Math 6000, Fall 2017 (Prof. Kinser), Final

Nicholas Camacho

December 11, 2017

Problem 1. *This problem introduces a type of dual for modules and further develops properties of projective modules.* Let R be a ring and M a left R-module. The *R-dual* of M is defined to be the right R-module $M^\vee := \operatorname{Hom}_R(M, R)$.

(a) Note that $M^{\vee\vee}$ is then a left R-module again. Give an example of a ring R and a finitely generated left R-module M such that $M^{\vee\vee} \not\cong M$.

Solution:

I stumbled upon this example while reading Dummit and Foote's "Examples" of projective and non-projective modules on pages 391-392. Take $R = \mathbb{Z}$ and $M = \mathbb{Z}/n\mathbb{Z}$ for any $n \geq 2$. Since there are no nonzero $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}$, then $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$. So

$$(\mathbb{Z}/n\mathbb{Z})^{\vee\vee} = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(0, \mathbb{Z}) = 0 \ncong \mathbb{Z}.$$

(b) Prove that if P is a finitely generated projective left R-module, then $P^\vee$ is a finitely generated projective R-module.

Proof. Since P is a finitely generated projective R-modules, we have $P \oplus Q \cong R^n$ for some R-module Q. This gives

$$P^\vee \oplus Q^\vee \cong (R^n)^\vee = \operatorname{Hom}_R(R^n, R) \cong \operatorname{Hom}_R(R, R)^n \cong R^n,$$

so $P^\vee$ is a finitely generated projective R-module. ☕

(c) Prove that a module P is projective if and only if there exist $\{x_i\}_{i \in I} \subseteq P$ and $\{f_i\}_{i \in I} \subseteq P^\vee$ such that, for all $x \in P$, the following hold: (i) $f_i(x) = 0$ for all but finitely many $i \in I$, and (ii) $x = \sum_i f_i(x) x_i$.

Proof. ($\Rightarrow$) Let $\{x_i\}_{i \in I} \subseteq P$ be a set of R-module generators for P. So we get a surjective map $\pi : \bigoplus_{i \in I} R \twoheadrightarrow P$, mapping $e_i \mapsto x_i$, where $\{e_i\}_{i \in I}$ are the standard basis elements of $\bigoplus_{i \in I} R$. Since P is projective, there exists an injective map $\sigma : P \hookrightarrow \bigoplus_{i \in I} R$ so that $\pi\sigma = \mathbb{1}_P$. For $x \in P$, we have a unique (since σ is injective) expression

$$\sigma(x) = \sum_i r_i e_i$$

where $\{r_i\}_i \subset R$ and $r_i = 0$ for all but finitely many $i \in I$. So for each $i \in I$, define $f_i(x) = r_i$. Now f is a well-defined R-module homomorphism since it is defined in terms of te injective homomorphism σ. Then

$$x = \pi\sigma(x) = \sum_i r_i \pi(e_i) = \sum_i f_i(x) x_i.$$

($\Leftarrow$) If $\pi : M \twoheadrightarrow P$ is any surjective R-module homomorphism, for each i, pick $m_i \in \pi^{-1}(x_i)$ and define $\sigma : P \to M$ by $x \mapsto \sum_i f_i(x) m_i$. Then for $r \in R$ and $x, y \in P$,

$$\begin{aligned}\sigma(x + ry) = \sigma\left(\sum_i f_i(x) x_i + r \sum_i f(y) x_i\right) = \sigma\left(\sum_i f_i(x + ry) x_i\right) &= \sum_i f_i(x + ry) m_i \\ &= \sum_i f_i(x) m_i + r \sum_i f_i(y) \\ &= \sigma(x) + r\sigma(y),\end{aligned}$$

so σ is indeed an R-module homomorphism. Moreover, $\pi\sigma = \mathbb{1}_P$, and so P is projective. ☕

These collections are called *dual bases*, just like for finite dimensional vector spaces.

(d) **Optional.** For any finitely generated projective left R-module P, construct a natural isomorphism $P \to P^{\vee\vee}$. What can you say if P is projective but not finitely generated?

Problem 2. Let G be a group, and p a prime dividing $|G|$, and K a field of characteristic p. The goal of this problem is to prove that the group ring KG is *not* semi-simple. This is essentially Rotman Exercise 8.37, which you may consult for additional hints if necessary.

Since $\{g \in G\}$ is a basis of KG, the *augmentation map*

$$\varepsilon\colon KG \to K, \qquad \varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$$

is well-defined. Let $I := \ker\varepsilon$. Recall that K can be regarded as a KG-module by the action $g \cdot x = x$ for all $x \in K$, called the *trivial* KG-module.

(a) Prove that ε is a morphism of KG-modules, and a ring homomorphism. Conclude that I is a 2-sided ideal of KG.

Proof. For $x = \sum_{h\in G} x_h h \in KG$, we have

$$\epsilon\left(\sum_{g\in G} a_g g + x\sum_{\ell\in G} b_\ell \ell\right) = \epsilon\left(\sum_{g\in G} a_g g + \sum_{g\in G}\sum_{h\ell=g} x_h b_\ell g\right) = \epsilon\left(\sum_{g\in G}\left(a_g + \sum_{h\ell=g} x_h b_\ell\right) g\right)$$
$$= \sum_{g\in G}\left(a_g + \sum_{h\ell=g} x_h b_\ell\right),$$

and on the other hand

$$\epsilon\left(\sum_{g\in G} a_g g\right) + x\epsilon\left(\sum_{\ell\in G} b_\ell \ell\right) = \sum_{g\in G} a_g + \sum_{h\in G} x_h h \sum_{\ell\in G} b_\ell = \sum_{g\in G} a_g + \sum_{h\in G} x_h \sum_{\ell\in G} b_\ell$$
$$= \sum_{g\in G}\left(a_g + \sum_{h\ell=g} x_h b_\ell\right).$$

We also have

$$\epsilon\left(\sum_{g\in G} a_g g \sum_{h\in G} b_h h\right) = \epsilon\left(\sum_{k\in G}\sum_{gh=k} a_g b_h k\right) = \sum_{k\in G}\sum_{gh=k} a_g b_h,$$

and on the other hand

$$\epsilon\left(\sum_{g\in G} a_g g\right)\epsilon\left(\sum_{h\in G} b_h h\right) = \sum_{g\in G} a_g \sum_{h\in G} b_h = \sum_{k\in G}\sum_{gh=k} a_g b_h.$$

Hence ϵ is a ring homomorphism and a morphism of KG-modules, and so I is a two-sided ideal of KG. ☕

(b) Let $v = \sum_{g\in G} g \in KG$. Prove that the one dimensional subspace $Kv \subseteq KG$ is the *only* submodule of KG isomorphic to the trivial module.

Proof. Any KG-submodule of KG isomorphic to the trivial module is one-dimensional and has the form Kw where $w = \sum_{g\in G} a_g g \in KG$. Since Kw is isomorphic to the trivial module, $hw = w$ for each $h \in G$. Then

$$\sum_{g\in G} a_g hg = hw = w = \sum_{g\in G} a_g g,$$

For $g_0 \in G$ there exists $g \in G$ so that $g_0 = hg$. The coefficient of g_0 is $a_{h^{-1}g_0}$ and a_{g_0}, on the LHS and RHS, respectively. Since the elements of g form a basis for KG, we have $a_{g_0} = a_{h^{-1}g_0}$ for all $h \in G$.

Since the left regular action of G is transitive, for each $g_1 \in G$, there exists $h^{-1} \in G$ with $g_1 = h^{-1}g_0$, so $a_{g_1} = a_{h^{-1}g_0} = a_{g_0}$. Hence $w \in Kv$, so $Kw = Kv$, and therefore Kv is the only KG-submodule of KG isomorphic to the trivial module. ☕

(c) Now prove that KG is *not* a semisimple ring by contradiction.

Proof. If KG is semisimple, then the short exact sequence $0 \to I \to KG \xrightarrow{\epsilon} K \to 0$ splits, i.e., there is a section $\sigma : K \to KG$ of ϵ. Since σ is injective, then $K \cong \sigma(K)$, so $\sigma(K)$ is a submodule of KG isomorphic to K, and hence $\sigma(K) = Kv$ by part (b). So then $KG = Kv \oplus I$, but $\epsilon(v) = |G| = 0$ since p divides $1 \cdot |G|$, and so $Kv \cap I \neq \varnothing$, which contradicts that KG is a direct sum of Kv and I. ☕

Therefore, we conclude (along with Maschke's theorem) that KG is a semi-simple ring *if and only if* $\text{char}K$ does not divide $|G|$.