathcal{C})$ by $a \mapsto R_a$ for $a \in Ob(\mathcal{C})$. For a morphism $h : a \to b$ in $\mathcal{C}$, we need to define a morphism in $PSh(\mathcal{C})$,

$$\psi h : R_a \implies R_b,$$

i.e., a natural transformation between the functors R_a and R_b. Define $\psi h = h_*$, where h_* is post-composition with h. For $y \in Ob(\mathcal{C})$, the component of h_* at y is the map $h_*^y : \mathrm{Hom}_{\mathcal{C}}(y, a) \to \mathrm{Hom}_{\mathcal{C}}(y, b)$ given by $\alpha \mapsto h \circ \alpha$.

$$\begin{array}{ccc} y & \xrightarrow{\alpha} & a \\ & \underset{h_*^y(\alpha)=h\circ\alpha}{\searrow} & \downarrow{\scriptstyle h} \\ & & b \end{array}$$

To check that this definition makes ψh a natural transformation, we need to show that for a morphism $f : x \to y$ in $\mathcal{C}$, the following diagram in Set commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(y, a) & \xrightarrow{R_a(f)=f_a^*} & \mathrm{Hom}_{\mathcal{C}}(x, a) \\ {\scriptstyle \psi h_y = h_*^y}\downarrow & & \downarrow{\scriptstyle \psi h_x = h_*^x} \\ \mathrm{Hom}_{\mathcal{C}}(y, b) & \xrightarrow{R_b(f)=f_b^*} & \mathrm{Hom}_{\mathcal{C}}(x, b) \end{array}$$

For $\alpha \in \text{Hom}_{\mathcal{C}}(y, a)$, we have

$$h_*^x(f_a^*(\alpha)) = h_*^x(\alpha \circ f) = h \circ \alpha \circ f = f_b^*(h \circ \alpha) = f_b^*(h_*^y(\alpha)).$$

Hence ψh is indeed a morphism in $PSh(\mathcal{C})$ (i.e., a natural transformation from R_a to R_b). Suppose now that $g : b \to c$ is another morphism in $\mathcal{C}$. We want to show $\psi(g \circ h) = \psi g \circ \psi h$, i.e. that

$$(gh)_*^y : R_a \implies R_c \quad \text{and} \quad g_*^y \circ h_*^y : R_a \implies R_b \implies R_c$$

are the same natural transformation for all $y \in Ob(\mathcal{C})$.

$$\text{Hom}_{\mathcal{C}}(y, a) \xrightarrow{h_*^y} \text{Hom}_{\mathcal{C}}(y, b) \xrightarrow{g_*^y} \text{Hom}_{\mathcal{C}}(y, c), \qquad \text{Hom}_{\mathcal{C}}(y, a) \xrightarrow{(gh)_*^y} \text{Hom}_{\mathcal{C}}(y, c)$$

For $\alpha : y \to a$ in $\text{Hom}_{\mathcal{C}}(y, a)$, we have

$$(gh)_*^y(\alpha) = g \circ h \circ \alpha = g_*^y\,(h \circ \alpha) = (g_*^y \circ h_*^y)(\alpha),$$

as desired. Finally, if $1_a : a \to a$ is an identity morphism in $\mathcal{C}$, then $\psi 1_a = (1_a)_*$ is the identity natural transformation on the functor R_a: For $y \in Ob(\mathcal{C})$ and $\alpha \in \text{Hom}_{\mathcal{C}}(y, a)$

$$(1_a)_*^y(\alpha) = 1_a \circ \alpha = \alpha = 1_{\text{Hom}_{\mathcal{C}}(y,a)}(\alpha) = 1_{R_a(y)}(\alpha),$$

altogether showing that ψ is a covariant functor. □

3. Prove that ψ is full and faithful. Is ψ an equivalence?

Proof. Fix $a, b \in Ob(\mathcal{C})$. We show that the map

$$\psi : \text{Hom}_{\mathcal{C}}(a, b) \to \text{Hom}_{PSh(\mathcal{C})}(R_a, R_b)$$

is both surjective (full) and injective (faithful). First, injectivity: Let $h, g \in \text{Hom}\,\mathcal{C}(a, b)$ and suppose $h_* = \psi h = \psi g = g_*$. Then in particular, their components at a are the same: $h_*^a = g_*^a : \text{Hom}_{\mathcal{C}}(a, a) \to \text{Hom}_{\mathcal{C}}(a, b)$. So

$$h = h \circ 1_a = h_*^a(1_a) = g_*^a(1_a) = g \circ 1_a = g,$$

showing that ψ is faithful.

Now suppose $\eta \in \text{Hom}_{PSh(\mathcal{C})}(R_a, R_b)$. We want to find $h \in Hom_{\mathcal{C}}(a, b)$ so that $h_* = \psi h = \eta$. Since η has a component at a, $\eta_a : \text{Hom}_{\mathcal{C}}(a, a) \to \text{Hom}\,\mathcal{C}(a, b)$, a natural (ha!) choice for h seems to be $h := \eta_a(1_a)$. To show $\eta = h_*$, we show their components are equal, $\eta_y = h_*^y$, for any $y \in Ob(\mathcal{C})$.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(a, a) & \xrightarrow{\alpha^*} & \text{Hom}_{\mathcal{C}}(y, a) \\ {\scriptstyle \eta_a}\big\downarrow{\scriptstyle h_*^a} & & {\scriptstyle \eta_y}\big\downarrow{\scriptstyle h_*^y} \\ \text{Hom}_{\mathcal{C}}(a, b) & \xrightarrow{\alpha^*} & \text{Hom}_{\mathcal{C}}(y, b) \end{array}$$

Then

$$h_*^a(\alpha) = h \circ \alpha = \eta_a(1_a) \circ \alpha = \alpha^*(\eta_a(1_a)) = \eta_y(\alpha^*(1_a)) = \eta_y(1_a \circ \alpha) = \eta_y(\alpha),$$

and so $h_* = \eta$, showing that ψ is full.

□

4. Show that if F is *any* presheaf on $\mathcal{C}$ then

$$\mathrm{Hom}_{PSh(\mathcal{C})}(R_a, F) \cong F(a)$$

Proof. Both $\mathrm{Hom}_{PSh(\mathcal{C})}(R_a, F)$ and $F(a)$ are objects in the category Set, where an isomorphism is a bijection of sets. Define set maps

$$\begin{aligned} \varphi_F : \mathrm{Hom}(R_a, F) &\longrightarrow F(a) \\ \eta &\longmapsto \eta_a(1_a), \\ \varphi_F^{-1} : F(a) &\longrightarrow \mathrm{Hom}(R_a, F) \\ s &\longmapsto \epsilon = \{\epsilon_x : \mathrm{Hom}_{\mathcal{C}}(x,a) \to F(x), g \mapsto F(g)(s)\}_{x\in\mathcal{C}}. \end{aligned}$$

We check that ϵ (in our definition of φ_F^{-1}) is indeed a natural transformation. So if $f : x \to y$ is a morphism in $\mathcal{C}$, we check that the diagram commutes:

$$\begin{array}{ccc} F(y) & \xrightarrow{\quad Ff \quad} & F(x) \\ {\scriptstyle \epsilon_y}\uparrow & & \uparrow{\scriptstyle \epsilon_x} \\ \mathrm{Hom}_{\mathcal{C}}(y,a) & \xrightarrow{f_a^*} & \mathrm{Hom}_{\mathcal{C}}(x,a) \end{array}$$

Let $g \in \mathrm{Hom}_{\mathcal{C}}(y,a)$. Then

$$Ff(\epsilon_y(g)) = Ff(F(g)(s)) = F(g \circ f)(s)$$

and on the other hand

$$\epsilon_x(f_a^*(g)) = \epsilon_x(g \circ f) = F(g \circ f)(s).$$

So η is a natural transformation. Now,

$$\varphi_F^{-1}\varphi_F(\eta) = \varphi_F^{-1}(\eta_a(1_a)) =: \gamma.$$

Then if $h : x \to a$ is a morphism in $\mathcal{C}$, we have the commutative diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\quad Fh \quad} & F(x) \\ {\scriptstyle \eta_a}\uparrow & & \uparrow{\scriptstyle \eta_x} \\ \mathrm{Hom}_{\mathcal{C}}(a,a) & \xrightarrow{h_a^*} & \mathrm{Hom}_{\mathcal{C}}(x,a) \end{array}$$

So,

$$\gamma_x(h) = F(h)(\eta_a(1_a)) = \eta_x(h_a^*(1_a)) = \eta_x(1_a \circ h) = \eta_x(h),$$

which means $\varphi_F^{-1}\varphi_F(\eta) = \eta$. Moreover,

$$\varphi_F\varphi_F^{-1}(s) = \varphi_F(\epsilon) = \epsilon_a(1_a) = F(1_a)(s) = 1_{F(a)}(s) = s,$$

so φ_F is a bijection. ∎

5. What does it mean for the isomorphism in the last problem to be natural?

   Solution:

   It means that the bijection $\varphi_F : \mathrm{Hom}_{PSh(\mathcal{C})}(R_a, F) \xrightarrow{\sim} F(a)$ extends to a natural isomorphism $\varphi : \mathrm{Hom}_{PSh(\mathcal{C})}(R_a, -) \implies (-)(a)$, where the functor $(-) : PSh(\mathcal{C}) \to \mathrm{Set}$ maps a presheaf $F \mapsto F(a)$, and if $\eta : F \implies G$ is a morphism in $PSh(\mathcal{C})$, then $(\eta)(a) : F(a) \to G(a)$ is the map η_a. So in other words, for all morphisms $\eta : F \implies G$ in $PSh(\mathcal{C})$, we have a commutative diagram,

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta(a)=\eta_a} & F(b) \\ {\scriptstyle \varphi_F}\big\uparrow & & \big\uparrow{\scriptstyle \varphi_G} \\ \mathrm{Hom}(R_a, F) & \xrightarrow{Hom(R_a,\eta)} & \mathrm{Hom}(R_a, G) \end{array}$$

   where $Hom(R_a, \eta)$ is the map η_*, i.e., $\epsilon \mapsto \eta_*(\epsilon) = \eta\epsilon := \{\tilde{\epsilon_b} : \mathrm{Hom}(b, a) \to G(b), \alpha \mapsto \eta_b \epsilon_b(\alpha)\}_{b \in \mathcal{C}}$.

6. Check that this isomorphism is natural.

   Solution:

   Let $\epsilon \in \mathrm{Hom}(R_a, F)$. Then
$$\eta_a(\varphi_F(\epsilon)) = \eta_a(\epsilon_a(1_a)),$$
   and on the other hand,
$$\varphi_G\left(\mathrm{Hom}(R_a, \eta)(\epsilon)\right) = \varphi_G(\eta\epsilon) = (\eta\epsilon)_a(1_a) = \eta_a(\epsilon_a(1_a)).$$

7. Prove that:
$$Hom(R_a, R_b) \cong Hom(a, b)$$

   Proof. We did all the work previously. Setting $R_b = F$ in problem 4 of this section, we get
$$\mathrm{Hom}(R_a, R_b) \cong R_b(a) = \mathrm{Hom}(a, b).$$
   □

If we think of $Hom(X, A)$ as the set of ways in which A is related to X in $\mathcal{C}$ then to understand A up to isomorphism it suffices to understand the sets $Hom(X, A)$ for all $X \in Ob(\mathcal{C})$.

2.2 Variations on the theme, i.e. perversions

Suppose A is a finite dimensional k-algebra. Decompose $1_A \in A$ as a sum:

$$1_A = \sum_{x \in \mathcal{S}} 1_x$$

where

$$1_x 1_x = \delta_{x,y} 1_x$$

The elements 1_x are called mutually orthogonal idempotents.

1. Show that this data is equivalent to a direct sum decomposition of A:

$$A \cong \bigoplus_{x,y} 1_x A 1_y$$

2. Define a category $\mathcal{A}$ with objects $\mathcal{S}$ and morphisms:

$$\mathrm{Hom}_{\mathcal{A}}(x, y) = 1_y A 1_x$$

   Prove that this is a category.

3. Show that there is a canonical correspondence between right A-modules and presheaves on $\mathcal{A}$. (Here, a presheaf on $\mathcal{A}$ is a contravariant functor $F : \mathcal{A} \to$ Abelian Groups, not $F : \mathcal{A} \to$ Set.)

4. Give an example of a presheaf which is not of the form R_a.

5. (BONUS) Under this correspondence projective modules correspond to direct sums of functors R_a.

3 Exercises in naturality are important not punny

3.1 When duality is spun from a natural web

Denote the category of finite dimensional vector spaces over a field k by $Vect_k$.

1. Is k an initial or final object in $Vect_k$?

   Proof. Neither. There are n ways to include k into $k^{\oplus_n}$, and n ways to project $k^{\oplus_n}$ onto k. We also proved previously that the 0 vector space is both initial and final in $k - Mod = Vect_k$, and that initial and final objects are unique up to isomorphism, but $k \ncong 0$. □

2. (BONUS) Prove that k is uniquely determined up to isomorphism by the property of being the unit with respect to the tensor product

$$\otimes_k : Vect_k \times Vect_k \to Vect_k.$$

3. Prove that for any two vector spaces V and W

$$\mathrm{Hom}_{Vect_k}(V, W) \in Vect_k$$

   Proof. The set of linear transformations from V to W is an additive abelian group. For $f \in \mathrm{Hom}_{Vect_k}(V, W)$ and $\lambda \in k$, we can define a vector space structure on $\mathrm{Hom}_{Vect_k}(V, W)$ by $\lambda \cdot f := (\lambda f : v \mapsto \lambda f(v))$. That this gives a well-defined k-vector space structure on $\mathrm{Hom}_{Vect_k}(V, W)$ is left as an exercise for the reader. □

4. Prove that the (contravariant) Yoneda functor: $L_k : Vect_k \to Vect_k$ determined by

$$L_k(V) = Hom_{Vect_k}(V, k)$$

   is naturally isomorphic to its own inverse:

$$L_k \circ L_k \xrightarrow{\sim} 1_{Vect_k}$$

   Proof. On a morphism f, the Yondeda functor is given by $L_k(f) = f^*$.

$$\begin{array}{ccccc} V & & \mathrm{Hom}(V,k) & & \mathrm{Hom}(\mathrm{Hom}(V,k),k) \\ \downarrow f & \xrightarrow{L_k} & \uparrow f^* & \xrightarrow{\quad L_k \quad} & \downarrow f^{**} \\ W & & \mathrm{Hom}(W,k) & & \mathrm{Hom}(\mathrm{Hom}(W,k),k) \end{array}$$

   Then $L_k L_k(f) = f^{**}$, where

$$f^{**}(\alpha : \mathrm{Hom}(V, k) \to k) \longmapsto (\alpha f^* : \mathrm{Hom}(W, k) \to k, g \mapsto \alpha(g \circ f))$$

   Define a natural transformation $\eta : 1_{Vect_k} \implies L_k L_k$ with components

$$\eta_V : V \to L_k L_k(V), v \mapsto (eval_v : \mathrm{Hom}(V, k) \to k, g \mapsto g(v))$$

   Each component is injective: If $eval_v \equiv 0$, then in particular $g(v) = 0$ for all $g \in \mathrm{Hom}(V, k)$ and so $v = 0$. Since V and $\mathrm{Hom}(\mathrm{Hom}(V, k))$ have the same dimension and η_V is injective, then η_V is an isomorphism. Now, naturality.

$$\begin{array}{ccc} V & \xrightarrow{\qquad f \qquad} & W \\ \downarrow \eta_V & & \downarrow \eta_W \\ \mathrm{Hom}(\mathrm{Hom}(V,k),k) & \xrightarrow{f^{**}} & \mathrm{Hom}(\mathrm{Hom}(W,k),k) \end{array}$$

Let $v \in V$. Then

$$\eta_W(f(v)) = eval_{f(v)} \quad \text{and} \quad f^{**}(\eta_V(v)) = f^{**}(eval_v) = eval_v f^*.$$

Now if $g \in \mathrm{Hom}(W, k)$, then

$$eval_{f(v)}(g) = g(f(v)) \quad \text{and} \quad (eval_v f^*)(g) = eval_v(g \circ f) = g(f(v)).$$

Hence $eval_{f(v)} = (eval_v f^*)$ and so η is natural. □

5. Prove that for every finite dimensional vector space V

$$Hom_{Vect_k}(V, k) \cong V$$

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for V and $\{v_1^*, \ldots, v_n^*\}$ its dual basis. The map $\varphi : V \to \mathrm{Hom}(V, k), v_i \mapsto v_i^*$ is an isomorphism. □

6. Prove that L_k is not naturally isomorphic to identity.

Proof. The identity functor is covariant and L_k is contravariant, hence these functors cannot be isomorphic. □

3.2 The opposite of my opposite is your friend

1. Show that $\mathcal{C} \mapsto \mathcal{C}^{op}$ determines a covariant functor from the category of small categories to itself.

Proof. Let $F : \mathrm{Cat} \to \mathrm{Cat}$ be the map given by $\mathcal{C} \mapsto \mathcal{C}^{op}$. For a (covariant) functor $f : \mathcal{C} \to \mathcal{D}$ in Cat, let $F(f) : \mathcal{C}^{op} \to \mathcal{D}^{op}$ be the functor given by $x \mapsto f(x)$ for objects $x \in \mathcal{C}$. If α is a morphism in $\mathcal{C}$, write α^{op} for its corresponding morphism in $\mathcal{C}^{op}$; similarly for morphisms in $\mathcal{D}$. Then define $Ff(\alpha^{op}) = (f(\alpha))^{op}$. For cleaner notation, we write $f(\alpha)^{op}$ in place of $(f(\alpha))^{op}$.

Let $g : \mathcal{D} \to \mathcal{E}$ be another (covariant) functor in Cat. Then

$$F(g \circ f)(\alpha^{op}) = (g \circ f)(\alpha)^{op}.$$

On the other hand,

$$(Fg \circ Ff)(\alpha^{op}) = Fg(f(\alpha)^{op}) = g(f(\alpha))^{op} = (g \circ f)(\alpha)^{op}$$

Moreover, if $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an identity functor and α is a morphism in $\mathcal{C}$, then

$$F1_{\mathcal{C}}(\alpha^{op}) = 1_{\mathcal{C}}(\alpha)^{op} = \alpha^{op} = 1_{\mathcal{C}^{op}}(\alpha^{op}) = 1_{F(\mathcal{C})}(\alpha^{op}),$$

and so F is a covariant functor. □

2. Show that $(\mathcal{C}^{op})^{op} \cong \mathcal{C}$

Proof. If α is a morphism in $\mathcal{C}$, then write α^{op} for its corresponding morphism in $\mathcal{C}^{op}$.

Let $F : \mathcal{C} \to (\mathcal{C}^{op})^{op}$ be the functor which is the identity on objects, and $\alpha \mapsto (\alpha^{op})^{op}$ on morphisms. Let $G : (\mathcal{C}^{op})^{op} \to \mathcal{C}$ be the functor which is the identity on objects, and $(\alpha^{op})^{op} \mapsto \alpha$. This last definition makes sense since if γ is a morphism in $(\mathcal{C}^{op})^{op}$, then there exists a morphism β in $\mathcal{C}^{op}$ so that $\beta^{op} = \gamma$. Similarly, there exists a morphism α in $\mathcal{C}$ so that $\alpha^{op} = \beta$. So $\gamma = \beta^{op} = (\alpha^{op})^{op}$.

Then FG and GF are equal to the identity functors. On objects, this is clear. On morphisms, $FG((\alpha^{op})^{op}) = F(\alpha) = (\alpha^{op})^{op}$ and $GF(\alpha) = G((\alpha^{op})^{op}) = \alpha$. Hence $(\mathcal{C}^{op})^{op} \cong \mathcal{C}$. □

Algebraic Topology

Homework 2

Name: 16977

1 You may recall the homotopy category

Fix a ring R. Recall that a chain complex $C_* = \{d_k^C : C_k \to C_{k-1}\}_{k\in\mathbb{Z}}$ is a collection of maps between R-modules C_k which satisfy $d_k^C d_{k+1}^C = 0$ for all $k \in \mathbb{Z}$. A map $f : C_* \to D_*$ between chain complexes of degree $|f| = n$ is a collection of R-module maps $f = \{f_k : C_k \to D_{n+k}\}_{k\in\mathbb{Z}}$. Denote the collection of maps by $Hom(C_*, D_*)$.

1. Show that this determines a category, $dgCH$.

Proof. The objects of this category $dgCH$ are chain complexes and morphisms are maps between chain complexes with degree as defined above. If $f = \{f_k : C_k \to D_{n+k}\}_{k\in\mathbb{Z}}$ and $g = \{g_k : D_k \to E_{m+k}\}_{k\in\mathbb{Z}}$ define composition in $dgCH$ by the rule

$$g \circ f = \{g_{n+k} \circ f_k : C_k \to E_{m+n+k}\}_{k\in\mathbb{Z}}.$$

We need to check that for all $C_* \in Ob(dgCH)$, there exists $1_{C_*} \in \mathrm{Hom}(C_*, C_*)$ so that $f \circ 1_{C_*} = f$ and $1_{C_*} \circ f = f$ (whenever the composition makes sense, of course), and that composition is associative.

Define $1 \in \mathrm{Hom}(C_*, C_*)$ by $1_{C_*} = \{1_{C_k} : C_k \to C_k\}_{k\in\mathbb{Z}}$, where each 1_{C_k} is the identity R-module morphism. Then $f \circ 1_{C_*} = \{1_{C_k} \circ f_k : C_k \to D_{0+n+k}\}_{k\in\mathbb{Z}} = \{f_k : C_k \to D_{n_k}\}_{k\in\mathbb{Z}} = f$, and similarly, $1_{C_*} \circ f = f$ (when $f \in \mathrm{Hom}(D_*, C_*)$).

Now let $h = \{h_k : E_k \to F_{\ell+k}\}_{k\in\mathbb{Z}}$. Then

$$\begin{aligned} h \circ (g \circ f) &= h \circ \{g_{n+k} \circ f_k : C_k \to E_{m+n+k}\}_{k\in\mathbb{Z}} \\ &= \{h_{m+n+k} \circ g_{n+k} \circ f_k : C_k \to F_{\ell+m+n+k}\}_{k\in\mathbb{Z}} \\ &= \{h_{m+k} \circ g_k : D_k \to F_{\ell+m+k}\}_{k\in\mathbb{Z}} \circ f \\ &= (h \circ g) \circ f. \end{aligned}$$

□

2. Show that for all C_*, D_* the abelian group $Hom(C_*, D_*)$ is a chain complex in such a way that a map $f : C_* \to D_*$ in $dgCh$ is a chain map if an only if it is a degree zero cycle.

Proof. For degree n maps in $\mathrm{Hom}(C_*, D_*)$, write $\mathrm{Hom}(C_*, D_{*+n})$. Let $f \in \mathrm{Hom}(C_*, D_{*+n})$.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{k+1} & \xrightarrow{d_{k+1}^C} & C_k & \xrightarrow{d_k^C} & C_{k-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & & \\ \cdots & \longrightarrow & D_{k+1+n} & \xrightarrow[d_{k+1+n}^D]{} & D_{k+n} & \xrightarrow[d_{k+n}^D]{} & D_{k-1+n} & \longrightarrow & \cdots \end{array}$$

For all n, define maps $\delta_n : \mathrm{Hom}(C_*, D_{*+n}) \to \mathrm{Hom}(C_*, D_{*+n-1})$ by

$$\delta_n : f \longmapsto \{d_{k+n}^D f_k + (-1)^{n+1} f_{k-1} d_k^C\}_{k\in\mathbb{Z}}.$$

Now let $|f| = 0$. Then f is a chain map if and only if $d_k^D f_k = f_{k-1} d_k^C$ for all k, if and only if

$$0 = \{d_k^D f_k - f_{k-1} d_k^C\}_{k\in\mathbb{Z}} = \delta_0(f).$$

The last statement is what it means for f to be a cycle. □

3. Prove that cycles form a subcategory $Z(dgCh)$.

Proof. Let $Z(dgCh)$ have the same objects as $dgCh$, and for chain complexes C_*, D_*, let $\mathrm{Hom}_{Z(dgCh)}(C_*, D_*)$ contain only cycles. We need to show that the composition of cycles is still a cycle:

Suppose $f : C_* \to D_*$ and $g : D_* \to E_*$ are cycles. Then[1]

$$0 = \delta f = d^D f + (-1)^{|f|+1} f d^C \implies -d^D f = (-1)^{|f|+1} f d^C$$
$$0 = \delta g = d^E g + (-1)^{|g|+1} g d^D \implies d^E g = (-1)^{|g|} g d^D$$

So

$$\begin{aligned}\delta(g \circ f) &= d^E(g \circ f) + (-1)^{|f|+|g|+1}(g \circ f) d^C \\ &= (-1)^{|g|} g d^D f - (-1)^{|g|} g d^D f \\ &= 0.\end{aligned}$$

So the composition of cycles is a cycle.

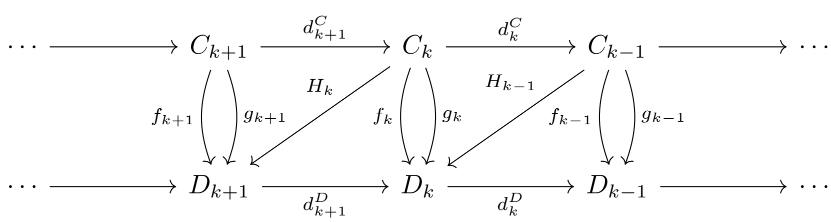
Associativity of morphisms in $Z(dgCh)$ is the same as that in $dgCh$, which we showed previously. Since

$$\delta(1_{C_*}) = \{d_k \circ 1_{C_k} - 1_{C_{k-1}} d_k\}_{k\in\mathbb{Z}} = \{d_k - d_k\}_{k\in\mathbb{Z}} = 0,$$

then all identity morphisms are cycles and hence $1_{C_*} \in \mathrm{Hom}_{Z(dgCh)}(C_*, C_*)$ for all C_*. Moreover, $f \circ 1_{C_*} = f$ and $1_{C_*} \circ g = g$ in $Z(dgCh)$ since we showed this in $dgCh$. Therefore, $Z(dgCh)$ forms a subcategory of $dgCh$. □

4. Prove that homotopy of chain maps is the same as two cycles differing by a boundary.

Proof. Suppose $f, g \in \mathrm{Hom}(C_*, D_{*+0})$ are chain maps and that $H \in \mathrm{Hom}(C_*, D_{*+1})$ is a homotopy between f and g, i.e. $f - g = d^D H + H d^C$.



Since

$$\delta(H) = d^D H + (-1)^{1+1} H d^C = d^D H + H d^C = f - g,$$

then f and g differ by a boundary. Conversely, if $f - g = \delta(\tilde{H})$ for some $\tilde{H} \in \mathrm{Hom}(C_*, D_{*+1})$, then $f - g = d^D \tilde{H} + \tilde{H} d^C$, so $\tilde{H}$ is a homotopy between f and g. □

5. Prove that homotopy equivalence is an equivalence relation.

Proof. *Reflexivity:* If $f \in \mathrm{Hom}(C_*, D_*)$, then 0 is a homotopy between f and itself since

$$d^D 0 + (-1)^{n+1} 0 d^C = 0 = f - f.$$

Symmetry: If $f \simeq g$ by H (so $f - g = d^D H + H d^C$), then $-H$ is a homotopy between g and f, since

$$d^D(-H) + (-H) d^C = -d^D H - H d^C = -(f - g) = g - f.$$

Transitivity: If $f \simeq g$ by H and $g \simeq h$ by J, then

$$f - k = f - g + g - k = d^D H + H d^C + d^D J + J d^C = d^D(H + J) + (H + J) d^C,$$

so $H + J$ is a homotopy between f and k. □

[1] From now on, we drop the subscripts on δ and on morphisms, since the computations hold for all indices.

6. Prove that homotopy classes of chain maps $H_0(dgCh)$ is a category. This is called the homotopy category of chain complexes.

Proof. As before, the objects of $H_0(dgCh)$ are chain complexes, but now the morphisms are equivalence classes of chain maps. Define $[g] \circ [f] = [g \circ f]$. If we can show this is well-defined, then associativity of morphisms in this category follows from our previous work. Also, since 1_{C_*} is a chain map, then $[1_{C_*}]$ is in $H_0(dgCh)$.

Before we show that our composition is well-defined, we need show that δ is an antiderivation of degree 1; that is, $\delta(g \circ f) = (\delta g)f + (-1)^{|g|}g(\delta f)$. For $f : C_* \to D_*$ and $g : D_* \to E_*$, we have

$$\delta f = d^D f + (-1)^{|f|+1} f d^C \implies \delta f - d^D f = (-1)^{|f|+1} f d^C$$
$$\delta g = d^E g + (-1)^{|g|+1} g d^D \implies d^E g = \delta g + (-1)^{|g|} g d^D$$

So

$$\begin{aligned}
\delta(g \circ f) &= d^E(g \circ f) + (-1)^{|f|+|g|+1}(g \circ f)d^C \\
&= (\delta g + (-1)^{|g|} g d^D)f + (-1)^{|g|} g(\delta f - d^D f) \\
&= (\delta g)f + (-1)^{|g|} g d^D f + (-1)^{|g|} g(\delta f) - (-1)^{|g|} g d^D f \\
&= (\delta g)f + (-1)^{|g|} g(\delta f),
\end{aligned}$$

as desired. Moreover, δ is linear: Suppose $h_1, h_2 : C_* \to D_*$ have the same degree. Then $|h_1 + h_2| = |h_1| = |h_2|$, and we have

$$\begin{aligned}
\delta(h_1 + h_2) &= d^D(h_1 + h_2) + (-1)^{|h_1+h_2|+1}(h_1 + h_2)d^C \\
&= d^D h_1 + d^D h_2 + (-1)^{|h_1+h_2|+1} h_1 d^C + (-1)^{|h_1+h_2|+1} h_2 d^C \\
&= \left(d^D h_1 + (-1)^{|h_1|+1} h_1 d^C\right) + \left(d^D h_2 + (-1)^{|h_2|+1} h_2 d^C\right) \\
&= \delta(h_1) + \delta(h_2).
\end{aligned}$$

Now, suppose $f_1 \simeq_H f_2$ and $g_1 \simeq_J g_2$. We want to show that $g_1 \circ f_1 \simeq g_2 \circ f_2$. We now use the result from problem 4, that a homotopy of chain maps is the same as two cycles differing by a boundary, and in particular, they differ by the boundary of their homotopy. In other words, we will use that

$$\delta H = f_1 - f_2 \quad \text{and} \quad \delta J = g_1 - g_2.$$

We claim that $g_1 H + J f_2$ is a homotopy between $g_2 f_1$ and $g_2 f_2$. Notice that $\delta g_1 = \delta f_2 = 0$ since g_1 and f_2 are chain maps. So

$$\begin{aligned}
\delta(g_1 H + J f_2) &= \delta(g_1 H) + \delta(J f_2) \\
&= (\delta g_1)H + (-1)^{|g_1|} g_1(\delta H) + (\delta J)f_2 + (-1)^{|J|} J(\delta f_2) \\
&= g_1(\delta H) + (\delta J)f_2 && \text{(Since } |g_1| = 0) \\
&= g_1(f_1 - f_2) + (g_1 - g_2)f_2 \\
&= g_1 f_1 - g_2 f_2.
\end{aligned}$$

So $g_1 f_1 \simeq g_2 f_2$ and so $[g_1] \circ [f_1] = [g_1 \circ f_1] = [g_2 \circ f_2] = [g_2] \circ [f_2]$. $\square$

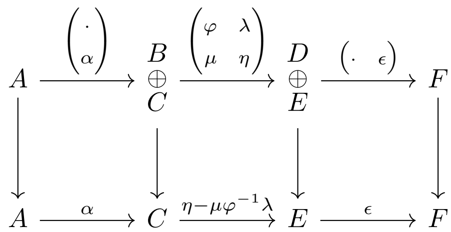
2 You don't flush after a Gaussian Elimination

1. Define the notion of subcomplex of a chain complex. Justify your answer in some way.

   Solution:

   A chain complex C_* is a subcomplex of another chain complex D_* if there exists an injective map of chains $\iota : C_* \to D_*$. This definition makes sense since we can identify the image of C_* in D_* with C_*, and consider C_* as sitting inside of D_*, hence C_* is a "subcomplex" of D_*.

2. Prove the following useful lemma: Let K_* be a chain complex of R-modules containing a subcomplex isomorphic to the top row below. If $\varphi : B \to D$ is an isomorphism there is a homotopy equivalence from K_* to a smaller complex containing the bottom row below.



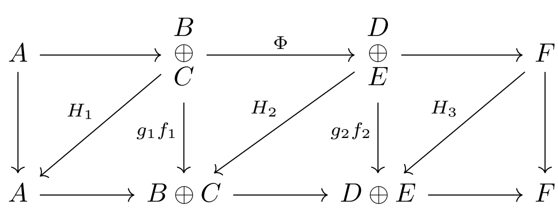
   Proof. We want to show that the top row is homotopy equivalent to the bottom row. So we need to define maps $f_1 : B \oplus C \rightleftarrows C : g_1$ and $f_2 : D \oplus E \rightleftarrows E : g_2$ such that

$$g_1 f_1 \simeq 1_{B\oplus C}, g_2 f_2 \simeq 1_{D\oplus E} \quad \text{and} \quad f_1 g_1 \simeq 1_C, f_2 g_2 \simeq 1_E.$$

   Define f_1 to be projection and g_2 to be inclusion, and

$$f_2 : D \oplus E \longrightarrow E, \quad (d,e) \longmapsto -\mu\varphi^{-1}(d) + e,$$
$$g_1 : C \longrightarrow B \oplus C, \quad c \longmapsto (-\varphi^{-1}\lambda(c), c),$$

   Now, $f_1 g_1 = 1_C$ and $f_2 g_2 = 1_E$ so certainly $f_1 g_1 \simeq 1_C$ and $f_2 g_2 \simeq 1_E$. To show $g_1 f_1 \simeq 1_{B\oplus C}$ and $g_2 f_2 \simeq 1_{D\oplus E}$, we need to define a homotopy $H = \{H_1, H_2, H_3\}$. Let $\Phi : B \oplus C \to D \oplus D$ be the map associated to the matrix $\left(\begin{smallmatrix} \varphi & \lambda \\ \mu & \eta \end{smallmatrix}\right)$, so $\Phi(b,c) = (\varphi(b) + \lambda(c), \mu(b) + \eta(c))$.



   Define $H_1 = H_3 = 0$, and $H_2 : (d,e) \mapsto (-\varphi^{-1}(d), 0)$. Then

$$(g_1 f_1 - 1_{B\oplus C})(b,c) = (-\varphi^{-1}\lambda(c) - b, 0) = H_2\Phi(b,c).$$

   and

$$(g_2 f_2 - 1_{D\oplus E})(d,e) = (-d, -\mu\varphi^{-1}(d)) = \Phi H_2(d,e),$$

   so $g_1 f_1 \simeq 1_{B\oplus C}$ and $g_2 f_2 \simeq 1_{D\oplus E}$. □

3. Use your words to explain the meaning of the lemma that you just proved. More precisely, what is the relationship between the most important operation in linear algebra (Gaussian elimination) and the homotopy theory of chain complexes?

   Solution: In the same way that we can row reduce a matrix if we have an invertible element in the $(1,1)$-entry, and essentially "forget" about the first column, we just showed that in a similar way, if we have an isomorphism in the $(1,1)$-entry, then we can essentially "forget" about the information in the first column, up to homotopy. We basically "forgot" about the modules B and D in the homotopy, since we had an isomorphism between them.

3 You don't need chain-ge when you've got supergroups

1. Show that there is a category of comodules over a coalgebra.
2. Show that the ring of functions on a group is a bialgebra.
3. The ring of functions on the space of automorphisms $Aut(\mathbb{A}^{0|1})$ of the 1-dimensional super line is given by the algebra $A = k[a, b, b^{-1}]/(a^2)$ with coproduct
$$\Delta(b) = b \otimes b \quad \text{and} \quad \Delta(a) = a \otimes 1 + b \otimes a$$
Show that the category of A-comodules is cochain complexes.
4. NB. If X is a space then $\mathrm{Hom}(\mathbb{A}^{0|1}, X)$ inherits a canonical coaction of A. This mapping space is the de Rham complex $\Omega^*(X)$ of X.
5. Research problem: find supergeometric interpretation of Gaussian elimination.

4 Ya don't whine about the Dold-Kan core despondence

1. Recall the category Δ of finite ordered sets and non-decreasing set maps. Prove that Δ is equivalent to the full subcategory with objects $[n] = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let $\tilde{\Delta}$ be the full subcategory. Define a functor $F : \Delta \to \tilde{\Delta}$ be $N \mapsto [n]$, where $|N| = n$. Let N_j denote the jth element in the ordered set N, and let $\varphi_N : N \to [n]$ be the map $N_j \mapsto j$, and $\varphi^{-1} : j \mapsto N_j$.

If $f : N \to M$ is in Δ, then define $Ff := \varphi_M f \varphi_N^{-1}$. So if $N \xrightarrow{f} M \xrightarrow{g} P$ is in Δ, then
$$F(g \circ f) = \varphi_P g f \varphi_N^{-1} = \varphi_P g \varphi^{-1} M \varphi_M f \varphi_N^{-1} = Fg \circ Ff.$$
If $1_N : N \to N$ is an identity morphism in Δ, then
$$F 1_N = \varphi_N 1_N \varphi^{-1} = 1_{[n]} = 1_{F(N)},$$
and so F is a functor. Now let $G : \tilde{\Delta} \to \Delta$ be the inclusion functor. We have $F \circ G = 1_{\tilde{\Delta}}$, and so we only need to show $G \circ F \cong 1_\Delta$. Define a natural transformation $\eta : GF \implies 1_\Delta$ with components $\eta_N := \varphi^{-1}$. Then
$$f \eta_N = f \varphi^{-1} = \varphi_M^{-1} \varphi_M f \varphi_N^{-1} = \eta_M F f,$$
so η is natural, and is an isomorphism since its components are bijections. Hence $G \circ F \cong 1_\Delta$, and therefore Δ and $\tilde{\Delta}$ are equivalent categories.

$$\begin{array}{ccc} GF(N) & \xrightarrow{\eta_N} & N \\ {\scriptstyle GFf = Ff}\downarrow & & \downarrow{\scriptstyle f} \\ GF(M) & \xrightarrow{\eta_N} & M \end{array}$$

□

2. If R is a ring then a *simplicial R-module* is a functor $F : \Delta^{op} \to R - mod$. Given such an F construct a chain complex $k[a, b, b^{-1}]/(a^2)(F) \in Ch^+(R - mod)$ such that
$$C(F)_n = F([n])$$

Proof. We want to define a chain complex
$$\cdots \longrightarrow F[n+1] \xrightarrow{d_{n+1}} F[n] \xrightarrow{d_n} F[n-1] \longrightarrow \cdots$$
Since we have the canonical inclusions $\iota_n : [n-1] \to [n]$, define $d_n := F(\iota_n)$. Then
$$d_n d_{n+1} = F(\iota_n) \circ F(\iota_{n+1}) = F(\iota_{n+1} \iota_n).$$

□

3. If we were able to construct a simplicial R-module $D(c)$ for a chain complex $c \in Ch^+(R - mod)$ then by Yoneda $D(c)([n]) = Hom(\Delta[n], D(c))$ where $\Delta[n]$ is the representable functor $\Delta[n](S) = R\langle Hom_\Delta(S,[n])\rangle$ the R-span of non-decreasing maps from S to $[n] = \{0 < 1 < \cdots < n\}$. Assuming D is right adjoint to C we get

$$D(c)([n]) = Hom(\Delta[n], D(c)) = Hom_{Ch^+}(C(\Delta[n]), c)$$

Prove that $D(c)$ is a simplicial R-module.

Proof. Write $D(c) = D_c$. We need to show that D_c is a functor. The above definition gives us how the map D_c acts on objects, and so we need to define it on morphisms and show this definition makes the map D_c a functor. Notice that

$$C(\Delta[n])_k = \Delta[n]([k]) = R\langle \mathrm{Hom}_\Delta([k],[n])\rangle = \bigoplus_\alpha R\alpha.$$

where the direct sum ranges over all maps $\alpha : [k] \to [n]$.

Now, for $f : [n] \to [m]$ in Δ, define

$$D_c(f) : Hom_{Ch^+}(C(\Delta[m]), c) \longrightarrow Hom_{Ch^+}(C(\Delta[n]), c)$$
$$g = \left\{ g_k : \bigoplus_\beta R\beta \longrightarrow c_k \right\} \longmapsto h = \left\{ g_k \left(\bigoplus -f(-) \right) : \bigoplus_\alpha R\alpha \longrightarrow c_k \right\},$$

where $\bigoplus -f(-)$ is the map

$$\bigoplus_{i=1}^{\ell} r_i\alpha_i \longmapsto \bigoplus_{i=1}^{\ell} r_i f(\alpha_i).$$

If $[n] \xrightarrow{f} [m] \xrightarrow{g} [p]$ is in Δ, then $D_c(g \circ f) : \{j_k\} \longmapsto \{j_k\left(\bigoplus -(g\circ f)(-)\right)\}$, and on the other hand

$$D_c(f) \circ D_c(g) : \{j_k\} \longmapsto \left\{ j_k \left(\bigoplus -g(-) \right) \right\} \longmapsto \left\{ j_k \left(\bigoplus -g(-) \right) \left(\bigoplus -f(-) \right) \right\} = \left\{ j_k \left(\bigoplus -(g \circ f)(-) \right) \right\},$$

where the last equality follows from the computation

$$j_k \left(\bigoplus -g(-) \right) \left(\bigoplus -f(-) \right) \left(\bigoplus_{i=1}^{\ell} r_i\alpha_i \right) = j_k \left(\bigoplus -g(-) \right) \left(\bigoplus_{i=1}^{\ell} r_i f(\alpha_i) \right) = j_k \left(\bigoplus_{i=1}^{\ell} r_i g(f(\alpha_i)) \right).$$

So $D(g \circ f) = D(f) \circ D(g)$. Moreover, if $1_{[n]} : [n] \to [n]$ is an identity morphism in Δ, then

$$D_c(1_{[n]}) : \{g_k\} \longmapsto \left\{ g_k \left(\bigoplus -1_{[n]}(-) \right) \right\} = \{g_k\},$$

and so $D_c(1_{[n]}) = 1_{D_c([n])}$. Therefore, $D_c : \Delta \to R - mod$ is a contravariant functor, i.e. D_c is a simplicial R-module. ☐

4. This sets up an equivalence between simplicial R-modules and (connected $\Leftrightarrow C_k = 0$ for $k < 0$) chain complexes called the Dold-Kan correspondence. You don't have to prove it. One can consider simplicial objects in any category.

5. Prove that there map $F \mapsto |F|$ from simplicial sets $F : \Delta^{op} \to Sets$ to topological spaces which associates to $F([n])$ a collection of n-simplices.

Proof. Send the simplicial R-module F to the simiplicial n simplex

$$F \mapsto \coprod_{n \geq 0} F([n]) \times \sigma_n / \sim$$

where σ_n is the standard n simplex. ☐

5 You-need-a Baer sum times

Suppose that A and B are R-modules and set of short exact sequences of R-modules:

$$Ext^1(B, A) = \{0 \to A \to M \to B \to 0\}/\sim$$

where two short exact sequences are related by $\sim$ when there is an isomorphism $f : M \to M'$ making the appropriate diagram commute.

1. Make $\sim$ a definition and show that it is an equivalence relation.

Proof. Reflexivity: If $\mathcal{E} : 0 \to A \xrightarrow{g} M \xrightarrow{f} B \to 0$ is short exact, then $1_M : M \to M$ is an isomorphism making the diagram commute:

$$\begin{CD}
\mathcal{E} : 0 @>>> A @>g>> M @>f>> B @>>> 0\\
@. @VVV @VV{1_M}V @VVV @.\\
\mathcal{E} : 0 @>>> A @>g>> M @>f>> B @>>> 0
\end{CD}$$

Symmetry: Let $\mathcal{E} : 0 \to A \xrightarrow{g} M \xrightarrow{f} B \to 0$ and $\mathcal{E}' : 0 \to A \xrightarrow{g'} M' \xrightarrow{f'} B \to 0$ be short exact sequences, and suppose $\mathcal{E} \sim \mathcal{E}'$ with isomorphism $\alpha : M \to M'$ making the diagram commute:

$$\begin{CD}
\mathcal{E} : 0 @>>> A @>g>> M @>f>> B @>>> 0\\
@. @VVV @VV{\alpha}V @VVV @.\\
\mathcal{E}' : 0 @>>> A @>g'>> M' @>f'>> B @>>> 0
\end{CD}$$

Then α^{-1} is an isomorphism making the diagram commute, so $\mathcal{E}' \sim \mathcal{E}$:

$$\begin{CD}
\mathcal{E}' : 0 @>>> A @>g'>> M' @>f'>> B @>>> 0\\
@. @VVV @VV{\alpha^{-1}}V @VVV @.\\
\mathcal{E} : 0 @>>> A @>g>> M @>f>> B @>>> 0
\end{CD}$$

Transitivity: Let $\mathcal{E} : 0 \to A \xrightarrow{g} M \xrightarrow{f} B \to 0$, $\mathcal{E}' : 0 \to A \xrightarrow{g'} M' \xrightarrow{f'} B \to 0$, and $\mathcal{E}'' : 0 \to A \xrightarrow{g''} M'' \xrightarrow{f''} B \to 0$ be short exact sequences, and suppose $\mathcal{E} \sim \mathcal{E}'$ and $\mathcal{E}' \sim \mathcal{E}''$, with respective isomorphisms $\alpha : M \to M'$ and $\beta : M' \to M''$ making the diagrams commute:

$$\begin{CD}
\mathcal{E} : 0 @>>> A @>g>> M @>f>> B @>>> 0\\
@. @VVV @VV{\alpha}V @VVV @.\\
\mathcal{E}' : 0 @>>> A @>g'>> M' @>f'>> B @>>> 0
\end{CD}$$

$$\begin{CD}
\mathcal{E}' : 0 @>>> A @>g'>> M' @>f'>> B @>>> 0\\
@. @VVV @VV{\beta}V @VVV @.\\
\mathcal{E}'' : 0 @>>> A @>g''>> M'' @>f''>> B @>>> 0
\end{CD}$$

Then $\beta \circ \alpha$ is an isomorphism making the diagram commute, so $\mathcal{E} \sim \mathcal{E}''$:

$$\begin{CD}
\mathcal{E} : 0 @>>> A @>g>> M @>f>> B @>>> 0\\
@. @VVV @VV{\beta\alpha}V @VVV @.\\
\mathcal{E}'' : 0 @>>> A @>g''>> M'' @>f''>> B @>>> 0
\end{CD}$$

□

2. The operation of Baer sum: given two short exact sequences:

$$0 \to A \to M \xrightarrow{f} B \to 0 \quad \text{and} \quad 0 \to A \to M' \xrightarrow{f'} B \to 0$$

Prove that there is short exact sequence:

$$0 \to A \to \nabla_* \Delta^* (M \oplus M') \to B \to 0$$

Proof. We use the definition of Baer sum given in Weibel's *Introduction to Homological Algebra*: First, the pullpack of f and f' is

$$\Delta_*(M \oplus M') = \{(m, m') \in M \oplus M' : f(m) = f(m')\}.$$

Next, the pushout of the pullback is the quotient

$$\nabla_* \Delta^* (M \oplus M') = \Delta^*(M \oplus M') \Big/ \{(g(a), -g'(a)) : a \in A\}.$$

We will use the bar notation to denote the quotient. Define maps [2]

$$\begin{aligned} G &: A \to \nabla_* \Delta^*(M \oplus M'), \quad a \mapsto \overline{(g(a), 0)} \\ F &: \nabla_* \Delta^*(M \oplus M') \to B, \quad \overline{(m, m')} \mapsto f(m) = f'(m'). \end{aligned}$$

Now we show that F is well-defined. If $\overline{(m_1, m_1')} = \overline{(m_2, m_2')}$, then there exists $a \in A$ so that $g(a) = m_1 - m_2$ and $g'(a) = m_1' - m_2'$. Now,

$$0 = fg(a) = f(m_1) - f(m_2) \implies f(m_1) = f(m_2),$$

and likewise $f'(m_1') = f'(m_2')$. So F is well defined.

Now if $G(a) = \overline{(0, 0)}$, then $g(a) = 0$ and so $a = 0$ since g is injective, thus G is injective. The surjectivity of F follows from the surjectivity of f. Moreover,

$$FG(a) = F\overline{(0, 0)} = f(0) = 0,$$

so $\operatorname{im} G \subseteq \ker F$.

If $F\overline{(m, m')} = 0$, then $f(m) = f'(m) = 0$, and so $m \in \ker f = \operatorname{im} g$ and $m' \in \ker f' = \operatorname{im} g'$. Hence there exists $a, a' \in A$ such that $g(a) = m$ and $g'(a') = m'$. Define $\tilde{a} = a + a'$. Since

$$(m - g(\tilde{a}), m') = (g(a - \tilde{a}), m') = (g(-a'), g'(a')) \in \{(g(c), -g'(c)) : c \in A\},$$

then $G(\tilde{a}) = \overline{(g(\tilde{a}), 0)} = \overline{(m, m')}$, so $\ker F \subseteq \operatorname{im} G$, and thus we have $\ker F = \operatorname{im} G$. Hence the sequence

$$0 \to A \xrightarrow{G} \nabla_* \Delta^*(M \oplus M') \xrightarrow{F} B \to 0$$

is exact. □

[2] These are the maps that we will use for the next problem as well.

3. Prove that $Ext^1(B, A)$ is an abelian group under Baer sum.

Proof. For this proof, in certain cases, we will write fm in place of $f(m)$, and fga in place of $f(g(a))$ for cleaner notation.

- <u>*Identity:*</u>

  Let $\mathcal{E} : 0 \to A \xrightarrow{g} M \xrightarrow{f} B \to 0$ be short exact, and let $\mathbf{0} : 0 \to A \to A \oplus B \to B \to 0$ be the trivial sequence. Then

  $$\mathcal{E} + \mathbf{0} : 0 \to A \xrightarrow{G} \nabla_* \Delta^*(M \oplus A \oplus B) \xrightarrow{F} B \to 0.$$

  Notice that $\Delta^*(M \oplus A \oplus B) = \{(m, (a, b)) \in M \oplus A \oplus B : fm = b\}$, so all elements of $\overline{\Delta^*(M \oplus A \oplus B)}$ have the form $(m, (a, fm))$. Define a map $\alpha : M \to \nabla_* \Delta^*(M \oplus A \oplus B)$ by $m \mapsto \overline{(m, (0, fm))}$.

  $$\begin{array}{rcccccccc}
  \mathcal{E}: & 0 & \longrightarrow & A & \xrightarrow{\quad g \quad} & M & \xrightarrow{\quad f \quad} & B & \longrightarrow 0 \\
  & & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow & \\
  \mathcal{E} + \mathbf{0} : & 0 & \longrightarrow & A & \xrightarrow{G} & \nabla_* \Delta^*(M \oplus A \oplus B) & \xrightarrow{F} & B & \longrightarrow 0
  \end{array}$$

  Let $m_1, m_2 \in M$ and $r \in R$. Then

  $$\alpha(m_1 + rm_2) = \overline{(m_1 + rm_2, (0, f(m_1 + rm_2)))} = \overline{(m_1, (0, fm_1))} + r\overline{(m_2, (0, fm_2))} = \alpha(m_1) + r\alpha(m_2),$$

  so α is indeed and R-module homomorphism. Since

  $$\alpha g(a) = \overline{(ga, (0, fga))} = \overline{(ga, (0, 0))} = G(a)$$

  and

  $$F\alpha(m) = F(m, (0, fm)) = fm,$$

  then α makes the diagram commute, and so α is an isomorphism by the Short 5 Lemma. Thus $\mathcal{E} + \mathbf{0} \sim \mathcal{E}$. Similarly, we have $\mathbf{0} + \mathcal{E} \sim \mathcal{E}$, and so $\mathbf{0}$ is the identity in $Ext^1(B, A)$.

- <u>*Inverses:*</u>

  Let $\mathcal{E}$ be as before, and define its inverse $-\mathcal{E} : 0 \to A \xrightarrow{-g} M \xrightarrow{f} B \to 0$. Then

  $$\mathcal{E} + (-\mathcal{E}) : 0 \to A \xrightarrow{G} \nabla_* \Delta^*(M \oplus M) \xrightarrow{F} B \to 0.$$

  To show that $\mathcal{E} + (-\mathcal{E}) \sim \mathbf{0}$, we will show that $\mathcal{E} + (-\mathcal{E})$ is split. Since

  $$\Delta^*(M \oplus M) = \{(m, m') \in M \oplus M : fm = fm'\},$$

  then for each (m, m') in $\Delta^*(M \oplus M)$, we have $m - m' \in \ker f = \operatorname{im} g$, so there exists a unique (since g is injective) $a \in A$ so that $ga = m - m'$. Define a map

  $$\pi : \nabla_* \Delta_*(M \oplus M) \to A, \quad \overline{(m, m')} \mapsto a.$$

  We show that this map is well-defined. First notice that

  $$\nabla_* \Delta^*(M \oplus M) = \Delta_*(M \oplus M) \Big/ \{(ga, ga) : a \in A\}.$$

  So if $\overline{(m_1, m_1')} = \overline{(m_2, m_2')}$, then there exists $a \in A$ so that $ga = m_1 - m_2 = m_1' - m_2'$. So $m_1 - m_1' = m_2 - m_2'$, which means there exists a unique $\tilde{a} \in A$ so that $g\tilde{a} = m_1 - m_1' = m_2 - m_2'$. Therefore,

  $$\pi\overline{(m_1, m_1')} = \tilde{a} = \pi\overline{(m_2, m_2')},$$

  and so π is well-defined. Moreover,

  $$\pi G(a) = \pi\overline{(ga, 0)} = a,$$

  or in other words, $\pi g = 1_A$, and so $\mathcal{E} + (-\mathcal{E})$ is split, showing that $\mathcal{E} + (-\mathcal{E}) \sim \mathbf{0}$. Similarly, we have $-\mathcal{E} + \mathcal{E} \sim \mathbf{0}$.

- *Associativity:*
  Let

$$\mathcal{E} : 0 \to A \xrightarrow{g} M \xrightarrow{f} B \to 0$$
$$\mathcal{E}' : 0 \to A \xrightarrow{g'} M' \xrightarrow{f'} B \to 0$$
$$\mathcal{E}'' : 0 \to A \xrightarrow{g''} M'' \xrightarrow{f''} B \to 0$$

be short exact sequences. Then

$$\mathcal{E} + \mathcal{E}' : 0 \to A \xrightarrow{\widetilde{G}} \nabla_* \Delta^*(M \oplus M') \xrightarrow{\widetilde{F}} B \to 0$$
$$\mathcal{E}' + \mathcal{E}'' : 0 \to A \xrightarrow{\widehat{G}} \nabla_* \Delta^*(M' \oplus M'') \xrightarrow{\widehat{F}} B \to 0$$

and so we have

$$(\mathcal{E} + \mathcal{E}') + \mathcal{E}'' : 0 \to A \xrightarrow{\widetilde{\widetilde{G}}} \nabla_* \Delta^*(\nabla_* \Delta^*(M \oplus M') \oplus M'') \xrightarrow{\widetilde{\widetilde{F}}} B \to 0$$
$$\mathcal{E} + (\mathcal{E}' + \mathcal{E}'') : 0 \to A \xrightarrow{\widehat{\widehat{G}}} \nabla_* \Delta^*(M \oplus \nabla_* \Delta^*(M' \oplus M'')) \xrightarrow{\widehat{\widehat{F}}} B \to 0$$

We're going to use a "double bar" notation, since we are passing to a quotient of a quotient. The context will make clear where we are taking the quotient at a given time.
Based on our definition of the Baer sum in the second problem of this section, the maps in these sums are defined as follows:

$$\widetilde{\widetilde{G}} : a \longmapsto \overline{(\widetilde{G}a, 0)} = \overline{(\overline{(ga, 0)}, 0)}$$
$$\widetilde{\widetilde{F}} : \overline{(\overline{(m, m')}, m'')} \longmapsto \widetilde{F}\ \overline{(m, m')} = fm$$
$$\widehat{\widehat{G}} : a \longmapsto \overline{(ga, \overline{(0, 0)})}$$
$$\widehat{\widehat{F}} : \overline{(m, \overline{(m', m'')})} \longmapsto fm.$$

Our goal is to give a well-defined morphism α that makes the following diagram commute, at which point we can apply the Short 5 Lemma to conclude that $(\mathcal{E} + \mathcal{E}') + \mathcal{E}'' \sim \mathcal{E} + (\mathcal{E}' + \mathcal{E}'')$.

$$\begin{array}{ccccccccc}
(\mathcal{E} + \mathcal{E}') + \mathcal{E}'' : 0 & \longrightarrow & A & \xrightarrow{\widetilde{\widetilde{G}}} & \nabla_* \Delta^*(\nabla_* \Delta^*(M \oplus M') \oplus M'') & \xrightarrow{\widetilde{\widetilde{F}}} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow & & \\
\mathcal{E} + (\mathcal{E}' + \mathcal{E}'') : 0 & \longrightarrow & A & \xrightarrow{\widehat{\widehat{G}}} & \nabla_* \Delta^*(M \oplus \nabla_* \Delta^*(M' \oplus M'')) & \xrightarrow{\widehat{\widehat{F}}} & B & \longrightarrow & 0
\end{array}$$

Define α by the rule $\overline{(\overline{(m, m')}, m'')} \longmapsto \overline{(m, \overline{(m', m'')})}$. Suppose that

$$\overline{(\overline{(m_1, m_1')}, m_1'')} = \overline{(\overline{(m_2, m_2')}, m_2'')}.$$

We want to find $c \in A$ so that

$$g(c) = m_1 - m_2 \quad \text{and} \quad \overline{(-g'(c), 0)} = -\widehat{G}(c) = \overline{(m_1' - m_2', m_1'' - m_2'')},$$

because then $\overline{(m_1, \overline{(m_1', m_1'')})} = \overline{(m_2, \overline{(m_2', m_2'')})}$, which will give that α is well-defined.
There exists $a \in A$ such that

$$\widetilde{G}(a) = \overline{(m_1 - m_2, m_1' - m_2')} \quad \text{and} \quad -g''(a) = m_1'' - m_2''.$$

So $\overline{(0, g'a)} = \overline{(ga, 0)} = \widetilde{G}(a) = \overline{(m_1 - m_2, m_1' - m_2')}$, and so there exists $\tilde{a} \in A$ such that

$$g(\tilde{a}) = -(m_1 - m_2) \quad \text{and} \quad -g'(\tilde{a}) = g'(a) - (m_1' - m_2').$$

Consider $c := -\tilde{a}$. Then the above equations give

$$g(c) = -g(\tilde{a}) = m_1 - m_2 \quad \text{and} \quad \overline{(-g'(c), 0)} = \overline{(m_1' - m_2', m_1'' - m_2'')},$$

where the last equality follows from the fact that

$$(-g'(c) - (m_1' - m_2'), -(m_1'' - m_2'')) = (g'(-a), -g''(-a)) \in \{(g'(d), -g''(d)) : d \in A\}.$$

Now let $r \in R$. Then

$$\begin{aligned} \alpha\ \overline{(\overline{(m_1 + rm_2, m_1' + rm_2')}, m_1'' + rm_2'')} &= \overline{(m_1 + rm_2, \overline{(m_1' + rm_2', m_1'' + rm_2'')})} \\ &= \overline{(m_1, \overline{(m_1' + m_1'')})} + \overline{(rm_2, \overline{(rm_2' + rm_2'')})} \\ &= \overline{(m_1, \overline{(m_1' + m_1'')})} + r\overline{(m_2, \overline{(m_2' + m_2'')})} \\ &= \alpha\ \overline{(\overline{(m_1, m_1')}, m_1'')} \ + \ r\ \alpha\ \overline{(\overline{(m_2, m_2')}, m_2'')}, \end{aligned}$$

and so α is indeed and R-module homomorphism. Finally,

$$\alpha\widetilde{\widetilde{G}}(a) = \alpha\ \overline{(\overline{(ga, 0)}, 0)} = \overline{(ga, \overline{(0, 0)})} = \widehat{\widehat{G}}$$

and

$$\widehat{\widehat{F}}\alpha\ \overline{(\overline{(m, m')}, m'')} = \widehat{\widehat{F}}\overline{(m, \overline{(m', m'')})} = fm = \widetilde{\widetilde{F}}\overline{(\overline{(m, m')}, m'')},$$

and so α makes the diagram commute and we are done. *Whew!*

- *Commutativity:*

  Let $\mathcal{E} : 0 \to A \xrightarrow{g} M \xrightarrow{f} B \to 0$ and $\mathcal{E}' : 0 \to A \xrightarrow{g'} M' \xrightarrow{f'} B \to 0$ be short exact sequences. Then

$$\mathcal{E} + \mathcal{E}' : 0 \to A \xrightarrow{\widetilde{G}} \nabla_*\Delta^*(M \oplus M') \xrightarrow{\widetilde{F}} B \to 0$$

$$\mathcal{E}' + \mathcal{E} : 0 \to A \xrightarrow{\widehat{G}} \nabla_*\Delta^*(M' \oplus M'') \xrightarrow{\widehat{F}} B \to 0$$

  Again we seek to give a well-defined morphism α which makes the following diagram commute, so that we can apply the Short 5 Lemma to conclude that $(\mathcal{E} + \mathcal{E}') \sim (\mathcal{E}' + \mathcal{E})$.

$$\begin{array}{ccccccccc} \mathcal{E} + \mathcal{E}' : 0 & \longrightarrow & A & \xrightarrow{\widetilde{G}} & \nabla_*\Delta^*(M \oplus M') & \xrightarrow{\widetilde{F}} & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ \mathcal{E}' + \mathcal{E} : 0 & \longrightarrow & A & \xrightarrow{\widehat{G}} & \nabla_*\Delta^*(M' \oplus M) & \xrightarrow{\widehat{F}} & B & \longrightarrow & 0 \end{array}$$

  Define α to be the map $\overline{(m, m')} \mapsto \overline{m', m}$. Suppose $\overline{(m_1, m_1')} = \overline{(m_2, m_2')}$. Then there exists $a \in A$ so that

$$g(a) = m_1 - m_2 \quad \text{and} \quad -g'(a) = m_1' - m_2'.$$

  So

$$(m_1' - m_2', m_1 - m_2) = (g'(-a), -g(-a)) \in \{(g'(c), -g(c)) : c \in C\}$$

  and hence $\overline{(m_1', m_1)} = \overline{(m_2', m_2)}$, i.e. α is well-defined. If $r \in R$, then

$$\alpha\ \overline{(m_1 + rm_2, m_1' + rm_2')} = \overline{(m_1' + rm_2', m_1 + rm_2)} = \overline{(m_1', m_1)} + r\ \overline{m_2', m_2} = \alpha\ \overline{(m_1, m_1')} + r\alpha\ \overline{(m_2, m_2')}$$

  so α is an R-module homomorphism. Finally,

$$\alpha\tilde{G}(a) = \alpha\ \overline{(g(a), 0)} = \overline{(0, g(a))} = \overline{g'(a), 0} = \widetilde{G}(a),$$

  and

$$\widetilde{F}\alpha\ \overline{(m, m')} = \widehat{F}\ \overline{(m', m)} = f'(m') = f(m) = \widetilde{F}\ \overline{(m, m')},$$

  and so α makes the diagram commute.

$\square$

4. Compute $Ext^1(\mathbb{Z}/2, \mathbb{Z}/2)$.

Solution:

Suppose $0 \to \mathbb{Z}/2 \to M \to \mathbb{Z}/2 \to 0$ is short exact. Then $\frac{M}{\mathbb{Z}/2} \cong \mathbb{Z}/2$, and so $|M| = 4$. Since there are only 2 groups of order 4, then we get the two following short exact sequences:

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \to \mathbb{Z}/2 \to 0$$
$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

Since $(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \ncong \mathbb{Z}/4$, then these are the only classes of sequences in $Ext^1(\mathbb{Z}/2, \mathbb{Z}/2)$.

Algebraic Topology
Homework 3
Name: 16977

1 Compactly supported generalized Mayer-Vietoris

Suppose that $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of a manifold M. Set

$$U_{\alpha_0\cdots\alpha_k} = \bigcap_{i=0}^{k} U_{\alpha_i} \quad \text{and} \quad C_i(\mathcal{U}, \Omega_c^*) = \prod_{\alpha_0<\cdots<\alpha_i} \Omega_c^*(U_{\alpha_0\cdots\alpha_i})$$

1. Construct a chain complex $\delta : C_i(\mathcal{U}, \Omega_c^*) \to C_{i-1}(\mathcal{U}, \Omega_c^*)$.

   Solution:

   The product should actually be a direct sum, i.e.

   $$C_i(\mathcal{U}, \Omega_c^*) = \bigoplus_{\alpha_0<\cdots<\alpha_i} \Omega_c^*(U_{\alpha_0\cdots\alpha_i})$$

   so that $\omega \in C_i(\mathcal{U}, \Omega_c^*)$ has only a finite number of nonzero components. This is needed to ensure that in our definition of δ below, the components of $\delta\omega$ have compact support. Define δ by the formula

   $$(\delta\omega)_{\alpha_0\cdots\alpha_{i-1}} = \sum_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{i-1}},$$

   where on the right hand side, we've extended $\omega_{\alpha_0\cdots\alpha_{i-1}}$ by zero to the form $\omega_{\alpha\alpha_0\cdots\alpha_{i-1}}$ on $U_{\alpha_0\cdots\alpha_{i-1}}$. This definition yields a chain complex, since $\delta^2 = 0$:

   $$(\delta^2\omega)_{\alpha_0\cdots\alpha_{i-2}} = \sum_\alpha (\delta\omega)_{\alpha\alpha_0\cdots\alpha_{i-2}} = \sum_\alpha \sum_\beta \omega_{\beta\alpha\alpha_0\cdots\alpha_{i-2}}.$$

   Then $\omega_{\beta\alpha\cdots} = \omega_{\alpha\beta\cdots}$ together with the fact that α and β run over the same indices, gives that $(\delta^2\omega)_{\alpha_0\cdots\alpha_{i-2}} = 0$.

2. Prove that there is a long exact sequence

   $$\Omega_c^*(M) \overset{r}{\leftarrow} C_0(\mathcal{U}, \Omega_c^*) \overset{\delta}{\leftarrow} C_1(\mathcal{U}, \Omega_c^*) \leftarrow \cdots$$

   Proof. First, note that we need the cover $\mathcal{U}$ to be locally finite. Define r to be the summing map, i.e. for $\omega \in C_0(\mathcal{U}, \Omega_c^*), r\omega = \sum_{\alpha_0} \omega_{\alpha_0}$. If $\{\rho_\alpha\}_{\alpha\in\Lambda}$ is a partition of unity subordinate to $\mathcal{U}$, and $\omega \in \Omega_c^*(M)$, then let $\tau_\alpha = \rho_\alpha\omega$. So $r\tau = \sum_\alpha \rho_\alpha\omega = \omega$, and so r is onto.

   Since we've shown that $\delta^2 = 0$, then $\operatorname{im}\delta \subseteq \ker\delta$. Going the other way, suppose $\delta\omega = 0$ for $\omega \in C_i(\mathcal{U}, \Omega_c^*)$. Then define

   $$\tau_{\alpha_0\cdots\alpha_{i+1}} = \sum_{k=0}^{i+1} (-1)^k \rho_{\alpha_k} \omega_{\alpha_0\cdots\hat{\alpha}_k\cdots\alpha_{i+1}}.$$

   Note that $\tau_{\alpha_0\cdots\alpha_{i+1}}$ has compact support since ω does. Moreover, since $\omega_{\alpha_0\cdots\alpha_i} \neq 0$ for finitely many indices $\alpha_0\cdots\alpha_i$ and since each $U_{\alpha_0\cdots\alpha_i} \subset U_{\alpha_0}$ intersects finitely many U_{α_k},[1], then $\rho_{\alpha_k}\omega_{\alpha_0\cdots\alpha_i} \neq 0$

[1] Since $\mathcal{U}$ is locally finite

for finitely many indices $\alpha_k\alpha_0 \cdots \alpha_i$. So τ has finitely many nonzero components and indeed we have $\tau \in \bigoplus \Omega_c^*(U_{\alpha_0\cdots\alpha_{i+1}})$. Then

$$\begin{aligned}
(\delta\tau) &= \sum_\alpha \tau_{\alpha\alpha_0\cdots\alpha_i} \\
&= \sum_\alpha \left(\rho_\alpha \omega_{\alpha_0\cdots\alpha_i} + \sum_k (-1)^{k+1} \rho_{\alpha_k} \omega_{\alpha\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_i} \right) \\
&= \omega_{\alpha_0\cdots\alpha_i} + \sum_k (-1)^{k+1} \rho_{\alpha_k} (\delta\omega)_{\alpha_0\cdots\hat{\alpha}_k\cdots\alpha_i} \\
&= \omega_{\alpha_0\cdots\alpha_i}
\end{aligned}$$

□

2 A spectral sequence of sorts

Suppose that $F \to E \xrightarrow{\pi} B$ is a fiber bundle of smooth manifolds and $\mathfrak{U}$ is a good cover of B.

1. Prove that $\pi^{-1}\mathfrak{U}$ is a (not necessarily good) cover of E.

   Proof. Since π is surjective and $\mathfrak{U}$ covers B, then $E \subseteq \pi^{-1}\mathfrak{U}$. □

2. Show that the total homology of the bicomplex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ is isomorphic to the de Rham cohomology of E via the map r discussed in class.

   Proof. Suppose $\mathfrak{U} = \{U, V\}$, and $E \subseteq \pi^{-1}\mathfrak{U} = \{\pi^{-1}U, \pi^{-1}V\}$. Recall that we have a differential $D : C^*(\mathfrak{U}, \Omega^*) \to C^*(\mathfrak{U}, \Omega^*)$ given by $D = \delta + (-1)^p d$. We have a map

$$r : \Omega^*(E) \to \Omega^*(\pi^{-1}U) \oplus \Omega^*(\pi^{-1}V)$$

given by the restriction of forms. First notice that r is a chain map, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \Omega^*(E) & \xrightarrow{\ r\ } & C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle D} \\ \Omega^*(E) & \xrightarrow[r]{} & C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \end{array}$$

This is because

$$Dr = (\delta + (-1)^p d)r = \delta r + dr = rd$$

since $p = 0$ and $\delta r = 0$. So we get a map on cohomology

$$r^* : H^*_{DR}(E) \to H_D\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\}$$

Recall that a q-cochain α in the double complex $C * (\mathfrak{U}, \Omega^*)$ has two components

$$\alpha = \alpha_0 + \alpha_1, \quad \alpha_0 \in C^0(\pi^{-1}\mathfrak{U}, \Omega^q), \quad \alpha_1 \in C^1(\pi^{-1}\mathfrak{U}, \Omega^{q-1}).$$

By the exactness of the Mayer-Vietoris sequence, there exists β so that $\delta\beta = \alpha_1$. So

$$\alpha - D\beta = \alpha_0 + \alpha_1 - \delta\beta - d\beta = \alpha_0 - d\beta,$$

and so α is D-cohomologous to a cochain with only the "top" component, i.e., a cochain in $C^0(\pi^{-1}\mathfrak{U}, \Omega^q)$.

To show that r^* is surjective, we use this fact, and so we may assume that a given cohomology class in $H_D\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\}$ is represented by a cocycle [2] $\phi = \phi_1 + \phi_2$ with only the top component. Since ϕ is a cocycle, $D\phi = 0$, and so

$$0 = D\phi = \delta(\phi_1 + \phi_2) + d(\phi_1 + \phi_2) \implies \delta(\phi_1 + \phi_2) = -d(\phi_1 + \phi_2).$$

From the definition of δ, we have $\delta(\phi_1 + \phi_2) = \phi_2|_{\pi^{-1}U} - \phi_1|_{\pi^{-1}V}$. Now, $\phi_2|_{\pi^{-1}U} - \phi_1|_{\pi^{-1}V}$ is a form in the intersection $\pi^{-1}U \cap \pi^{-1}V$, but $-d(\phi_U + \phi_V)$ is a form in $\Omega^{q+1}(\pi^{-1}U) \oplus \Omega^{q+1}(\pi^{-1}V)$. So, these must be zero, i.e.

$$\delta(\phi_U + \phi_V) = -d(\phi_U + \phi_V) = 0,$$

which implies $\phi_2|_{\pi^{-1}U} = \phi_1|_{\pi^{-1}V}$. This says that the first component of ϕ restricted to $\pi^{-1}V$ is the same as the second component of ϕ restricted to $\pi^{-1}U$. Hence ϕ is a global form on $\pi^{-1}\mathfrak{U}$, and so $r\phi = \phi|_{\pi^{-1}U} + \phi|_{\pi^{-1}V} = \phi_1 + \phi_2 = \phi$.

To show that r^* is injective, suppose $r\omega = D\phi$ for some cochain ϕ in $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$. By the remark above, we may write $\phi = \phi' + D\phi''$, where ϕ' has only the top component. Then

$$r\omega = D\phi = D\phi' + DD\phi'' = d\phi', \quad \delta\phi' = 0.$$

So ω is the exterior derivative of a global form on E. □

[2] we can write ϕ this way since $\phi \in \Omega^q(\pi^{-1}U) \oplus \Omega^q(\pi^{-1}V)$

3. Show that if $\mathcal{U}$ is good then the vertical cohomology can be computed by the cochain complex $C^*(\mathcal{U}, \overline{H^*(F)})$ where $\overline{H^*(F)}$ is a locally constant sheaf on B (non-canonically) isomorphic to the cohomology of the fiber F.

Proof. For each intersection $U_{\alpha_0 \cdots \alpha_k}$, we have $\pi^{-1}(U_{\alpha_0 \cdots \alpha_k}) \cong U_{\alpha_0 \cdots \alpha_k} \times F$ since π is a bundle map. Then since the cover $\mathcal{U}$ is good, we get that $U_{\alpha_0 \cdots \alpha_k} \simeq R^n \simeq \{pt\}$ and so in fact $H^*(\pi^{-1}(\mathcal{U})) \simeq H^*(F)$. So $C^*(\mathcal{U}, \overline{H^*(F)}) \cong H^*(F)$. □

3 Topologists like tori

Let T^n be the n-torus.

1. Compute $H^*(T^n)$.

Proof. We prove by induction on n that $H^k(T^n) = \mathbb{R}^{\binom{n}{k}}$. For $n = 0$, this is obvious, and for $n = 1$, $T^1 = S^1$, and we know

$$H^*(S^1) = \begin{cases} \mathbb{R} & \text{if } * = 0, 1 \\ 0 & \text{if } * > 1 \end{cases}.$$

Suppose the result is true for T^{n-1}. Notice that $T^n = T^{n-1} \times S^1$. By the Kunneth formula,

$$H^k(T^{n-1} \times S^1) = \bigoplus_{k=p+q} H^p(T^{n-1}) \otimes_{\mathbb{R}} H^q(S^1).$$

So,

$$\begin{aligned}
H^k(T^n) &= \bigoplus_{k=p+q} H^p(T^{n-1}) \otimes H^q(S^1) \\
&= \Big(H^k(T^{n-1}) \otimes H^0(S^1)\Big) \oplus \Big(H^{k-1}(T^{n-1}) \otimes H^1(S^1)\Big) \oplus \Big(H^{k-2}(T^{n-1}) \otimes H^2(S^1)\Big) \oplus \cdots \\
&= \Big(H^k(T^{n-1}) \otimes H^0(S^1)\Big) \oplus \Big(H^{k-1}(T^{n-1}) \otimes H^1(S^1)\Big) \\
&= \Big(H^k(T^{n-1}) \otimes \mathbb{R}\Big) \oplus \Big(H^{k-1}(T^{n-1}) \otimes \mathbb{R}\Big) \\
&= H^k(T^{n-1}) \oplus H^{k-1}(T^{n-1}) \\
&= \mathbb{R}^{\binom{n-1}{k}} \oplus \mathbb{R}^{\binom{n-1}{k-1}} \\
&= \mathbb{R}^{\binom{n-1}{k} + \binom{n-1}{k-1}} \\
&= \mathbb{R}^{\binom{n}{k}}.
\end{aligned}$$

□

2. Determine Poincare duals $\eta_S \in H^k(T^n)$ of the 2^n standard subtori $S \subset T^n$, i.e. classes which satisfy

$$\int_S i_S^*(\omega) = \int_M \omega \wedge \eta_S$$

4 Oh, the nerve of that cover!

1. To each category $\mathcal{C}$ there is a set $N_n(\mathcal{C})$ consisting of n-fold sequences of composable morphisms in $\mathcal{C}$. Show that the assignment $[n] \mapsto N_n(\mathcal{C})$ determines a simplicial set $\Delta^{op} \to Set$. This is called the *nerve* of the category $\mathcal{C}$.

   Proof. Let $F : \Delta^{op} \to Set$ be given by the assignment $[n] \mapsto N_n(\mathcal{C})$. If $d_i : [n-1] \to [n]$ is the ith face map, define $Fd_i : N_n(\mathcal{C}) \to N_{n-1}(\mathcal{C})$ by

   $$(C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} C_n) \longmapsto (C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \to C_{i-1} \xrightarrow{f_i f_{i-1}} C_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} C_n)$$

   whenever $0 < i < n$, and we remove the ith component of the sequence when $i = 0$ or k. If $s_i : [n+1] \to [n]$ is the ith degeneracy map, define $Fs_i : N_n(\mathcal{C}) \to N_{n+1}(\mathcal{C})$ by

   $$(C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} C_n) \longmapsto (C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_i \xrightarrow{\mathbb{1}_{C_i}} C_i \xrightarrow{f_i} \cdots \xrightarrow{f_{n-1}} C_n)$$

   Since every morphism in $\mathcal{C}$ is a composition of face and degeneracy maps, this makes F functor. □

2. Suppose that M is a smooth manifold and $\mathcal{U}$ is a cover. Show that there is a category $\mathcal{C}(\mathcal{U})$ with objects given by open sets $U \in \mathcal{U}$ and morphisms $V \hookrightarrow U$ given by inclusions among open sets in the cover.

   Proof. If $V \hookrightarrow U$ and $U \hookrightarrow W$ are two morphisms in $\mathcal{C}(\mathcal{U})$, then define their composition to be the inclusion $V \hookrightarrow W$. For any $U \in \mathcal{U}$, since $U \subseteq U$, then there is an identity morphism $1_U \in \text{Hom}(U, U)$ which has the desired properties:

   $$(U \overset{1_U}{\hookrightarrow} U) \circ (U \hookrightarrow W) = U \hookrightarrow W \quad \text{and} \quad (V \hookrightarrow U) \circ (U \overset{1_U}{\hookrightarrow} U) = V \hookrightarrow U$$

   Composition is associative:

   $$\begin{aligned}(V \hookrightarrow U) \circ \big((U \hookrightarrow W) \circ (W \hookrightarrow Y)\big) &= (V \hookrightarrow U) \circ (U \hookrightarrow Y) \\ &= V \hookrightarrow Y \\ &= (V \hookrightarrow W) \circ (W \hookrightarrow Y) \\ &= \big((V \hookrightarrow U) \circ (U \hookrightarrow W)\big) \circ (W \hookrightarrow Y).\end{aligned}$$

   □

3. Suppose that M is a smooth compact manifold and $\mathcal{U}$ is a good cover which is closed under intersection ($U, V \in \mathcal{U}$ and $U \cap V \neq \emptyset$ implies $U \cap V \in \mathcal{U}$). Then it is true that $|N_*(\mathcal{C}(\mathcal{U}))| \simeq M$. Draw an example in the case of the torus.

   Solution:

4. One can use abstract simplicial complexes too. If $\mathcal{U}$ is a cover then show that the set of non-empty n-fold (for some n) intersections is a simplicial complex. How is this related to the Cech construction appearing in Bott and Tu?

   Solution:

   Bott and Tu give a way to correspond n-fold nonempty intersections of a good cover to a simplicial complex, exactly the way that was done in the previous exercise. Each open set corresponds to a vertex, each 2-fold intersection corresponds to an edge between two vertices, each 3-fold intersection corresponds to a "filled in" standard 2-simplex, etc.