Homework for General Topology

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Most exercises are from
Topology (2nd Edition) by Munkres.
For example, "18.4", "18-4", or "§18, #4 "
each mean exercise 4 from section 18 in Munkres.
Beware: Some solutions may be incorrect!

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General Topology
Homework 1
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Monday Exercises

1. If U is the universal set and $A, B \subset U$ then,

$$(A \cap B)^c = A^c \cup B^c \tag{1}$$

and

$$(A \cup B)^c = A^c \cap B^c \tag{2}$$

Proof of (1).

$$\begin{aligned} x \in (A \cap B)^c &\Leftrightarrow x \notin (A \cap B) \\ &\Leftrightarrow (x \notin A) \text{ or } (x \notin B) \\ &\Leftrightarrow (x \in A^c) \text{ or } (x \in B^c) \Leftrightarrow x \in (A^c \cup B^c) \end{aligned}$$

■

Proof of (2).

$$\begin{aligned} x \in (A \cup B)^c &\Leftrightarrow x \notin (A \cup B) \\ &\Leftrightarrow (x \notin A) \text{ and } (x \notin B) \\ &\Leftrightarrow (x \in A^c) \text{ and } (x \in B^c) \Leftrightarrow x \in (A^c \cap B^c) \end{aligned}$$

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2. Let $f : A \to B$ be a function, $C_1, C_2 \subset A$ and $D_1, D_2 \subset B$.

$$f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2) \tag{3}$$

Proof of (3).

$$\begin{aligned} x \in f^{-1}(D_1 \cup D_2) &\Leftrightarrow f(x) \in D_1 \cup D_2 \\ &\Leftrightarrow (f(x) \in D_1) \text{ or } (f(x) \in D_2) \\ &\Leftrightarrow (x \in f^{-1}(D_1)) \text{ or } (x \in f^{-1}(D_2)) \\ &\Leftrightarrow x \in (f^{-1}(D_1) \cup f^{-1}(D_2)) \end{aligned}$$

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$$f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2) \tag{4}$$

Proof.

$$\begin{aligned} x \in f^{-1}(D_1 \cap D_2) &\Leftrightarrow f(x) \in D_1 \text{ and } D_2 \\ &\Leftrightarrow (f(x) \in D_1) \text{ and } (f(x) \in D_2) \\ &\Leftrightarrow (x \in f^{-1}(D_1)) \text{ and } (x \in f^{-1}(D_2)) \\ &\Leftrightarrow x \in (f^{-1}(D_1) \cap f^{-1}(D_2)) \end{aligned}$$

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$$f^{-1}(D_1^c) = (f^{-1}(D_1))^c \tag{5}$$

Proof of (5).

$$\begin{aligned} x \in f^{-1}(D_1^c) &\Leftrightarrow f(x) \in D_1^c \\ &\Leftrightarrow f(x) \notin D_1 \\ &\Leftrightarrow x \notin f^{-1} D_1 \\ &\Leftrightarrow x \in (f^{-1}(D_1))^c \end{aligned}$$

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$$f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2) \tag{6}$$

Proof.

$$\begin{aligned} y \in f(C_1 \cap C_2) &\implies \exists x \in (C_1 \cap C_2) \ s.t.\ f(x) = y \\ &\implies (x \in C_1) \text{ and } (x \in C_2) \text{ where } f(x) = y \\ &\implies y \in f(C_1) \text{ and } y \in f(C_2) \\ &\implies y \in (f(C_1) \cap f(C_2)) \end{aligned}$$

A counter example where the two sides of (6) are not equal is $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$, with $C_1 = [-1, 0], C_2 = [0, 1]$. In this case,

$$f(C_1 \cap C_2) = 0,$$

and

$$f(C_1) \cap f(C_2) = [0, 1] \cap [0, 1] = [0, 1],$$

and so

$$0 = f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2) = [0, 1].$$

But,

$$[0, 1] = f(C_1) \cap f(C_2) \not\subset f(C_1 \cap C_2) = 0$$

■

$$f(C_1 \cup C_2) = f(C_1) \cup f(C_2) \tag{7}$$

Proof of (7).

$$\begin{aligned} y \in f(C_1 \cup C_2) &\implies \exists x \in (C_1 \cup C_2) \ s.t. \ f(x) = y \\ &\implies (x \in C_1) \text{ and } (x \in C_2) \text{ where } f(x) = y \\ &\implies (y \in f(C_1)) \text{ or } (y \in f(C_2)) \\ &\implies y \in (f(C_1) \cup f(C_2)) \end{aligned}$$

$$\begin{aligned} y \in (f(C_1) \cup f(C_2)) &\implies y \in f(C_1) \text{ or } y \in f(C_2) \\ &\implies \exists x \in C_1 \cup C_2 \ s.t. \ f(x) = y \\ &\implies y \in f(C_1) \text{ or } y \in f(C_2) \\ &\implies y \in (f(C_1) \cup f(C_2)) \end{aligned}$$

■

3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

   Proof.

$$\begin{aligned} (x, y) \in (A \times B) \cap (C \times D) &\Leftrightarrow ((x, y) \in (A \times B)) \text{ and } ((x, y) \in (C \times D)) \\ &\Leftrightarrow (x \in A \cap C) \ and \ (y \in B \cap D) \\ &\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D) \end{aligned}$$

■

Wednesday Exercises

1. - What is $A \times \emptyset$?
     Solution: $A \times \emptyset$ is the set of all functions $\boldsymbol{x} : \{1, 2\} \to A \cup \emptyset$ such that $\boldsymbol{x}(1) \in A$ and $\boldsymbol{x}(2) \in \emptyset$. Since $\emptyset$ contains no elements, $\boldsymbol{x}(2)$ cannot be in $\emptyset$. Thus, no such functions $\boldsymbol{x}$ exists. Thus, $A \times \emptyset = \emptyset$.
   - Does there exist a map $f : A \to \emptyset$? If so, how many? Are they 1-1? Onto?
     Solution: In this case, the set of possible functions would be all subsets of the cartesian product $A \times \emptyset = \emptyset$. And since the empty set has only one subset, only one function f exists. This map is in fact 1-1 and onto because there are no elements contained in the empty set which are able to directly contradict the definitions of 1-1 and onto. In other words, $f : A \to \emptyset$ satisfy the definitions of 1-1 and onto.
   - Does there exists a map $f : \emptyset \to A$? If so, how many? Are they 1-1? Onto?
     Solution: Similar answer as above

2. Let $I_X = \{(x, x) \in X \times X\}$. Is I_X a function? Is it 1-1? Is it onto?
   Solution: I_x is indeed a function. This function assigns each element x in X to itself, and so it is well-defined. It is certainly 1-1, since $x \neq x$ (in the domain) trivially implies $x \neq x$ (in the range). It is also onto, since given any $x \in X$, we know $(x, x) \in I_x$.

3. Suppose that $f : A \to B$, and $g : B \to C$ are functions. Prove the following:

   - If f, g are 1-1, then $g \circ f$ is 1-1.

     Proof. Let f, g be 1-1. Suppose $a_1, a_2 \in A, a_1 \neq a_2$. As f is 1-1

     $$f(a_1) \neq f(a_2)$$

     Similarly, since g is 1-1,

     $$g(f(a_1)) \neq g(f(a_2))$$

     Thus,

     $$a_1 \neq a_2 \implies (g \circ f)(a_1) \neq (g \circ f)(a_2)$$

     and so $g \circ f$ is 1-1. ■

   - If f, g are onto, then $g \circ f$ is onto.

     Proof. Let f, g be onto. Suppose $c \in C$. As g is onto, there exists a $b \in B$ such that $g(b) = c$. Similarly, as f is onto, there exists an $a \in A$ such that $f(a) = b$. So, $g(f(a)) = c$ and thus, $g \circ f$ is onto. ■

   - If $g \circ f$ is onto then g is onto.

     Proof. Let $g \circ f$ be onto and $c \in C$. Since $g \circ f$ is onto, there exists an $a \in A$ so that $g(f(a)) = c$. Let $f(a) = b$. Then, $g(b) = c$ and so g is onto. ■

   - If $g \circ f$ is 1-1, then f is 1-1.

     Proof. Let $g \circ f$ be 1-1 and $a_1, a_2 \in A, a_1 \neq a_2$. Let $b_1 = f(a_1)$ and $b_2 = f(a_2)$. As $g \circ f$ is 1-1, $g(b_1) \neq g(b_2)$. This immediately implies $b_1 \neq b_2$. Otherwise, g would send a single point in B to two different elements in C and thus, g would not be a function. So, $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$. ■

4. Let 2^X denote the set of all subsets of X. Prove that there exists a function $f : 2^X \setminus \{0\} \to X$ so that for any $B \in 2^X$, $f(B) \in B$. *Hint: Axiom of Choice*

   Proof. Let $\mathscr{B} = 2^X \setminus \{\emptyset\}$ and notice that

   $$X = \bigcup_{B \in \mathscr{B}} B$$

   Lemma 2.9 ensures the existence of a function

   $$c : \mathscr{B} \to \bigcup_{B \in \mathscr{B}} B$$

   such that $c(B)$ is an element of B, for each $B \in \mathscr{B}$. ■

5. Let $X = \mathbb{N}$ and $Y = \mathbb{N}$. Suppose however that X and Y are disjoint copies of the natural numbers. Let $f : X \to Y$ be $f(n) = n + 1$. Let $g : Y \to X$ be $g(n) = 2n$. These are clearly 1-1.

- Prove that $X_\infty = \emptyset$ and $Y_\infty = \emptyset$. *Hint: Inverse images get smaller and smaller.*

  Proof. To get a contradiction, assume $x \in X_\infty$. This means that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$, defined by

  $$\begin{aligned} x_1 &:= x, \\ x_{2k} &:= \frac{x_{2k-1}}{2}, \\ x_{2k+1} &:= x_{2k} - 1 \end{aligned}$$

  for $k \in \mathbb{N}$. This means that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. This implies that there is a subset of the natural numbers which has no minimal element — a contradiction to the well-ordering principle. Thus, $X_\infty = \emptyset$. Similarly, let $y \in Y$. Then, there is a sequence $\{y_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ defined by

  $$\begin{aligned} y_1 &:= y, \\ y_{2k} &:= y_{2k} - 1, \\ y_{2k+1} &:= \frac{y_{2k-1}}{2} \end{aligned}$$

  for $k \in \mathbb{N}$. This means that $y_{n+1} < y_n$ for all $n \in \mathbb{N}$ and the same contradiction is reached as before. Thus, $Y_\infty = \emptyset$. ■

- Prove that $Y_Y = \{2^m - 1 | m \in \mathbb{N}\}$. Prove that $X_Y = \{2^{m+1} - 2 | m \in \mathbb{N}\}$.

  Proof. Notice that all elements in Y_Y have ultimate ancestor 1. This is because 1 is the only element in Y that is not in the image of f. Then, in an effort to find a general form of all the elements in Y_Y, we define a recurrence relation:

  $$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= 2a_n + 1 \end{aligned}$$

  To see how this was constructed, first note that the first element in Y_Y is 1, as explained above. Then, let $y \in Y_Y$, $(y \neq 1)$. The closest ancestor of y that is in Y, say y', must have passed from the set Y to the set X first through the function g — yielding $g(y') = 2y'$ — and then passed from the set X to the set Y through f— yielding $f(g(y')) = 2y' + 1$. We now proceed by induction on n to prove that the n-th term of the sequence produced by this recurrence relation is $a_n = 2^n - 1$.
  For the base case, let $n = 1$. Clearly,

  $$a_{n+1} = 2a_1 + 1 = 2 \cdot 1 + 1 = 3$$
  $$\text{and}$$
  $$2^1 + 1 = 3$$

  Now, assume that the conclusion holds for $n-1$. That is, assume $a_{n-1} = 2^{n-1} - 1$. Then,

  $$a_n = 2a_{n-1} + 1 = 2 \cdot (2^{n-1} + 1) - 1 = 2^n + 1 - 1 = 2^n - 1.$$

  Therefore, by the principle of mathematical induction, $a_n = 2^n - 1$ for all $n \in \mathbb{N}$. Since the recurrence relation was constructed in such a way so as to include all the points in

Y that have original ancestors in Y, we conclude that $Y_Y = \{2^m - 1 | m \in \mathbb{N}\}$.
Next, we consider X_Y. Since all elements of Y have ultimate ancestor $1 \in Y$ (as we saw above), then $x \in X_Y$ has ultimate ancestor $1 \in Y$ as well. In light of this, all ancestors of $x \in X_Y$ must be of the form $2^m - 1$ for some $m \in \mathbb{N}$ and so, assuming $y \in Y$ is the parent of $x \in X_Y$, we see that $x = g(2^m - 1) = 2^{m+1} - 2$. Thus, $X_Y = \{2^{m+1} - 2 | m \in \mathbb{N}\}$. ■

- Use the fact that the sets partition X and Y to find X_X and Y_X.

  Proof. Since $X = X_X \cup X_Y \cup X_\infty = X_X \cup X_Y$, then

  $$X_X = \{x \in X | x \neq 2^{m+1} - 2 \text{ for } m \in \mathbb{N}\}.$$

  Similarly, since $Y = Y_Y \cup Y_X \cup Y_\infty = Y_Y \cup Y_X$, then

  $$Y_X = \{y \in Y | y \neq 2^m - 1 \text{ for } m \in \mathbb{N}\}.$$

  ■

- Following the recipe in the proof, write out the formula for $h : X \to Y$ that is 1-1 and onto.

  Proof. $f|_{X_X} \cup (g|_{Y_Y})^{-1} \cup f|_{X_\infty} = h(n) = \begin{cases} n+1 & \text{if } n \neq 2^{m+1} - 2 \\ n/2 & \text{if } n = 2^{m+1} - 2 \end{cases}$ ■

Friday Exercises

1. Prove that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.

   Proof. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by

   $$f(m, n) = 2^{m-1}(2n - 1)$$

   We claim the f is bijective and hence $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ have the same cardinality. To show that f is 1-1, assume $f(a, b) = f(c, d)$. Then,

   $$2^{a-1}(2b - 1) = 2^{c-1}(2d - 1) \implies (2b - 1) = 2^{c-a}(2d - 1) \qquad (*)$$

   If $a < c$, then $(*)$ would imply that $2b - 1$ is divisible by 2, which is nonsense. Likewise, if $a > c$, then $(*)$ would imply that $2d - 1$ is divisible by 2. Again: nonsense. Thus $a = c$ and so,

   $$(2b - 1) = 2^{c-a}(2d - 1) \implies b = d$$

   To show that f is onto, let $\ell \in \mathbb{N}$. If ℓ is odd, we can write $\ell = 2p - 1$, $p \in \mathbb{N}$ and so $f(1, p) = \ell$. If ℓ is even, we can write $\ell = 2^k(2p - 1)$ for $k, p \in \mathbb{N}$, and so $f(k, p) = \ell$. ■

2. Prove that if A and B have the same cardinality, then A is infinite if and only if B is infinite.

Proof. Let A be infinite and A and B have the same cardinality.
($\Rightarrow$)Since A is infinite, then there exists a function $f : A \to A$ that is 1-1 and not onto. Since A and B have the same cardinality, then there exists a function $g : A \to B$ that is 1-1 and onto. Consider $f \circ g^{-1}$. Since g^{-1} and f are 1-1, but f is not onto, then $f \circ g^{-1}$ is 1-1 but not onto. Then, since g and $f \circ g^{-1}$ are both 1-1, then $g \circ f \circ g^{-1}$ is 1-1. We claim that $g \circ f \circ g^{-1}$ is not onto. By way of contradiction, suppose $g \circ f \circ g^{-1}$ is onto and let $b \in B$. Since $g \circ f \circ g^{-1}$ is onto, there exists $a_1 \in A$ so that $g(a_1) = b$. Similarly, there exists $a_2 \in A$ so that $f(a_2) = a_1$. But, since f is not onto, we cannot ensure that such an element a_2 exists and so $g \circ f \circ g^{-1}$ is not onto. Therefore, $g \circ f \circ g^{-1} : B \to B$ is 1-1 but not onto and thus B is infinite.
($\Leftarrow$) Switch the roles of A and B. ■

Monday Exercises

1. Prove that $\mathbb{Z}$ is countable.

   Proof. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

   $$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2(-x)+1 & \text{if } x < 0 \\ 2x & \text{if } x > 0 \end{cases}$$

   Clearly, f is injective. Now, let $g : \mathbb{N} \to \mathbb{Z}$ be the injective function $g(x) = x$. Thus, by the Schroeder-Bernstien Theorem, there exists a bijection between $\mathbb{Z}$ and $\mathbb{N}$, and so $\mathbb{Z}$ is countable. ■

2. Prove by induction that the k-fold cartesian product of $\mathbb{Z}$ with itself is countable. You can use that $\mathbb{N} \times \mathbb{Z}$ is countable.

   Proof. Since we know $\mathbb{N} \times \mathbb{Z}$ is countable, there exists a bijection between $\mathbb{N} \times \mathbb{Z}$ and $\mathbb{N}$:

   $$f : \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$$

   We proceed by induction to show that $\mathbb{Z}^k$ is countable for all k. For the base case, we proved that $\mathbb{Z}$ is countable in problem 1. For the induction hypothesis, assume $\mathbb{Z}^k$ is countable. That is, there exists a function

   $$g : \mathbb{N} \to \mathbb{Z}^k$$

   which is a bijection. Now, consider the function

   $$h : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}^k \times \mathbb{Z}$$

   defined by $h(x, y) = (g(x), y)$. Notice that $(g(x_1), y_1) = (g(x_2), y_2)$ implies $y_1 = y_2$ and $g(x_1) = g(x_2)$, and since g is 1-1, $x_1 = x_2$, and thus h is 1-1. Also, given $(a, b) \in \mathbb{Z}^k \times \mathbb{Z}$, we can let $x = g^{-1}(a)$ and $y = b$ so that $h(x, y) = h(g(g^{-1}(a)), y) = (a, b)$ and thus h is surjective.

   Consider the function

   $$\phi = h \circ f : \mathbb{N} \to \mathbb{Z}^{k+1}$$

   since f and h are both bijective, then so is ϕ, and thus $\mathbb{Z}^{k+1}$ is countable. ■

3. Prove that the set of polynomials with integer coefficients is countable.

   Proof. For any polynomial of degree n with integer coefficients, we can define a bijective function
   $$\phi_{n+1} : \mathbb{Z}^n \to \mathbb{N}$$
   by the previous exercise. So, given a polynomial of degree k with integer coefficients,
   $$p(x) = z_k x^k + z_{k-1}x^{k-1} + \cdots + z_1 x + z_0$$
   we can define a function
   $$\Phi(p) := \phi_{k+1}(p)$$
   Clearly, Φ is bijective, and so the set of polynomials with integer coefficients is countable. ∎

4. *Recall:* Rolle's Theorem: If $f : [a, b] \to \mathbb{R}$ is pointwise continuous, and $f'(x)$ exists for all $x \in (a, b)$, then if $f(a) = f(b)$, there exists $c \in (a, b)$ so that $f'(c) = 0$. Recall if f is a polynomial of degree n then f' is a polynomial of degree $n - 1$. Prove that if a polynomial of degree n has more than n zeroes, then it is the 0 polynomial.

   Proof. Let f be a polynomial of degree n with more than n zeroes. By Rolle's Theorem, we know that f' is a polynomial with more than $n - 1$ zeroes, which means f'' have more than $n - 2$ zeroes. Then, the constant function $f^{(n-1)}$ has more than 1 zero, which means $f^{(n-1)}$ is the zero function. This means that $f^{(n-2)}$ is a constant function — since its derivative is 0 — that has more than 2 zeroes. Again, this means that $f^{(n-2)}$ is the zero function. Continuing this pattern, we see that f is a constant function — since its derivative is 0 — with more than n zeroes. So, f is the zero function. ∎

5. $x \in \mathbb{R}$ is *algebraic* if there exists a polynomial with integer coefficients so that $f(x) = 0$. Prove that the set of algebraic real numbers is countable.

   Proof. Let $\mathcal{P}$ be the set of all polynomials with integer coefficients. From a previous exercise, we know that $\mathcal{P}$ is countable. Let $\{x_p\}$ be the set of all algebraic numbers for a polynomial $p \in \mathcal{P}$. Since the degree of any polynomial is finite, the set of algebraic numbers, $\{x_p\}$, for a polynomial p is of finite order. Thus,
   $$\bigcup_{p \in \mathcal{P}} \{x_p\}$$
   is the countable union of countable sets, which is countable. Thus, the set of all algebraic numbers is countable. ∎

6. A subset $C \subset \mathbb{R}$ is *convex* if $a, b \in C$ and $a < c < b$, then $c \in C$. Convex subsets of $\mathbb{R}$ are points, intervals, rays and $\mathbb{R}$.
Find an efficient way of proving that if a convex subset of $\mathbb{R}$ has at least two points, then it has the same cardinality as $\mathbb{R}$.

Proof. Any convex set will contain an open interval. If Thus, given a convex set containing (a, b), we can define a 1-1 map $f : (a, b) \to \mathbb{R}$ by $f(x) = x$. Then, we can define a map
$$g : \mathbb{R} \to (0, 1) \quad \text{by} \quad g(x) = \frac{1}{1 + e^x}$$
which is bijective since it has an inverse. Then, define
$$h : (a, b) \to (0, 1) \quad \text{by} \quad h(x) = (x - a)/(b - a)$$
If $(x_1 - a)/(b - a) = (x_2 - a)/(b - a)$, then simple cancellation shows $x_1 = x_2$, and thus h is injective. For any $c \in (0, 1)$, let $x = c(b - a) + a$ and then $h(x) = c$, and thus h is surjective. So, the function
$$\hat{f} = h^{-1} \circ g : \mathbb{R} \to (a, b)$$
is bijective since it is the composition of bijective functions. Thus, $\mathbb{R}$ has the same cardinality as any open interval (a, b), which is contained in any convex set. ■

7. Prove that the cardinality of $\mathbb{R}^2$ is the same as the cardinality of $\mathbb{R}$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(r) = (r, r)$. Clearly, f is injective. Now, let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$g(\ldots a_3 a_2 a_1 . b_1 b_2 b_3 \ldots, \ldots c_3 c_2 c_1 . d_1 d_2 d_3 \ldots) = \ldots c_2 a_2 c_1 a_1 . b_1 d_1 b_2 d_2 \ldots$$
(the elements are given by their decimal expansion) which is injective. Thus, by the Schroeder-Bernstien Theorem, there exists a bijection between $\mathbb{R}^2$ and $\mathbb{R}$ and thus $|\mathbb{R}^2| = |\mathbb{R}|$ ■

Wednesday Exercises

8. Prove that every proper ideal in a commutative ring is contained in a maximal ideal.

Proof. We first prove that the arbitrary union of proper ideals is also an ideal. Let $\{I_\alpha\}$ be an indexed family of ideals. Then, given $x \in \bigcup I_\alpha$, then $x \in I_x$ for one of the ideals in $\bigcup I_\alpha$. Let $r \in R$. Then, $xr \in I_x$ since I_x is an ideal, which means $xr \in \bigcup I_\alpha$ and so $\bigcup I_\alpha$ is an ideal.

Let $\mathcal{I}$ be the set of proper ideals of the commutative ring R. The set $\mathcal{I}$ is partially ordered by inclusion. Given a chain of proper ideals, the union of these ideals, say I, is also an ideal, and is an upper bound for the chain. So, since every totally ordered chain of proper ideals has an upper bound, then $\mathcal{I}$ has a maximal ideal by Zorn's Lemma. So, every proper ideal in $\mathcal{I}$ is contained in the maximal ideal. ■

Friday Exercises

Pg83-3 Show that the collection $\mathcal{T}_c$ given in Example 4 of section 12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U | X - U \text{ is finite or empty or all of } X\}$$

a topology on X?

Proof. The set $\mathcal{T}_c$ is the collection of all subsets U of X such that $X - U$ either is countable or is all of X. The emptyset is in $\mathcal{T}_c$ since $X - \emptyset = X$. The set X is in $\mathcal{T}_c$ since $X - X = \emptyset$, which is countable. If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_c$, to show that $\bigcup U_\alpha$ is in $\mathcal{T}_c$, we compute

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

The latter set is countable because each $X - U_\alpha$ is countable. If $U_1 \dots U_n$ are nonempty elements of $\mathcal{T}_c$, to show that $\bigcap U_i$ is in $\mathcal{T}_c$, we compute

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

The latter set is countable since it is the union of a countable number of countable sets.

For the collection $\mathcal{T}_\infty$, clearly $\emptyset$ and X are both in $\mathcal{T}_\infty$, but, an arbitrary union of sets in $\mathcal{T}_\infty$ need not be infinite, empty, nor all of X. For example, let $X = \mathbb{R}$ and let $U_1 = \{x \in \mathbb{R} | x \neq -n \ \forall n \in \mathbb{N}, x \neq 0\}$ and $U_2 = \{x \in \mathbb{R} | x \neq n \ \forall n \in \mathbb{N}, x \neq 0\}$. Then, $U_1, U_2 \in \mathcal{T}_\infty$, since

$$\mathbb{R} - U_1 = \{-n | n \in \mathbb{N}\} \cup \{0\} \quad \text{and} \quad \mathbb{R} - U_2 = \{n | n \in \mathbb{N}\} \cup \{0\}$$

are infinite. However,

$$\mathbb{R} - (U_1 \cup U_2) = (\mathbb{R} - U_1) \cap (\mathbb{R} - U_2) = \{0\}$$

is neither infinite, empty, nor all of X. ■

Pg83-4 (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

Proof. Since each $\mathcal{T}_\alpha$ is a topology on X, each contains X and $\emptyset$, and thus $X, \emptyset \in \bigcap \mathcal{T}_\alpha$. Unions of sets from $\bigcap \mathcal{T}_\alpha$ all lie in each $\mathcal{T}_\alpha$, since each $\mathcal{T}_\alpha$ is a topology, and so, the unions lie in $\bigcap \mathcal{T}_\alpha$. Intersections of finite sets in $\bigcap \mathcal{T}_\alpha$ all lie in each $\mathcal{T}_\alpha$ since each $\mathcal{T}_\alpha$ is a topology, and so, the intersections lie in $\bigcap \mathcal{T}_\alpha$.

The collection of sets $\bigcup \mathcal{T}_\alpha$ is not necessarily a topology because there could be sets in $\bigcup \mathcal{T}_\alpha$ whose union is not in $\bigcup \mathcal{T}_\alpha$. For example,

$$\{\emptyset, \{1,2,3\}, \{3\}\} \cup \{\emptyset, \{1,2,3\}, \{1\}\} = \{\emptyset, \{1,2,3\}, \{1\}, \{3\}\}$$

but the latter set is not a topology since it does not contain $\{1,3\}$. ∎

(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Proof. Consider $\mathcal{B} = \bigcup \mathcal{T}_\alpha$. We claim that the topology generated by $\mathcal{B}$ is the unique smallest topology on X that contains all of the $\mathcal{T}_\alpha$'s. To see this, consider that the topology generated by $\mathcal{B}$ clearly contains all of the $\mathcal{T}_\alpha$'s, and it is the smallest since any other topology containing all the $\mathcal{T}_\alpha$'s must contain unions of elements of $\mathcal{B}$, i.e., must contain the topology generated by $\mathcal{B}$.

The unique largest topology contained in all $\mathcal{T}_\alpha$ will be $\bigcap \mathcal{T}_\alpha$, (which we proved is a topology in the previous exercise). This is because $\bigcap \mathcal{T}_\alpha$ is certainly contained in all the $\mathcal{T}_\alpha$'s and it is the largest since any topology contained in each $\mathcal{T}_\alpha$ would necessarily be contained in $\bigcap \mathcal{T}_\alpha$. ∎

(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a,b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b,c\}\}$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.
Solution: The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_3 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}.$$

The largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_4 = \{\emptyset, X, \{a\}\}$$

Pg83-5 Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof. Each topology on X that contains $\mathcal{A}$ also contains all unions of elements in $\mathcal{A}$, by definition of being topologies, and so the intersection of these topologies is all unions of elements of $\mathcal{A}$ — and all unions of all elements of $\mathcal{A}$ is precisely the definition of the topology generated by $\mathcal{A}$.

If $\mathcal{A}$ is a subbasis, then each topology on X that contains $\mathcal{A}$ also contains all finite intersections of elements in $\mathcal{A}$, and also contains all unions of these intersections by definition of being a topology on X, and this is precisely the definition of the topology generated by a subbasis. ■

13-7 Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{ the standard topology on } \mathbb{R}, \\
\mathcal{T}_2 &= \text{ the topology of } \mathbb{R}_K, \\
\mathcal{T}_3 &= \text{ the finite complement topology,} \\
\mathcal{T}_4 &= \text{ the upper limit topology, having all sets } (a,b] \text{ as basis,} \\
\mathcal{T}_5 &= \text{ the topology having all sets } (-\infty, a) = \{x \mid x < a\} \text{ as basis}
\end{aligned}$$

Determine, for each of these topologies, which of the others it contains.
Solution:

(1) — $\mathcal{T}_1 \subset \mathcal{T}_4$ — Let (a,b) be a basis element of $\mathcal{T}_1$, and let $x \in (a,b)$. Then, the basis element $(a,x]$ of $\mathcal{T}_4$ contains x and is contained in (a,b). Thus, $\mathcal{T}_4$ is finer than $\mathcal{T}_1$.
Conversely, let $(a,b]$ be a basis element of $\mathcal{T}_4$ and let $x = b$. Then, there does not exist a basis element of $\mathcal{T}_1$ that contains b and is a subset of $(a,b]$. To see this, we assume by contradiction that such a basis element of $\mathcal{T}_1$ exists, say (g,h). Then, as $b \in (g,h)$ the point $(h-b)/2$ is in (g,h), but is not in $(a,b]$, since $(h-b)/2 > b$. Thus, $(g,h) \not\subset (a,b]$. Therefore, $\mathcal{T}_4$ is strictly finer that $\mathcal{T}_1$.

(2) — $\mathcal{T}_1 \subset \mathcal{T}_2$ — Let (a,b) be a basis element of $\mathcal{T}_1$, and let $x \in (a,b)$. Then, the basis element (a,b) in $\mathcal{T}_2$ contains x and is contains in (a,b). Thus, $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.
Conversely, given the basis element $B = (-1,1) - K$ in the K-topology, and the point $0 \in B$, $\mathcal{T}_1$ has no basis element which contains 0 and lies in B. By way of contradiction, suppose such a basis elements exists, say (a,b). Since $0 \in (a,b)$, then $0 < b$ and by the Archimedean Property, there exists $n \in \mathbb{Z}^+$ so that $0 < 1/n < b$. So, $1/n \in (a,b)$ which means $1/n \in (-1,1) - K$, a contradiction. Therefore, $\mathcal{T}_2$ is strictly finer than $\mathcal{T}_1$.

(3) — $\mathcal{T}_2 \not\subset \mathcal{T}_4$ — Given the basis element $B = (-1,1) - K$ in the K-topology, and the point $0 \in B$, the $\mathcal{T}_4$ has no basis element which contains 0 and lies in B, using the same line of argumentation as in (3). Thus, $\mathcal{T}_2 \not\subset \mathcal{T}_4$.
— $\mathcal{T}_4 \not\subset \mathcal{T}_2$ — Let $(a,b]$ be a basis element of $\mathcal{T}_4$ and let $x = b$. Then, There does not exists a basis element of $\mathcal{T}_2$ that contains x and lies in $(a,b]$ using the same line of argumentation as in (1).
Thus, $\mathcal{T}_2$ and $\mathcal{T}_4$ are not comparable.

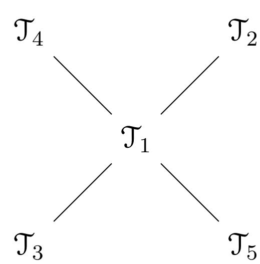
(4) — $\mathcal{T}_5 \subset \mathcal{T}_1$ — Let $(-\infty, a)$ be a basis element in $\mathcal{T}_5$ and let $x \in (-\infty, a)$. Then, let $\epsilon = |x - a|$ and notice that $(x - \epsilon, x + \epsilon)$ is an element of $\mathcal{T}_1$ which contains x and lies in $(-\infty, a)$. Thus, $\mathcal{T}_1$ is finer than $\mathcal{T}_5$.
Conversely, given a basis element (a,b) in $\mathcal{T}_1$, and a point $x \in (a,b)$, we can find a basis element of $\mathcal{T}_5$ which contains x, but not one which also is contained in (a,b). Thus, $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_5$.

(5) — $\mathcal{T}_3 \not\subset \mathcal{T}_5$ — Let U be an element of $\mathcal{T}_3$ so that $\mathbb{R} - U = \{-1\}$. Pick $x = 0 \in U$. Then, no element of $\mathcal{T}_5$ that contains 0 is also contained in U. To see this, assume

by contradiction that such a basis element, $(-\infty, a)$ exists. Since $0 \in (-\infty, a)$, then $0 < a$ which means $-1 \in (\infty, a)$ and so $-1 \in U$, a contradiction. Thus, $\mathcal{T}_3 \not\subset \mathcal{T}_5$.
— $\mathcal{T}_5 \not\subset \mathcal{T}_3$ — Since the complements of the basis elements of $\mathcal{T}_5$ are uncountable, none of them can be in $\mathcal{T}_3$. Therefore, $\mathcal{T}_5 \not\subset \mathcal{T}_3$, and so $\mathcal{T}_3$ and $\mathcal{T}_5$ are not comparable.

(6) — $\mathcal{T}_3 \subset \mathcal{T}_1$ — Let $U \in \mathcal{T}_3$ and $\mathbb{R} - U = \{x_1, x_2, \ldots, x_n\}$. Let $a \in U$ and let $\epsilon = |a - x_k|$ where x_k is the closest point of $\mathbb{R} - U$ to a. Then, $(a - \epsilon, a + \epsilon)$ is a basis element of $\mathcal{T}_1$ that contains a and is contained in U. Thus, $\mathcal{T}_1$ is finer than $\mathcal{T}_3$.
—$\mathcal{T}_1 \not\subset \mathcal{T}_3$ — Conversely, the complements of basis elements of $\mathcal{T}_1$ are uncountable and so none of them are in $\mathcal{T}_3$. Therefore, $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_3$.

In the graph below, an edge connecting two vertices means the topologies are comparable, with higher vertex being finer. By transitivity, $\mathcal{T}_4$ and $\mathcal{T}_2$ are both finer than $\mathcal{T}_3$ and $\mathcal{T}_5$.



16-1 Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. The set $A \cap B$ is the general basis element for the topology A inherits as a subspace of Y, where B is a basis element of Y. Since B is a basis element in Y, then $B = Y \cap B'$ for some basis element B' of X. So, $A \cap (Y \cap B') = A \cap B'$ is the general basis element for the subspace A of Y. This is precisely the general basis element for the topology A inherits as a subspace X. ■

16-4 A map $f : X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof. Let W be open in $X \times Y$. Then, let $x \in \pi_1(W)$. So, $x \times y \in W$ for some $y \in Y$. Since W is open in $X \times Y$, there is a basis element $U \times V$ containing $x \times y$ so that $U \times V \subset W$. Since U is open in X, then, $x \in U \subset \pi_1(W)$ and thus, $\pi_1(W)$ is open in Y. Similarly for π_2. ■

16-6 Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

Proof. Given any point $x \times y \in \mathbb{R}^2$, we can find rationals a, b, c, and d so that

$$a < x < b \quad \text{and} \quad c < y < d$$

So, $x \times y \in (a, b) \times (c, d)$. This is the first condition necessary to be a basis. Now, suppose

$$x \times y \in (a, b) \times (c, d) \quad \text{and} \quad x \times y \in (a', b') \times (c', d')$$

Then, pick rationals a'', b'', c'', d'' so that

$$a, a' < a'' < x < b'' < b, b' \quad \text{and} \quad c, c' < c'' < y < d'' < d, d'$$

So, we have

$$x \times y \in (a'', b'') \times (c'', d'') \subset (a, b) \times (c, d) \cap (a', b') \times (c', d')$$

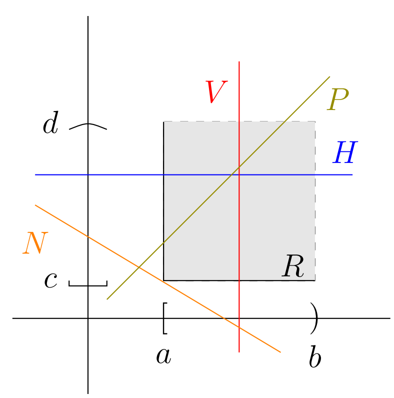
which is the second condition necessary to be a basis. So, the set is a basis. ■

16-7 Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or ray in X?
Solution:
If $X = \mathbb{R}$, and $Y = \{y\}$ for some $y \in \mathbb{R}$. Then Y is convex, but is neither an interval nor a ray in X.

16-8 If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.
Solution:
Let H, V, P, and N denote the horizontal lone, vertical line, positive-sloped line, and negative-sloped line subsets of the plane, respectively. We consider the subspace each line inherits from $\mathbb{R}_\ell \times \mathbb{R}_\ell$ and from $\mathbb{R}_\ell \times \mathbb{R}$.

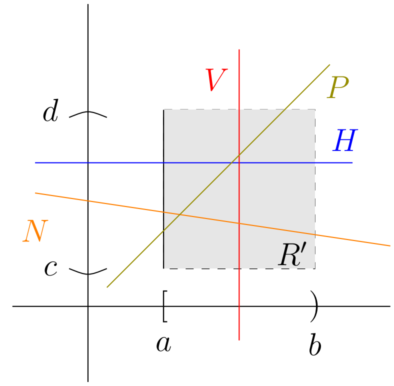
$\underline{\mathbb{R}_\ell \times \mathbb{R}_\ell}$:

Let $R = [a, b) \times [c, d)$ be a basis element for $\mathbb{R}_\ell \times \mathbb{R}_\ell$ (a rectangle in the plane). The line H always intersects R in intervals — that is, intervals which are *on the line* — of the form $[x, y)$. Thus, H inherits the lower limit topology as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Similarly, the lines V and P also inherit the lower limit topology as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Now, N intersects R in intervals of the form $(x, y), [x, y), (x, y]$ and $[x, y]$. However, N also intersects R at the point $a \times c$. Thus, the subspace topology of N is in fact the discrete topology.



$\mathbb{R}_\ell \times \mathbb{R}$:

Let $R' = [a, b) \times (c, d)$ be a basis element for $\mathbb{R}_\ell \times \mathbb{R}$ (a rectangle in the plane). The lines H, P, and N intersect R' in intervals of the form $[x, y)$ and thus each inherit the lower limit topology as a subspaces of $\mathbb{R}_\ell \times \mathbb{R}$. Now, the line V intersects R' in intervals of the form (x, y) and thus V inherits the standard topology as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$.



16-9 Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Proof. The general basis element of the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ are intervals of the form $(a \times b, a \times d)$. The general basis element for $\mathbb{R}_d \times \mathbb{R}$ are products of the form $\{x\} \times (y, z)$. Since this product represents the interval $(x \times y, x \times z)$, then clearly the basis elements for the order topology on $\mathbb{R} \times \mathbb{R}$ and the product topology on $\mathbb{R}_d \times \mathbb{R}$ are the same. ■

17-2 Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. A is closed in Y if and only if $A = U \cap Y$, where U is closed in X. So, A is closed in X since it is the intersection of two closed sets of X. ■

17-3 Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Since A is closed in X and B is closed in Y, then $X - A$ and $Y - B$ are open in X and Y, respectively. So, $(X - A) \times (Y - B)$ is open in $X \times Y$, which means

$$X \times Y - ((X - A) \times (Y - B)) = A \times B$$

is closed in $X \times Y$. ■

17-4 Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Proof. Since U is open in X, then $X - U$ is closed in X, which means $(X - U) \cap A$ is closed in X. So,

$$\begin{aligned} X - ((X - U) \cap A) &= (X - (X - U)) \cap (X - A) \\ &= U \cap (X - A) \\ &= U - A \end{aligned}$$

is open in X. Since A is closed in X, then $X - A$ is open in X, which means $(X-A) \cap U$ is open in X. So,

$$\begin{aligned} X - ((X - A) \cap U) &= (X - (X - A)) \cap (X - U) \\ &= A \cap (X - U) \\ &= A - U \end{aligned}$$

is closed in X. ■

17-5 Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does the equality hold?

Proof. Notice that $[a,b]$ contains (a,b), and also that $[a,b]$ is a closed set since $X \setminus [a,b] = (\infty, a) \cup (b, \infty)$ is the union of open sets and so open. Now, Since the set $\overline{(a,b)}$ is defined as the intersection of all closed sets containing (a,b), then $[a,b]$ participates in this intersection and so $\overline{(a,b)} \subset [a,b]$.
We know that $(a,b) \subsetneq [a,b]$. Thus, since $\overline{(a,b)} = (a,b) \cup (a,b)'$, then $[a,b] \subset \overline{(a,b)}$ if and only if $[a,b] \subset (a,b)'$, i.e., if a and b are limit points of (a,b). ■

17-6 Let A, B and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proof. If $x \in \overline{A}$, then every neighborhood U of x intersects A and since $A \subset B$, then every neighborhood U of x intersects B, which means $x \in \overline{B}$. ■

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. $x \in \overline{A \cup B} \iff$ every neighborhood U of x intersects $A \cup B \iff$ every neighborhood U of x intersects A or intersects $B \iff x \in \overline{A} \cup \overline{B}$. ■

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$; give an example where equality fails.

Proof. If $x \in \bigcup \overline{A_\alpha}$, then there exits α such that $x \in \overline{A_\alpha}$, i.e., every neighborhood of x intersects A_α. So, every neighborhood of x intersects $\bigcup A_\alpha$, i.e., $x \in \overline{\bigcup A_\alpha}$. ■

17-7 Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A_\alpha}$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus, U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \overline{A_\alpha}$.

Solution:

The proof says that "U must intersect some A_α", which is true. However, this does not guarantee that x is in the closure of that particular A_α. One would need to show that every neighborhood of x intersects that same A_α. We could have neighborhoods of x which all intersect different A_α's in the union, and therefore x may not be in the closure of any particular A_α.

17-9 Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

Proof. $a \times b \in \overline{A \times B} \iff$ every basis element $U \times V$ of $X \times Y$ containing $a \times b$ intersects $A \times B$. $\iff \emptyset \neq (U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B) \iff U$ contains a and intersects A, and likewise, V contains b and intersects $B \iff a \times b \in \overline{A} \times \overline{B}$. ■

17.8 Let A, B and A_α denote subsets of a space X. Determine whether the following equations hold; if equality fails, determine whether one of the inclusions $\subset$ or $\supset$ holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Proof.

$$\begin{aligned} x \in \overline{A \cap B} &\iff U \cap (A \cap B) \neq \emptyset, \text{for any } U \ni x \\ &\iff (U \cap A) \cap (U \cap B) \neq \emptyset \\ &\implies (U \cap A) \neq \emptyset \text{ and } (U \cap B) \neq \emptyset \\ &\iff x \in \overline{A} \text{ and } x \in \overline{B} \\ &\iff x \in \overline{A} \cap \overline{B} \end{aligned}$$

Thus, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ ☕

(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$.

Proof.

$$\begin{aligned} x \in \overline{\bigcap} A_\alpha &\iff U \cap \left(\bigcap A_\alpha \right) \neq \emptyset, \text{for any } U \ni x \\ &\iff \bigcap (U \cap A_\alpha) \neq \emptyset \\ &\implies (U \cap A_\alpha) \neq \emptyset \text{ for all } A_\alpha \\ &\iff x \in \overline{A_\alpha} \text{ for all } A_\alpha \\ &\iff x \in \bigcap \overline{A_\alpha} \end{aligned}$$

Thus, $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}$ ☕

(c) $\overline{A - B} = \overline{A} - \overline{B}$.

Proof.

$$\begin{aligned} x \in \overline{A} - \overline{B} &\iff x \in \overline{A}, x \notin \overline{B} \\ &\iff U \cap A \neq \emptyset \text{ for any } U \ni x, \text{ but } \exists V \ni x, V \cap B = \emptyset \text{ for any } U \ni x \\ &\implies U \cap (A - B) \neq \emptyset \\ &\iff x \in \overline{A - B} \end{aligned}$$

Thus, $\overline{A - B} \supset \overline{A} - \overline{B}$. ☕

17.13 Show that X is Hausdorff if and only if the *diagonal* $\triangle = \{x \times x | x \in X\}$ is closed in $X \times X$.

Proof. ($\Rightarrow$) Suppose X is Hausdorff and let $x \times y \in X \times X - \triangle$, i.e., let $x \times y \in \triangle^c$. Then, $x \neq y$ since $x \times y \notin \triangle$. There exists open, disjoint neighborhoods U and V of x and y, respectively. So, $x \times y \in U \times V$ and $U \times V \subset \triangle^c$, i.e., $\triangle^c$ is open in $X \times X$, which means $\triangle$ is closed in $X \times X$.
($\Leftarrow$) Suppose $\triangle^c$ is open in $X \times X$. Let $x, y \in X$ be distinct. So, $x \times y \in \triangle^c$. Since $\triangle^c$ is open, there is a basic element $U \times V$, a neighborhood of $x \times y$ so that $x \in U, y \in V$ and $x \times y \in U \times V \subset \triangle^c$. Since $U \times V \subset \triangle^c$, then $U \times V \not\subset \triangle$, which means $U \cap V = \emptyset$. Thus, X is Hausdorff.

☕

18.1 Prove that for functions $\mathbb{R} \to \mathbb{R}$, the $\epsilon - \delta$ definition of continuity implies the open set definition.

Proof. Given $\epsilon > 0$, we have a $\delta > 0$ so that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. So, for any $\epsilon > 0$ and $x_0 \in \mathbb{R}$, we have that for each open set $V_\epsilon = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ in our range space $\mathbb{R}$, there exists an open set $U = (x_0 - \delta, x_0 + \delta)$ in our domain space $\mathbb{R}$ so that $f^{-1}(V_\epsilon) = U$ is open. ☕

18.4 Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Proof. In order to show that f and g are imbeddings, we need to first show that f, g are injective and continuous. Then, we need to show that the maps $f_1 : X \to f(X)$ and $g_1 : Y \to g(Y)$ are homeomorphisms.

f, g are injective: This is clear:

$$x_1 \times y_0 = x_2 \times y_0 \implies x_1 = x_2 \quad \text{and} \quad x_0 \times y_1 = x_0 \times y_2 \implies y_1 = y_2$$

f, g are continuous: Let U be open in X. Then as $y_0 \in Y$, there exists a basic element B of Y containing y_0 so that $U \times y_0 \subset U \times B$, and thus $U \times y_0$ is open in $X \times Y$. Then,

$$\begin{aligned} f^{-1}(U \times y_0) &= \{x \in X \mid f(x) \in U \times y_0\} \\ &= \{x \in X \mid x \times y_0 \in U \times y_0\} \\ &= \{x \in X \mid x \in U\} \\ &= U \end{aligned}$$

which is open in X. Thus, f is continuous. Similarly, let V be open in Y. Then, as $x_0 \in X$, there is a basic element B' of X containing x_0. so that $x_0 \times V \subset V \times B'$, and thus $x_0 \times V$ is open in $X \times Y$. Then,

$$\begin{aligned} f^{-1}(x_0 \times V) &= \{y \in Y \mid g(y) \in \times y_0\} \\ &= \{y \in Y \mid x_0 \times y \in x_0 \times V\} \\ &= \{y \in Y \mid y \in V\} \\ &= V \end{aligned}$$

which is open in Y. Thus, g is continuous.

f_1, g_1 are homeomorphisms: Notice that $f(X)$ is a subset of $X \times Y$. So, $f(X) \cap X \times Y = X \times y_0$ is a subspace of $X \times Y$. Consider the map $f_1 : X \to X \times y_0$. By construction, f_1 is bijective.
Now, we need to show that $f_1(U)$ is open if and only if U is open. Suppose $U \times y_0$ is open in $X \times y_0$ and let $u \times y_0 \in U \times y_0$. Then, there exists a basic element $B \times y_0$ of $X \times y_0$ so that $u \times y_0 \subset B \times y_0 \subset U \times y_0$. So, $u \in B \subset U$ and thus, U is open in X.
Conversely, suppose U be open in X and let $u \in U$. Then, $f(u) = u \times y_0$. Since U is open in X, there exists a basic element B of X containing u so that $u \in B \subset U$. Thus, $u \times y_0 \subset B \times y_0 \subset U \times y_0$, which means $u \times y_0$ is open in $X \times y_0$.

Very similar argument for g_1.

☕

18.6 Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.
Solution:
The function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is only continuous at the point $x = 0$. Since we showed in Exercise 18.1 that the $\epsilon - \delta$ definition of continuous is equivalent to the open set definition, we use it here. Let $\epsilon > 0$ and $x = 0$. Then let $\delta = \epsilon$. If $y \in \mathbb{Q}$,

$$|y - 0| < \delta \implies |f(x) - f(y)| = |f(y)| = 0 < \delta = \epsilon$$

If $y \in \mathbb{R} \setminus \mathbb{Q}$, then

$$|y - 0| < \delta \implies |f(x) - f(y)| = |f(y)| = |y| < \delta = \epsilon$$

and so f is continuous at zero. Now suppose $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, for any $\delta > 0$, let $\epsilon = |x|/2$. For any $y \in \mathbb{Q} \cap (x - \delta, x + \delta)$, we have

$$|f(x) - f(y)| = |x - 0| = |x| > |x|/2 = \epsilon$$

If $x \in \mathbb{Q}, x \neq 0$, then for any $\delta > 0$, let $\epsilon = |x|/2$. Then, for $y \in \mathbb{R} \setminus \mathbb{Q} \cap (x - \delta, x + \delta)$

$$|f(x) - f(y)| = |0 - y| = |y| > |x|/2 = \epsilon$$

18.9 Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. let $f : X \to Y$; suppose that $f|_{A_\alpha}$ is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

Proof. Suppose $X = \bigcup_{i=1}^n A_i$ and let F be closed in Y. Then,

$$f^{-1}(F) = \bigcup_{i=1}^n (f|_{A_i})^{-1}(A_i)$$

Since each $f|_{A_i}$ is continuous, then each $(f|_{A_i})^{-1}(A_i)$ is closed. Since a finite union of closed sets is closed, then $f^{-1}(F)$ is closed, which means f is continuous. ☕

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

Solution: (Thanks Jesse for this solution!)

Define

$$A_n = \left\{ \left(r\cos\left(\frac{(n-1)\pi}{2n}\right), r\sin\left(\frac{(n-1)\pi}{2n}\right)\right) \middle| r \in [-1,1] \right\}$$

Then, $X = \bigcup_n A_n \subseteq \mathbb{R}^2$ with the subspace topology. Define, $f : X \to [-1,1]$ by

$$f\left(r\cos\left(\frac{(n-1)\pi}{2n}\right), r\sin\left(\frac{(n-1)\pi}{2n}\right)\right) = nr.$$

If $(a,b) \in [-1,1]$ and $x \in (a,b)$, then

$$f|_{A_n}^{-1}(x) = \left(\frac{x}{n} \cdot \cos\left(\frac{(n-1)\pi}{2n}\right), \frac{x}{n} \cdot \sin\left(\frac{(n-1)\pi}{2n}\right)\right)$$

and so each $f|_{A_n}$ is continuous. Notice that $(0,0) = \bigcap_n A_n$. Given and open interval $V \in [-1,1]$, then $f^{-1}(V) \ni (0,0)$, but there does not exist a basis element $B((0,0),r) \cap X$ in the subspace topology so that $B((0,0),r) \cap X \subset f^{-1}(V)$. Thus, f is not continuous.

18.10 Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

Proof. Let U, V be open sets in B, D, respectively. Since f and g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open in A and C, respectively. So, $f^{-1}(U) \times g^{-1}(V)$ is open in $A \times C$. Since $U \times V$ is open in $B \times D$ and

$$(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V),$$

is open in $A \times C$, then $f \times g$ is continuous. ☕

18.13 Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Proof. Suppose $h : \overline{A} \to Y$ is another extension of f. Since h and g are maps from $\overline{A}$ to Y, if we show that h and g agree for all points of $\overline{A}$, then, $h = g$ and so g is uniquely determined by f. Define the set E to be all the points of $\overline{A}$ where h and g agree, i.e.,

$$E = \{x \in \overline{A} \mid g(x) = h(x)\}.$$

We claim $E = \overline{A}$. First, notice that since g and h are extensions of f, then $g(x) = h(x)$ for all $x \in A$, and so $A \subset E$. Also, by definition of E, we have $E \subset \overline{A}$. Now, define the map

$$\varphi : A \to Y \times Y \quad \text{by} \quad \varphi(a) = (g(a), h(a))$$

Since both h and g are continuous, then φ is continuous by Theorem 18.4. Remember that by problem 17.13, since Y is Hausdorff, then the diagonal $\triangle$ of $Y \times Y$ is closed. Then,

$$\begin{aligned}
\varphi^{-1}(\triangle) &= \{a \in A \mid \varphi(a) \in \triangle\} \\
&= \{a \in A \mid (g(a), h(a)) \in \triangle\} \\
&= \{a \in A \mid g(a) = h(a)\} \\
&= E
\end{aligned}$$

Since $\triangle$ is closed and φ is continuous, then $\varphi^{-1}(\triangle) = E$ is closed. Thus, E is a closed set containing A. Since $\overline{A}$ is defined to be the intersection of all closed sets which contain A, then $\overline{A} \subset E$. Thus, $E = \overline{A}$. ☕

19.1 Prove Theorem 19.2.

Theorem (Theorem 19.2). *Suppose the topology on each space X_α is given by a basis $\mathfrak{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathfrak{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

Proof. Let J be an indexing set. First note that we will write (x_α) to denote a general J-tuple in $\prod_{\alpha \in J} X_\alpha$— and that we will write $\prod X_\alpha$ in place of $\prod_{\alpha \in J} X_\alpha$
Let $(x_\alpha) \in \prod X_\alpha$. Since each space X_α is given by a basis $\mathfrak{B}_\alpha$, then for each $x \in X_\alpha$, there is a basis element $B'_\alpha \in \mathfrak{B}_\alpha$ containing x. So,

$$(x_\alpha) \in \prod B'_\alpha$$

Now, suppose

$$(x_\alpha) \in \prod B'_\alpha \quad \text{and} \quad (x_\alpha) \in \prod B''_\alpha$$

Notice that this means for each coordinate x_ϵ in (x_α) we have $x_\epsilon \in B'_\epsilon$ and $x_\epsilon \in B''_\epsilon$. Since $B'_\epsilon, B''_\epsilon \in \mathfrak{B}_\epsilon$, there must be a basis element $B'''_\epsilon \in \mathfrak{B}_\epsilon$ so that $x_\epsilon \in B'''_\epsilon \subset B'_\epsilon \cap B''_\epsilon$. Then,

$$\left(\prod B'_\alpha\right) \bigcap \left(\prod B''_\alpha\right) = \prod (B'_\alpha \cap B''_\alpha) \supset \prod B'''_\alpha \ni (x_\alpha)$$

Thus, $\prod_{\alpha\in J} B_\alpha$ will serve as a basis for the box topology on $\prod X_\alpha$. ☕

19.2 Prove Theorem 19.3

Theorem (Theorem 19.3)**.** *Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Proof. First suppose both products are given the box topology. Then, $\prod U_\alpha$ is the general basis element for $\prod X_\alpha$, where U_α is open in X_α for each $\alpha \in J$. So,

$$\left(\prod U_\alpha\right) \cap \left(\prod A_\alpha\right)$$

is the general basis element for the subspace $\prod A_\alpha$. Now,

$$\left(\prod U_\alpha\right) \cap \left(\prod A_\alpha\right) = \prod (U_\alpha \cap A_\alpha)$$

Since $U_\alpha \cap A_\alpha$ is the general open set for the subspace A_α, then

$$\prod (U_\alpha \cap A_\alpha)$$

is the general basis element for the box topology on $\prod A_\alpha$. Thus, the bases for the box topology and subspace topology on $\prod A_\alpha$ are the same. So, $\prod A_\alpha$ is a subspace of $\prod X_\alpha$.

Now suppose both products are given the product topology. We use almost an identical argument as above, (except we now remind the reader at each step that $U_\alpha = X_\alpha$ except for finitely many values of α). The general basis element for the subspace $\prod A_\alpha$ is

$$\left(\prod U_\alpha\right) \cap \left(\prod A_\alpha\right)$$

where U_α is equal to X_α for all but finitely many values of α. We now note that $U_\alpha \cap A_\alpha$ is the general open set for the subspace A_α, except we now keep in mind that $U_\alpha = X_\alpha$ for all but finitely many values of α. (This means $U_\alpha \cap A_\alpha = A_\alpha$ is a the general open set for the subspace A_α for all but finitely many values of α). So,

$$\prod (U_\alpha \cap A_\alpha)$$

is the general basis element for the product topology on $\prod A_\alpha$, where we remember that all but finitely many U_α equal X_α. Then, we make the same observation as before that

$$\left(\prod U_\alpha\right) \cap \left(\prod A_\alpha\right) = \prod (U_\alpha \cap A_\alpha).$$

So, the bases for the product topology and subspace topology on $\prod A_\alpha$ are the same. ☕

19.3 Prove Theorem 19.4

Theorem. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Proof. Let $(x_\alpha), (y_\alpha) \in \prod X_\alpha$ be distinct. Since each space X_α is Hausdorff, there exists disjoint neighborhoods U_α, V_α in X_α of each pair of coordinates x_α, y_α, respectively. So,

$$\prod U_\alpha \quad \text{and} \quad \prod V_\alpha.$$

are neighborhoods of (x_α) and (y_α), respectively. Then,

$$\left(\prod U_\alpha\right) \cap \left(\prod V_\alpha\right) = \prod (U_\alpha \cap V_\alpha) = \emptyset$$

because $U_\alpha \cap V_\alpha = \emptyset$ for each α in the box topology. Since (x_α) and (y_α) are distinct, then $U_\alpha \cap V_\alpha = \emptyset$ for at least one value of α. In either case, we get that $\prod U_\alpha$ and $\prod V_\alpha$ are disjoint neighborhoods of (x_α) and (y_α), respectively, and so $\prod X_a$ is Hausdorff. ☕

19.7 Let R^∞ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \dots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Proof. Since we know $\mathbb{R}^\infty \subset \overline{\mathbb{R}^\infty}$, our interest is in points that are in $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Let

$$\boldsymbol{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty.$$

Notice that since $\boldsymbol{x} \notin \mathbb{R}^\infty$, it must have infinitely many nonzero coordinates. First, suppose $\mathbb{R}^\infty$ and $\mathbb{R}^\omega$ are given the box topology. Let $\Lambda \subset \mathbb{Z}^+$ so that if $\lambda \in \Lambda$ then $x_\lambda \neq 0$. Consider a basic neighborhood of $\boldsymbol{x}$:

$$U = \prod_{i \in Z^+} U_i$$

where $U_k = (-1, 1)$ for all $k \notin \Lambda$ and $U_\lambda = \mathbb{R} \setminus \{0\}$ if $\lambda \in \Lambda$. Then, since any point in $\mathbb{R}^\infty$ has only finitely many nonzero coordinates, we will eventually have $(\mathbb{R} \setminus \{0\}) \cap \{0\} = \emptyset$ for infinitely many parts of $U \cap \mathbb{R}^\infty$ and so this intersection is empty. Hence, no point in $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$ is a point of closure for $\mathbb{R}^\infty$. Thus, $\overline{\mathbb{R}^\infty}$.
Now, suppose $\mathbb{R}^\infty$ and $\mathbb{R}^\omega$ are given the product topology. Then,

$$U' = U_1 \times U_2 \times U_3 \times \cdots \times U_N \times \mathbb{R} \times \mathbb{R} \times \dots$$

is the form of *any* basic neighborhood of $\boldsymbol{x}$, where we assume $x_k \in U_k$ for all $k \in \{1, \dots, N\}$, (of course). Then, $U' \cap \mathbb{R}^\infty \neq \emptyset$. So, $\overline{R^\infty} = \mathbb{R}^\omega$. ☕

20.1 (a) In $\mathbb{R}^n$, define

$$d'(\boldsymbol{x}, \boldsymbol{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

Proof. We first show that d' is a metric:

i. $d'(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ since $|x_i - y_i| \geq 0$ for all $i \in \{1, \dots, n\}$, and

$$d'(\boldsymbol{x}, \boldsymbol{y}) = 0 \iff |x_i - y_i| = 0 \; \forall i \iff x_i = y_i \; \forall i \iff \boldsymbol{x} = \boldsymbol{y}$$

ii. $d'(\boldsymbol{x}, \boldsymbol{y}) = d'(\boldsymbol{y}, \boldsymbol{x})$ because $|x_i - y_i| = |y_i - x_i|$ for all $i \in \{1, \dots, n\}$ and so

$$|x_1 - y_1| + \cdots + |x_n - y_n| = |y_1 - x_1| + \cdots + |y_n - x_n|$$

iii. $d'(\boldsymbol{x}, \boldsymbol{z}) \leq d'(\boldsymbol{x}, \boldsymbol{y}) + d'(\boldsymbol{y}, \boldsymbol{z})$ because

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \quad \text{for all } i \in \{1, \dots, n\},$$

and so

$$|x_1 - z_1| + \cdots + |x_n - z_n| \leq |x_1 - y_1| + \cdots + |x_n - y_n| + |y_1 - z_1| + \cdots + |y_n - z_n|$$

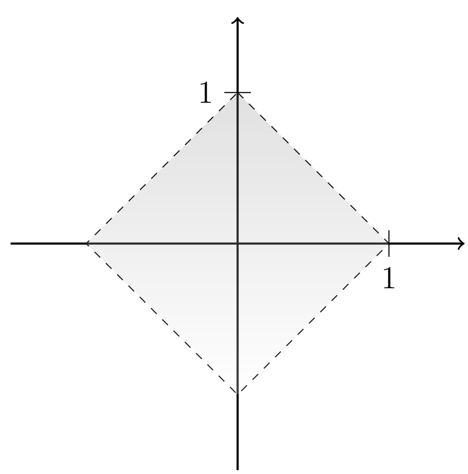
In order to show that d' induces the usual topology on $\mathbb{R}^n$, we need to show that basis elements of the topology induced by d' are contained in basis elements of the usual topology, and vice versa. By Theorem 20.3, the topology on $\mathbb{R}^n$ induced by the euclidean metric d is the same as the usual topology on $\mathbb{R}^n$. So, let $\boldsymbol{x} \in \mathbb{R}^n, \epsilon > 0$ and $\boldsymbol{y} \in B_{d'}(x, \epsilon)$. Notice that for $\delta = \epsilon - d(\boldsymbol{x}, \boldsymbol{y})$, we get

$$B_d(y, \delta) \subset B_{d'}(x, \epsilon)$$

On the other hand let $\boldsymbol{y} \in B_d(x, \epsilon)$. Then, let $\delta = \epsilon - d(\boldsymbol{x}, \boldsymbol{y})$ and we get

$$B'_d(y, \delta) \subset B_d(x, \epsilon)$$

and thus d' induces the usual topology of $\mathbb{R}^n$. When $n = 2$, the basis elements under d' look like rhombuses.



The shaded region in the figure shows all points contained in $B_{d'}(0, 1)$.

☕

(b) More generally, given $p \geq 1$, define

$$d'(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Proof. We show that d' is the same as the square metric ρ, and by Theorem 20.3, we conclude d' is the same as the usual topology on $\mathbb{R}^n$.
Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be two points of $\mathbb{R}^n$. It is simple algebra to check

$$\rho(\boldsymbol{x}, \boldsymbol{y}) \leq d'(\boldsymbol{x}, \boldsymbol{y}) \leq \sqrt[p]{n}\ \rho(\boldsymbol{x}, \boldsymbol{y})$$

The first inequality shows that

$$B'_d(\boldsymbol{x}, \epsilon) \subset B_\rho(\boldsymbol{x}, \epsilon)$$

for all $\boldsymbol{x}$ and ϵ since if $d'(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$, then $\rho(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$ also. Similarly, the second inequality shows that

$$B_\rho(\boldsymbol{x}, \epsilon/\sqrt{n}) \subset B'_d(\boldsymbol{x}, \epsilon)$$

for all $\boldsymbol{x}$ and ϵ. Thus, the two metric topologies d' and ρ are the same. ☕

20.3 Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

Proof. ***(From Online Solution Manual)***
Let U be an open subset of $\mathbb{R}$, and $x \times y \in d^{-1}(U)$. Then $a = d(x \times y) \in U$, and there is an open interval $(a - \epsilon, a + \epsilon) \subset U$. Take any point $x' \times y' \in B_d(x, \epsilon/2) \times B_d(y, \epsilon/2)$. Then,

$$|d(x \times y) - d(x \times y)| \leq d(x' \times x) + d(y' \times x) < \epsilon$$

and so

$$d(x' \times y') \in (a - \epsilon, a + \epsilon) \subset U,$$

and $x' \times y'$ is contained in $d^{-1}(U)$. Therefore, the set $B = B_d(x, \epsilon/2) \times B_d(y, \epsilon/2)$, open in $X \times X$, is such that $x \times y \in B \subset d^{-1}(U)$, and $d^{-1}(U)$ is open. ☕

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X. (One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous).

Proof. (Thanks for the help on this one Jesse!)

Let $x_0 \in X$ and $B_d(x_0, r)$ be a basis element of the metric topology on X. Define $i_{x_0} : X' \to X' \times X'$ by $y \mapsto (x_0, y)$. By Exercise 18.4, this map is continuous. Then, $d \circ i_{x_0} : X' \to \mathbb{R}$ is continuous since the composition of continuous functions is continuous. Then, given $\epsilon > 0$,

$$(d \circ i_{x_0})^{-1}(-\epsilon, \epsilon) = B_d(x_0, \epsilon)$$

which is open in X'. Take $\epsilon = r$. Then $B_d(x, r)$ is open in X'. Thus, the topology on X' is finer than the topology of X. ☕

20.9 Show that the equclidean metric d on $\mathbb{R}^n$ is a metric.

(a) Show that $\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{y}) + (\boldsymbol{x} \cdot \boldsymbol{z})$

Proof.

$$\begin{aligned}
\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) &= \boldsymbol{x}(y_1 + z_1, \ldots, y_n + z_n) \\
&= (\boldsymbol{x}(y_1 + z_1), \ldots, \boldsymbol{x}(y_n + z_n)) \\
&= (\boldsymbol{x}y_1 + \boldsymbol{x}z_1, \ldots, \boldsymbol{x}y_n + \boldsymbol{x}z_n) \\
&= (\boldsymbol{x}y_1, \ldots, \boldsymbol{x}y_n) + (\boldsymbol{x}z_1, \ldots, \boldsymbol{x}z_n) \\
&= \boldsymbol{x}(y_1, \ldots, y_n) + \boldsymbol{x}(z_1, \ldots, z_n) \\
&= (\boldsymbol{x} \cdot \boldsymbol{y}) + (\boldsymbol{x} \cdot \boldsymbol{z})
\end{aligned}$$

☕

(b) Show that $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}||\ ||\boldsymbol{y}||$.

Proof. Without loss of generality, suppose $\boldsymbol{x} = 0$. Then,

$$0 = |0| \leq ||0|| \cdot ||\boldsymbol{y}|| = 0 \cdot ||\boldsymbol{y}|| = 0$$

If $\boldsymbol{x}, \boldsymbol{y} \neq 0$, let $a = 1/||\boldsymbol{x}||$ and $b = 1/||\boldsymbol{y}||$. Since, $||a\boldsymbol{x} \pm b\boldsymbol{y}|| \geq 0$, then

$$\begin{aligned}
||a\boldsymbol{x} \pm b\boldsymbol{y}|| &\geq 0 \\
||a\boldsymbol{x} \pm b\boldsymbol{y}|| \cdot ||a\boldsymbol{x} \pm b\boldsymbol{y}|| &\geq 0 \\
a^2||\boldsymbol{x}||^2 \pm 2ab(\boldsymbol{x} \cdot \boldsymbol{y}) + b^2||\boldsymbol{y}||^2 &\geq 0 \\
2 \pm 2ab(\boldsymbol{x} \cdot \boldsymbol{y}) &\geq 0
\end{aligned}$$

Thus,

$$|(\boldsymbol{x} \cdot \boldsymbol{y})| \leq \frac{1}{ab} = ||\boldsymbol{x}||\ ||\boldsymbol{y}||$$

☕

(c) Show that $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$

Proof. Using the hint and part (b),

$$\begin{aligned} 0 &\leq ||\boldsymbol{x}+\boldsymbol{y}||^2 \\ &= ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + 2(\boldsymbol{x}\cdot\boldsymbol{y}) \\ &\leq ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + 2|\boldsymbol{x}\cdot\boldsymbol{y}| \\ &\leq (||\boldsymbol{x}|| + ||\boldsymbol{y}||)^2 \end{aligned}$$

Thus,

$$||\boldsymbol{x}+\boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$$

☕

(d) Verify that d is a metric.

Proof. Clearly, $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$. Also, $d(\boldsymbol{x},\boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. Part (c) shows the triangle inequality since $d(\boldsymbol{x},\boldsymbol{y}) = ||\boldsymbol{x}-\boldsymbol{y}||$. ☕

21.1 Let $A \subset X$. If d is a metric for the topology on X, show that $d|_{A\times A}$ is a metric for the subspace topology on A.

Proof. Let $\epsilon > 0$ and U be open in the subspace topology on A. If $u \in U$. Then there exists a basis element $B = A \cap B_d(x, \epsilon)$ so that $u \in B \subset U$ for some $x \in X$. Then, let $\delta = \epsilon - d(x, u)$. Then, $B_d(u, \delta) \subset A$ and $B_d(u, \delta) \subset B_d(x, \epsilon)$ which means

$$B_d(U, \delta) \subset B.$$

Thus, the metric topology on A is finer than the subspace topology on A. Now, let W be open in A in the metric topology and let $w \in W$. Then, there exists a basis element, $B_d(a, \epsilon)$ so that $w \in B_d(a, \epsilon) \subset W$. Let $\delta = \epsilon - d(w, a)$. Then

$$B_d(w, \delta) \subset B_d(a, \epsilon),$$

and so the subspace topology on A is finer that the metric topology. Thus, the metric and subspace topologies on A are the same, i.e., $d|_{A\times A}$ is a metric on the subspace topology on A.

21.2 Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an *isometric imbedding* of X in Y.

Proof. We need to show that f is injective, continuous, and open onto its image. For injectivity, simply notice that if $f(x_1) = f(x_2)$ then

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = 0 \implies x_1 = x_2.$$

To show that f is continuous, we let $\epsilon > 0$ and see that when $\delta = \epsilon$, we have

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) < \delta = \epsilon.$$

Now, we need to show that the map $f' : X \to f(X)$ is a homeomorphism. If U is an open set in X, then given $u \in U$, we can find $\epsilon > 0$ so that $B_{d_X}(u, \epsilon) \subset U$. If $d_X(u', u) < \epsilon$, then $d_Y(f(u'), f(u)) = d_X(u', u) < \epsilon$. Thus, $f(u') \in B_{d_Y}(f(u), \epsilon) \cap f(X)$. So $f(U)$ is open in $f(X)$. Therefore, f' is a homeomorphism, which means f is an imbedding.

21.5

Theorem. *Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then*

$$x_n + y_n \to x + y, \quad x_n - y_n \to x - y, \quad x_n y_n \to xy,$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n / y_n \to x/y$$

[Hint: Apply Lemma 21.4; recall from the exercises of section 19 that if $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to x \times y$].

Proof. Consider the following operation functions:

$$\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x, y) \mapsto x + y \qquad \text{(addition)}$$
$$\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x, y) \mapsto x - y \qquad \text{(subtraction)}$$
$$\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x, y) \mapsto xy \qquad \text{(multiplication)}$$
$$\delta : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R},\ (x, y) \mapsto x/y \qquad \text{(division)}$$

By Lemma 21.4, each of these functions is continuous. By exercise 6 in section 19, $x_n \times y_n \to x \times y$. So, by Theorem 21.3,

$$\alpha(x_n + y_n) = \alpha(x_n \times y_n) \to \alpha(x \times y) = x + y$$
$$\sigma(x_n - y_n) = \sigma(x_n \times y_n) \to \sigma(x \times y) = x - y$$
$$\mu(x_n y_n) = \mu(x_n \times y_n) \to \mu(x \times y) = xy$$
$$\delta(x_n/y_n) = \delta(x_n \times y_n) \to \delta(x \times y) = x/y$$

☕

21.8 let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points in X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Proof. Let d be the metric on Y and $\epsilon > 0$. Since the convergence of (f_n) to f is uniform, then f is continuous by Theorem 21.6. As f is continuous, there exists a neighborhood U of x so that

$$f(U) \subset B_d(f(x), \epsilon/2).$$

Since $(x_n) \to x$, we pick N_1 large enough so that $x_n \in U$ for all $n > N_1$. Then, we have

$$d(f(x_n), f(x)) < \frac{\epsilon}{2}.$$

Now, since f_n converges uniformly to f, we pick N_2 so that for all $x_k \in U$, whenever $n > N_2$, we have

$$d(f_n(x_k), f(x_k)) < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then by the triangle inequality,

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon.$$

☕

Quotient Topology Exercises

Complex Projective Spaces Let $\mathbb{C}$ denote the complex numbers. As a topological space, you can treat them as $\mathbb{R}^2$ with the product topology.

1. (a) Prove that complex multiplication $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by $(a + b\boldsymbol{i})(c + d\boldsymbol{i}) = (ac - bd) + (ad + bc)\boldsymbol{i}$ is continuous, as is complex conjugation $\overline{a + b\boldsymbol{i}} = a - b\boldsymbol{i}$.

Proof. We consider the complex number $(a + b\boldsymbol{i})$ as the point $(a, b) \in \mathbb{R}^2$; likewise for $(c + d\boldsymbol{i})$. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (f_1(x, y), f_2(x, y))$$

where $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$f_1((a, b), (c, d)) = ac - bd \quad \text{and} \quad f_2((a, b), (c, d)) = ad + bc.$$

Since multiplication, addition, and subtraction in $\mathbb{R}$ is continuous (Lemma 21.4, Munkres), then both f_1 and f_2 are continuous, as they are each the composition of continuous functions. Since each of the component functions of f are continuous, then so is f.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $g((a, b)) = (g_1(a, b), g_2(a, b))$. Where $g_1 : \mathbb{R}^2 \to \mathbb{R}$ $g_1((a, b)) = a$ and $g_2((a, b)) = -b$. Clearly, g_1 and g_2 are continuous, and so g is continuous. ☕

(b) Prove that inversion is a continuous map on $\mathbb{C} - \{0\}$. This map sends $a + b\boldsymbol{i}$ to

$$\frac{a - b\boldsymbol{i}}{a^2 + b^2}.$$

Proof. Let $f : \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{R}^2 - \{\mathbf{0}\}$ be defined by

$$f((a, b)) = (f_1(a, b), f_2(a, b))$$

where $f_1 : \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{R} - \{0\}$ and $f_2 : \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{R} - \{0\}$ are given by

$$f_1(a, b) = \frac{a}{a^2 + b^2} \quad \text{and} \quad f_2(a, b) = \frac{-b}{a^2 + b^2}$$

Since both f_1 and f_2 are continuous, so is f.

☕

2. We say $\vec{z}, \vec{w} \in \mathbb{C}^{n+1} - \{\vec{0}\}$ are equivalent, denoted $\vec{z} \sim \vec{w}$, if there is $\lambda \in \mathbb{C} - \{0\}$ so that $\lambda\vec{z} = \vec{w}$. If $\vec{z} = (z_0, \dots, z_n)$ we denote the equivalence class of $\vec{z}$ by $[z_0, \dots, z_n]$. These are *homogeneous coordinates*. Let $\mathbb{C}P(n)$ denote the set of equivalence classes with the quotient topology from

$$q : \mathbb{C}^{n+1} - \{\vec{0}\} \to \mathbb{C}P(n)$$

given by $q(\vec{z}) = [\vec{z}]$. Prove that the quotient map q is open.

Proof. We first show that the relation

$$R = \{(\vec{v}, \vec{w}) \in (\mathbb{C}^{n+1} - \{0\}) \times (\mathbb{C}^{n+1} - \{0\}) \mid \vec{v} = \lambda \vec{w}, \text{ for some } \lambda \in \mathbb{C}\}$$

comes from a continuous group action. Let $\mu : \mathbb{C}^* \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ given by $\mu(\lambda, \vec{x}) = \lambda \vec{x}$ be a group action. Then for each fixed $\lambda \in \mathbb{C}^*$, consider the map

$$\mu_\lambda : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\} \quad \text{defined by} \quad \mu_\lambda(\vec{x}) = \lambda \vec{x} = (\lambda x_0, \ldots, \lambda x_n).$$

Recall from problem 1 that complex multiplication is continuous. Since each coordinate function of μ_λ is simply complex multiplication at each coordinate of $\vec{x}$, then μ_λ is continuous. In fact, μ_λ is a homeomorphism, since the inverse map of μ_λ is $\mu_{\lambda^{-1}}$, which is continuous.
Now, let U be open in $\mathbb{C}^{n+1} - \{\vec{0}\}$. We need to show that $q(U)$ is open in $\mathbb{C}P(n)$. To that end, consider

$$q^{-1}(q(U)) = q^{-1}\left(\bigcup_{\vec{z} \in U} [\vec{z}]\right) = \bigcup_{\vec{z} \in U} q^{-1}[\vec{z}] = \bigcup_{\vec{z} \in U, \lambda \in C^*} \lambda U = \bigcup_{\vec{z} \in U, \lambda \in C^*} \mu_\lambda(U).$$

Since μ_λ is a homeomorphism for all $\lambda \in \mathbb{C}^*$, then $\mu_\lambda(U)$ is open for all $\lambda \in \mathbb{C}^*$. Thus, $q^{-1}(q(U))$ is a union of open sets, and so itself open. Hence, q is open. ☕

3. Prove that $\mathbb{C}P(n)$ is Hausdorff.

Proof. It is sufficient to show that the relation R (as defined in the previous problem) is closed. To that end, let $\vec{v}, \vec{w} \in (\mathbb{C}^{n+1} - \{0\})$ and consider the following matrix:

$$\begin{pmatrix} v_0 & w_0 \\ v_1 & w_1 \\ \vdots & \vdots \\ v_n & w_n \end{pmatrix}.$$

The vectors $\vec{v}$ and $\vec{w}$ are linearly dependent (i.e., if $\vec{v} R \vec{w}$) if and only if every 2×2 minor of the above matrix is zero. Let

$$D_{ij} = \det \begin{pmatrix} v_i & w_i \\ v_j & w_j \end{pmatrix} \quad \text{for all} \quad 0 \leq i \leq n-1 \text{ and for all } 1 \leq j \leq n.$$

Then, define a map $D : (\mathbb{C}^{n+1} - \{0\}) \times (\mathbb{C}^{n+1} - \{0\}) \to \mathbb{C}$ given by the equation

$$D(\vec{v}, \vec{w}) = \sum |D_{ij}|^2.$$

Then, D is continuous since it is the composition of continuous functions. Since $\{0\}$ is closed in $\mathbb{C}$ and D is continuous, then

$$D^{-1}(\{0\}) = R$$

is closed. ☕

4. Let $U_i = \{[z_0, \ldots, z_n] \in \mathbb{C}P(n) \mid z_i \neq 0\}$. Prove that U_i is open.

Proof. To show that U_i is open in $\mathbb{C}P(n)$, we need show that $q^{-1}(U_i)$ is open. First, notice that
$$q^{-1}(U_i) = \{(z_0, \ldots, z_n) \mid z_i \neq 0\}.$$
To show that this set is open, we show that its complement is closed. To that end, let $T_i = \mathbb{C}^{n+1} \setminus q^{-1}(U_i)$ and so $T_i = \{(z_0, \ldots, z_i = 0, \ldots, z_n)\}$. If $\pi_i : \mathbb{C}^{n+1} \to \mathbb{C}$ is the usual projection map onto the i-th coordinate then notice that
$$\pi_i^{-1}(\{0\}) = T_i.$$
Since π_i is continuous and $\{0\}$ is closed in $\mathbb{C}$, then T_i is closed in $\mathbb{C}^{n+1}$. ☕

5. Prove that the map $\phi_i : U_i \to \mathbb{C}^n$ given by
$$\phi_i([z_0, \ldots, z_n]) = \left(\frac{z_0}{z_1}, \ldots, \hat{z}_i, \ldots \frac{z_n}{z_i}\right)$$
is a homeomorphism.

Proof. Claim: The map $\tilde{\phi}_i : q^{-1}(U_i) \to \mathbb{C}^n$ defined by
$$(z_0, \ldots, z_n) \mapsto \left(\frac{z_0}{z_i}, \ldots, \hat{z}_i, \ldots \frac{z_n}{z_i}\right)$$
is a quotient map and $\tilde{\phi}_i = \phi_i \circ q$. If we can show this, then by Corollary 22.3 (§22, Munkres), the map ϕ_i is a homeomorphism.
We first show that $\tilde{\phi}_i$ is well defined: If $\vec{v}, \vec{w} \in q^{-1}(U_i)$ are such that $\vec{w} = \lambda\vec{v}$, then
$$\tilde{\phi}_i(\vec{v}) = \left(\frac{v_0}{v_i}, \ldots, \hat{v}_i, \ldots \frac{v_n}{v_i}\right) = \left(\frac{\lambda v_0}{\lambda v_i}, \ldots, \hat{v}_i, \ldots \frac{\lambda v_n}{\lambda v_i}\right) = \left(\frac{w_0}{w_i}, \ldots, \hat{w}_i, \ldots \frac{w_n}{w_i}\right) = \tilde{\phi}_i(\vec{w}).$$
Hence, $\tilde{\phi}_i$ is well-defined. Notice that $\tilde{\phi}_i(\vec{v}) = (\psi_0(\vec{v}), \ldots, \hat{v}_i, \ldots, \psi_n(\vec{v}))$ where
$$\psi_k(\vec{v}) = \frac{v_k}{v_i} \quad \text{for each } k \in \{0, 1, \ldots, i-1, i+1, \ldots, n\}.$$
Each ψ_k is continuous since $v_i \neq 0$, which means $\tilde{\phi}_i$ is continuous.
Now, to see that $\tilde{\phi}_i$ is surjective, given $\vec{y} = (y_0, \ldots, \hat{y}_i, \ldots y_n) \in \mathbb{C}^n$, let
$$\vec{v} = (v_0, v_1, \ldots, v_i, \ldots, v_n)$$
so that
$$v_i = y_i, v_0 = y_0 y_i, v_1 = y_1 y_i, \ldots, y_n = y_n y_i.$$
Then $\tilde{\phi}_i(\vec{v}) = \vec{y}$. Finally, we show that $\tilde{\phi}_i$ send saturated open sets to open sets. If S is an open set of $q^{-1}(U_i)$ which is saturated with respect to $\tilde{\phi}_i$, then $\tilde{\phi}_i^{-1}(\tilde{\phi}_i(S)) = S$. Notice that $S = \{[(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)]\}$, and so
$$\tilde{\phi}_i(S) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right)$$
which is open in $\mathbb{C}^n$. ☕

6. Conclude that the result of removing a single point from $\mathbb{C}P(1)$ is homeomorphic to the complex plane.

Proof. Notice that $U_1 = \{[z_i, z_j] \mid z_i \neq 0\} = \mathbb{C}P(1) \backslash \{[0, z_j]\}$. Since U_1 is homeomorphic to $\mathbb{C}$ by the previous exercise, then $\mathbb{C}P(1) \setminus \{[0, z_j]\}$ is homeomorphic to $\mathbb{C}$. ☕

7. Prove that $GL_2(\mathbb{C}) \times \mathbb{C}P(1) \to \mathbb{C}P(1)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [z, w] = [az + bw, cz + dw]$$

gives a well defined group action of $GL_2(\mathbb{C})$ on $\mathbb{C}P(1)$.

Proof. Simply observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} [z, w] = [1z + 0w, 0z + 1w] = [z, w],$$

and so the identity acts as it should and also

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} [z, w] \right) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} [a'z + b'w, c'z + d'w] \\
&= [a(a'z + b'w) + b(c'z + d'w), c(a'z + b'w) + d(c'z + d'w)] \\
&= [(aa' + bc')z + (ab' + bd')w, (ca' + dc')z + (cb' + dd')w] \\
&= \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} [z, w] \\
&= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) [z, w].
\end{aligned}$$

which means associativity is held. Thus, we have a group action. ☕

Munkres Exercises

§23, # 1 Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$, what does connectedness of X in one topology imply about connectedness in the other?

Proof. Claim: If X is connected in $\mathcal{T}'$, then X is connected under $\mathcal{T}$.
By way of contradiction, suppose C, D is a separation of X in $\mathcal{T}$. Then, as $C, D \in \mathcal{T}$ and $\mathcal{T} \subset \mathcal{T}'$, then $C, D \in \mathcal{T}'$. Since C is both closed and open and X is connected under $\mathcal{T}'$, then either $C = X$ or $C = \emptyset$. In either case, C is not a proper, nonempty subset of X, and so C, D is not a separation of X. Thus, X is connected under the topology $\mathcal{T}$. ☕

§23, # 2 Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\cup A_n$ is connected.

Proof. Suppose C, D is a separation of $\cup A_n$. Since A_1 is connected, then $A_1 \subset C$ or $A_1 \subset D$. Without loss of generality, suppose $A_1 \subset C$. Then, as $A_1 \cap A_2 \neq \emptyset$, then $A_2 \subset C$ also. By induction, $A_n \subset C$ for all n. Thus, $D = \emptyset$ which implies C, D is not a separation and so $\cup A_n$ is connected. ☕

§23, # 4 Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cap (\cup A_\alpha)$ is connected.

Proof. Let C, D be a separation of $A \cap (\cup A_\alpha)$. Since A is connected, suppose, without loss of generality, that $A \subset C$. Then, as $A \cap A_\alpha \neq \emptyset$ for all α, then $A_\alpha \subset C$ for all α. Thus, $D = \emptyset$, and so $A \cap (\cup A_\alpha)$ is connected. ☕

§23, # 7 Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

Proof. Notice that

$$\bigcup_{a \in \mathbb{R}^-} [a, 0) = (-\infty, 0)$$

and

$$\bigcup_{b>0} [0, b) = [0, \infty).$$

These two intervals are unions of elements of $\mathbb{R}_\ell$ and so are open in $\mathbb{R}_\ell$, and they form a separation of $\mathbb{R}$. Thus, $\mathbb{R}_\ell$ is not connected. ☕

§23, # 9 Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

Proof. By Theorem 23.6, $X \times Y$ is connected since both X and Y are connected. Note that

$$(X \times Y) - (A \times B) = (X - A) \times (X - B).$$

We show that each $(X - A)$ and $(X - B)$ are connected, and then applying Theorem 23.6 again, their cartesian product is connected. To get a contradiction, suppose $(X - A)$ admits the separation C, D. Now, A may be connected or not connected. We consider both cases: If A is connected, then X admits a separation, namely $(C \cup A), D$ because

$$(C \cup A) \cup D = C \cup D \cup A = (X - A) \cup A = X$$

and

$$(C \cup A) \cap D = (C \cap D) \cup (A \cap D) = \emptyset \cup \emptyset.$$

Thus we get a contradiction since X is connected. If A is not connected, let U, V be a separation of A. Then again, X admits a separation $(C \cup U \cup V), D$ because

$$(C \cup U \cup V) \cup D = C \cup D \cup V \cup U = X - A \cup A = X$$

and

$$(C \cup U \cup V) \cap D = (C \cap D) \cup (U \cap D) \cup (V \cap D) = \emptyset \cup \emptyset \cup \emptyset = \emptyset.$$

And again we get a contradiction, since X is connected. Thus, if $X - A$ admits a separation, then we can construct a separation of X, a contradiction, which means $X - A$ must be connected. Similarly for $X - B$. ☕

§24, # 1 (a) Show that no two of the spaces $(0,1), (0,1]$, and $[0,1]$ are homeomorphic. [Hint: What happens if you remove a point from each of these spaces?]

Proof. If two spaces are homeomorphic, then they remain homeomorphic if we remove a single point from each of them. Also, if X is homeomorphic to Y, then if X is connected, so is Y. Using the hint, if we remove any point from $(0,1)$, then it becomes disconnected — if we take x from $(0,1)$ then a separation of this set is $(0,x) \cup (x,1)$. If we remove the point 1 from $(0,1]$, it remains connected. Thus, $(0,1)$ and $(0,1]$ are not homeomorphic.
If we remove 0 from $[0,1]$ then it is connected. But $(0,1]$ is not homeomorphic to $(0,1)$ and so $[0,1]$ is not homeomorphic to $(0,1)$.
If we remove 1 from $[0,1]$ then the set remains connected. But $[0,1)$ is not homeomorphic to $(0,1)$ (by a similar argument as above), which means $[0,1]$ is not homeomorphic to $(0,1)$.
Finally if we remove 0 and 1 from $[0,1]$, then it equals $(0,1)$, which is not homeomorphic to $(0,1]$. Thus, $[0,1]$ is not homeomorphic to $(0,1]$. ☕

(b) Suppose that there exists embeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an examples that X and Y need not be homeomorphic.

Proof. Using information gained from part (a), we let $X = (0,1)$ and $Y = [0,1]$. Then, consider the maps $f(x) = x$ and $g(x) = x/2 + 1/2$. These maps are clearly injective, continuous, and open onto their respective images, i.e., f and g are imbeddings. But we know from part (a) that X and Y are not homeomorphic. ☕

(c) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Proof. By Example 4 in §24, the *punctured euclidean space* $\mathbb{R}^n - \{\mathbf{0}\}$ is path connected, and hence connected, when $n > 1$. However, $\mathbb{R} - \{0\}$ is not connected, since $(-\infty, 0) \cup (0, \infty)$ is a separation of this space. Thus, $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic. ☕

§24, # 3 Let $f : X \to X$ be continuous. Show that if $X = [0,1]$, there is a point x such that $f(x) = x$. The point x is called a *fixed point* of f. What happens if X equals $[0,1)$ or $(0,1)$?

Proof. Let $g(x) = f(x) - x$; note that g is continuous as it is the difference of continuous functions. Then, $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$. So,

$$g(1) \leq g(0).$$

If $g(1) = 0$ then $f(1) = 1$ and we are done. Similarly, if $g(0) = 0$ then $f(0) = 0$ and we are done. If $g(1)$ and $g(0)$ are not zero, then

$$g(1) < 0 < g(0).$$

By the Intermediate Value Theorem, there must be a point $x \in (0, 1)$ for which $g(x) = 0$, i.e., $0 = g(x) = f(x) - x$ and so $f(x) = x$ for some $x \in (0, 1)$.

Our proof above rested on the fact that f was defined at the points 0 and 1, and so we were able to invoke the Intermediate Value Theorem. However, $[0, 1)$, and $(0, 1)$ are not defined at one or both of the points 0 and 1, and so we cannot draw the same conclusions as above. ☕

§26, # 2 (a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact.

Proof. Let Y be a subspace of $\mathbb{R}$ in the finite complement topology. If $Y = \emptyset$ then we are done. Suppose $Y \neq \emptyset$ and let $\{A_\alpha\}_{\alpha\in J}$ a collection of open sets in $\mathbb{R}$ which cover Y. Not all A_α are such that $\mathbb{R} - A_\alpha = \emptyset$, otherwise $\{A_\alpha\}_{\alpha\in J}$ would not be a cover of Y. So, there exists $A_\alpha \in \{A_\alpha\}_{\alpha\in J}$ so that $\mathbb{R} - A_\alpha$ is finite. Then,

$$Y - A_\alpha \subset \mathbb{R} - A_\alpha$$

and so $Y - A_\alpha$ is finite. Let $Y - A_\alpha = \{x_1, \dots, x_n\}$. For each $i \in \{1, \dots, n\}$, there exists $A_{\alpha_i} \in \{A_\alpha\}_{\alpha\in J}$ so that $x_i \in A_{\alpha_i}$. So,

$$A_\alpha \cup \bigcup_{i=1}^{n} A_{\alpha_i} \supseteq Y$$

and so Y is compact. ☕

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} - A$ is either countable of all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

Proof. In the countable complement topology, $[0, 1]$ is not compact. Let

$$\begin{aligned} A_1 &= [0,1] \setminus \{1, 1/2, 1/3, \dots, 1/n, 1/(n+1), \dots\} \\ A_2 &= [0,1] \setminus \{1/2, 1/3, \dots, 1/n, 1/(n+1), \dots\} \\ A_3 &= [0,1] \setminus \{1/3, \dots, 1/n, 1/(n+1), \dots\} \\ &\vdots \\ A_n &= [0,1] \setminus \{1/n, 1/(n+1), \dots\} \\ &\vdots \end{aligned}$$

These sets are open in the countable complement topology, and cover $[0, 1]$. For contradiction, suppose there exists a finite subcover $\{A_i\}_{i=1}^{k}$ of $[0, 1]$. However, all of the points $\{1, 1/2, \dots 1/(k-1)\}$ are not in this finite subcover, and so $[0, 1]$ is not compact. ☕

§26, # 3 Show that a finite union of compact subspaces of X is compact.

Proof. Let $\{Y_1, \dots, Y_n\}$ be a finite collect of compact subspaces of X. If $\bigcup_{i=1}^{n} Y_i = \emptyset$ then we are done. Suppose $\bigcup_{i=1}^{n} Y_i \neq \emptyset$ and let $\mathcal{A}$ be a collection of open sets of X which covers $\bigcup_{i=1}^{n} Y_i$. Then, $\mathcal{A}$ covers each Y_i and since each Y_i is compact, there is a finite subset $\mathcal{A}_i \subset \mathcal{A}$ so that $\mathcal{A}_i$ covers Y_i for each i. Then, $\bigcup_{i=1}^{n} \mathcal{A}_i$ is a finite subset of $\mathcal{A}$ since it is the finite union of finite subsets of $\mathcal{A}$. Then,

$$\bigcup_{i=1}^{n} Y_i \subseteq \bigcup_{i=1}^{n} \mathcal{A}_i$$

and so $\bigcup_{i=1}^{n} Y_i$ is compact. ☕

§26, # 4 Show that every compact subspace of a metric space is bounded in the metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Proof. Since every metric space is a Hausdorff space[1], and every compact subspace of a Hausdorff space is closed (Theorem 26.3), then every subspace of a metric space is closed. If Y were not bounded, then we could construct an open cover of Y which has no finite subcover, i.e., Y would not be compact. To see this, let $y_0 \in Y$ and consider the collection $\mathbf{B} = \{B_d(y_0, n)\}_{n \in \mathbb{Z}+}$. Any finite subcollection of $\mathbf{B}$ would not contain every point of Y, since Y is unbounded. So, Y is not compact, which cannot happen. Thus Y must be bounded.

If we consider any set which is infinite and give it the discrete topology, then every subspace is closed. Then we can cover any closed subspace in this topology with the collection of all one-point sets, which has no finite subcover of the subspace.

§26, # 5 Let A and B be disjoint compact subspaces of the Hausdorff space S. Show that there exists disjoint open sets U and V containing A and B, respectively.

Proof. Since A and B are disjoint, then for every $a \in A$, we can choose disjoint open sets U_a and V_a containing a and B, respectively. Then $\bigcup_{a \in A} U_a \supseteq A$. Since A is compact, there exists a finite subcollection $\{U_{a_1}, \ldots, U_{a_n}\}$ which covers A. Define

$$U = U_{a_1} \cup \cdots \cup U_{a_n} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_n}.$$

Then, U and V are disjoint open sets which contain A and B, respectively.

§26, # 7 Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Proof. Let $F \subset X \times Y$ be closed and $a \notin \pi_1(F)$. Then, $a \times Y$ does not intersect $\pi_1(F)$. Since F is closed, then $U = X \times Y - F$ is open. Since Y is compact and U contains $a \times Y$, then by the tube lemma U contains some tube $W \times Y$ about $a \times Y$, where W is a neighborhood of a. So, $a \in W$ and $W \cap \pi_1(F) = \emptyset$. Thus, $\pi_1(F)$ is closed and so π_1 is a closed map.

§26, # 8 *Theorem.* Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the **graph** of f,

$$G_f = \{x \times f(x) \mid x \in X\}$$

is closed in $X \times Y$.

Proof. Suppose f is continuous and let $(a, b) \notin G_f$. Since Y is Hausdorff, we can find disjoint neighborhoods A and B of $f(a)$ and b, respectively. Since f is continuous $f^{-1}(A) = U$ is open and is a neighborhood of a. So, $U \times B$ is an open neighborhood of (a, b) which is disjoint from G_f. Other wise, $(x, y) \in G_f \cap U \times B$ implies $f(x) \in A$ and $f(x) \in B$, which

[1] If (X, d) is a metric space and x and y are distinct points of X, let $\epsilon = d(x, y)$. Then $B_d(x, \epsilon/2)$ and $B_d(y, \epsilon/2)$ are disjoint neighborhoods of x and y, respectively, and so X is Hausdorff.

cannot happen since A and B are disjoint. Hence G_f is closed.
Conversely, suppose G_f is closed and let V be an open set of Y. If V does not intersect G_f, then $f^{-1}(V) = \emptyset$ which is open and so f is continuous. So, suppose V intersects G_f at the point $f(x_0)$. Using the hint, since V is open, $Y - V$ is closed and so $F = G_f \cap X \times (Y - V)$ is closed. By exercise 7, $\pi_1(F) = f^{-1}(V)^c$ is closed. Thus $f^{-1}(V)$ is open so that f is continuous. ☕

§27, # 2 Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

Proof. Notice

$$\begin{aligned} d(x,A) = 0 &\implies \exists\ a \in A \ \ s.t.\ \ d(x,a) = 0 \\ &\implies x = a \\ &\implies x \in A \\ &\implies x \in \overline{A} \end{aligned}$$

And conversely,

$$\begin{aligned} x \in \overline{A} &\implies B_d(x, 1/n) \cap A \neq \emptyset \quad \forall n \in \mathbb{Z}^+ \\ &\implies \exists\ a_n \in B_d(x, 1/n) \cap A \quad \forall n \in \mathbb{Z}^+ \\ &\implies \inf\{d(x, a_n) \mid a_n \in A\} \to 0 \\ &\implies d(x, A) = 0 \end{aligned}$$

☕

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

Proof. Let $x \in X$ and let $f_x : A \to \mathbb{R}$ be defined by $a \mapsto d(x, a)$. Since f is defined in terms of a continuous function, it is continuous. So, f must attain a minimum value on its compact domain A by the Extreme Value Theorem, i.e., $d(x, a) = d(x, A)$ for some $a \in A$. ☕

(c) Define the $\epsilon - \mathit{neighborhood}$ of A in X to be the set

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

Proof. Let $\epsilon > 0$. Then,

$$\begin{aligned} x \in U(A, \epsilon) &\iff d(x, A) < \epsilon \\ &\iff x \in B_d(a_0, \epsilon) \text{ for some } a_0 \in A. \\ &\iff x \in \bigcup_{a \in A} B_d(a, \epsilon) \end{aligned}$$

☕

(d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

Proof. *From Online Solution Manual*
For $a_j \in A$, let r_j be such that $B_d(a_j, 2r_j) \subset U$. Then $\bigcup_{a_j \in A} B_d(a_j, r_j) \supseteq A$. Since A is compact, there is a finite list $a_{j_1}, \ldots a_{j_n}$ whose associated r_j-balls cover A. Let ϵ be the minimum r_{j_k} for all $k \in \{1, 2 \ldots, n\}$. Then if y is in any $B_d(a_r, r_i)$ for $1 \leq i \leq n$, then $B_d(y, \epsilon) \subseteq B_d(a_i, r_i + \epsilon) \subset B_d(a_i, 2r_i) \subset U$. ☕

(e) Show the result in (d) need not hold if A is closed but not compact.

Proof. *From Online Solution Manual*
Consider $x \times 1/x$ for $x > 0$ in $\mathbb{R}_+ \times \mathbb{R}_+$. Then, the only open set which contains the subset $x \times 1/x$ is $\mathbb{R}_+ \times \mathbb{R}_+$, but no ϵ-neighborhood in $\mathbb{R}_+ \times \mathbb{R}_+$ contains $x \times 1/x$ because we can always choose $x \in \mathbb{R}_+$ large enough which lives outside of any ϵ-neighborhood of $x \times 1/x$. ☕

1. Recall that
$$S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid ||\vec{x}||^2 = 1\}.$$

(a) Prove that S^n is compact.

Proof. Since $f(x) = ||\vec{x}||^2$ is a continuous function and $\{1\}$ is closed, $S^n = f^{-1}(\{1\})$ is closed. Since $[-1,1]$ is compact, $[-1,1]^{n+1}$ is compact. Since $S^n \subseteq [-1,1]^{n+1}$ is closed subspace of a compact space, S^n is compact.

Alternate proof:
Since $f(x) = ||\vec{x}||^2$ is a continuous function and $\{1\}$ is closed, $S^n = f^{-1}(\{1\})$ is closed. Let d be the standard metric on $\mathbb{R}^{n+1}$. Let $\vec{x}, \vec{y} \in S^n$. Then, $d(\vec{x}, \vec{y}) \leq d(\vec{x}, 0) + d(0, \vec{y}) = 2$. So S^n is closed and bounded and thus compact.

(b) Use this to prove that $\mathbb{R}P(n)$ is compact.

Proof. Define the standard quotient map $q : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P(n)$ where each vector $\vec{z}$ is sent to it's equivalence class $[\vec{z}] = \{\vec{w} \in \mathbb{R}^{n+1} - \{0\} \mid \vec{w} = \lambda\vec{z}, \lambda \in \mathbb{R} - \{0\}\}$. Then define $i : S^n \to \mathbb{R}^{n+1}$ be the restriction of the identity map to S^n, i.e., i is the inclusion map. Then, $q \circ i$ is surjective and continuous since q is surjective and both q and i are continuous. Since the domain of $q \circ i$ is compact by part (a), then the image of $q \circ i$ is compact, namely $\mathbb{R}P(n)$.

(c) We say that $\vec{x}, \vec{y} \in S^n$ are antipodal if $\vec{x} = -\vec{y}$. Define an equivalence relation on S^n be letting $\vec{x} \sim \vec{y}$ is they are equal or antipodal. Prove that the quotient space from the relation is homeomorphic to $\mathbb{R}P(n)$.

Proof. Let X be the quotient space induced from the "antipodal" relation by the quotient map $p : S^n \to X$. Since S^n is compact and p is continuous and surjective, then X is compact. Define another quotient map $g : S^n \to \mathbb{R}P(n)$ by $g(\vec{x}) = [\vec{x}]'$ where $[\vec{x}]' = \{\vec{w} \in \mathbb{R}^{n+1} - \{0\} \mid \vec{w} = \lambda\vec{x}, \lambda \in \mathbb{R} - \{0\}\}$. Notice that
$$X = \{g^{-1}(\{[\vec{x}]\}) \mid [\vec{x}] \in \mathbb{R}P(n)\}.$$

Then, we are guaranteed the existence of a bijective continuous map $f : X \to \mathbb{R}P(n)$ by Corollary 22.3 (Munkres). Since $\mathbb{R}P(n)$ is Hausdorff (by previous homework), and f is a bijective continuous function whose domain is compact and whose range is Hausdorff then f is a homeomorphism.

(d) Let $\vec{e}_{n+1} \in \mathbb{R}^{n+1}$ be the vector all of whose entries is zero except the last, which is 1. It is the north pole of S^n. Embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as all vectors whose last entry is 0. There is a map
$$St_N : S^n - \{\vec{e}_{n+1}\} \to \mathbb{R}^n$$
that sends each point $\vec{w}$ on the sphere that is not the north pole to the point where the line spanned by $\vec{e}_{n+1}$ and $\vec{w}$ intersects the copy of $\mathbb{R}^n$ embedded so that

the last coordinate is zero. Find a formula for St_N and its inverse to prove that $S^n - \{\vec{e}_{n+1}\}$ is homeomorphic to $\mathbb{R}^n$. Use this to conclude that S^n is the one point compactification of $\mathbb{R}^n$.

Proof. **Most of this proof came from working with Alex Bates**
Let $\vec{p} = (p_0, p_1, \dots, p_n)$ be a point on S^n and $\vec{e}_{n+1} = (0, 0, \dots, 0, 1)$. Let

$$L_{\vec{p}}(t) = (1-t)\vec{e}_{n+1} + t\vec{p}$$

be the equation for the line between $\vec{e}_{n+1}$ and $\vec{p}$. Let $t_0 \in \mathbb{R}$. Since $\mathbb{R}^n$ is embedded into $\mathbb{R}^{n+1}$ as all vectors with last coordinate 0, we need to find the value for t_0 for which $L_{\vec{p}}$ is 0. To that end, observe that

$$L_{\vec{p}}(t_0) = (t_0 p_0, t_0 p_1, \dots, t_0 p_{n-1}, t_0 p_n + (t - t_0))$$

which means $t_0 p_n + (1 - t_0)$ must be 0, so that

$$t_0 = \frac{1}{1 - p_n}.$$

Therefore, given any point $\vec{p}$ on S^n, we define

$$St_N(\vec{p}) = L_{\vec{p}}\left(\frac{1}{1 - p_n}\right).$$

Let $\vec{x} = (x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1}$ so that $\vec{x}$ is a point that has been embedded into $\mathbb{R}^{n+1}$ from $\mathbb{R}^n$. To find the inverse map for St_N, we consider the equation for the line between$\vec{e}_{n+1}$ and $\vec{x}$

$$L_{\vec{x}}(t) = (1-t)\vec{e}_{n+1} + t\vec{p}$$

and determine the value for t_0 when $||L_{\vec{x}}(t_0)|| = 1$. This will be the value for t_0 when the line from $\vec{e}_{n+1}$ to $\vec{x}$ will pass through a point on S^n. So,

$$\begin{aligned} ||L_{\vec{x}}(t_0)|| &= ||(t_0 x_1, t_0 x_2, \dots t_0 x_{n-1}, 1 - t_0)|| \\ &= t_0^2 x^2 + t_0^2 x_2^2 + \cdots + t_0^2 x_{n-1}^2 + (1 - t_0)^2. \end{aligned}$$

Solving for t_0, we get

$$t_0 = \frac{2}{x_1^2 + x_2^2 + \cdots + x_n^2 + 1}.$$

Therefore,

$$St_N^{-1}(\vec{x}) = L_{\vec{x}}\left(\frac{2}{x_1^2 + x_2^2 + \cdots + x_n^2 + 1}\right).$$

Notice that St_N and its inverse are continuous since they are each the composition of continuous functions (namely, vector addition, scalar and vector multiplication). Since $S^n - \vec{e}_{n+1}$ is compact, $\mathbb{R}^n$ is Hausdorff, and St_N is a bijective continuous map, we have that $S^n - \vec{e}_{n+1}$ and $\mathbb{R}^n$ are homeomorphic. Since $R^n - (S^n - \vec{e}_{n+1}) = \vec{e}_{n+1}$ (i.e., a single point), then S^n is the one point compactification of $\mathbb{R}^n$. ☕

2. A topological group is a topological space G that is a group so that the multiplication on G is a continuous map
$$\mu : G \times G \to G$$
and taking the inverse defines a continuous map
$$\iota : G \to G$$

(a) Prove that $SL_2(\mathbb{C})$ equipped with matrix multiplication and matrix inversion is a topological group.

Proof. Let $\mathbf{A}, \mathbf{B} \in SL_2(\mathbb{C})$. Then for $1, j \in \{1, 2\}$, the ij-entry of the product $\mathbf{AB}$ is
$$(\mathbf{A}, \mathbf{B})_{i,j} = A_{i1}B_{1j} + A_{i2}B_{2j}.$$
Thus each component of the product is obtained through complex multiplication and addition, which are both continuous. Similarly, since the inverse of $\mathbf{A}$
$$\mathbf{A}^{-1} = [1/\det(A)]\,(Tr(A)I - A)$$
is the composition of continuous functions, then inversion is continuous. ☕

(b) Suppose $H \leq G$ is a subgroup. We define the quotient space G/H by saying $a, b \in G$ are equivalent if $a^{-1}b \in H$. Prove that if H is closed, then G/H is Hausdorff.

Proof. Notice that we can create a map $f : G \times G \to G$ by $f(a, b) = \mu(\iota(a), b)$. Since both μ and ι are continuous, so is f. Notice that the relation on G/H is precisely $f^{-1}(H)$. Since H is closed, so is it's continuous preimage under the continuous function f. Ans since the relation of G/H is closed, then G/H is Hausdorff. ☕

(c) Prove that $Stab(x)$ is a closed subgroup of G.

Proof. Fix $x \in X$. Since the map $\sigma_x : G \to X$ where $g \mapsto g.x$ is a homeomorphism, then
$$\sigma_x^{-1}(\{x\}) = \{g \in G \mid \sigma_x(g) = x\} = \{g \in G \mid g.x = x\} = Stab(x)$$
is closed since $\{x\}$ is closed and σ_x is continuous. ☕

(d) Prove that if G is compact and $\tau : G \times X \to X$ is continuous, transitive group action and X is Hausdorff, then for any $x \in X$, $G/Stab(x)$ is homeomorphic to X.

Proof. Define the map

$$f : G/Stab(x) \to X \quad \text{where} \quad [g] \mapsto g.x$$

We claim f is a homeomorphism. Notice that f is well-defined and injective because

$$\begin{aligned}[g] = [h] \iff h^{-1}g \in Stab(x) &\iff (h^{-1}g).x = x \\ &\iff g.x = h.x \\ &\iff f([g]) = f([h]).\end{aligned}$$

Since X is transitive, given any $y \in X$, we can find $g \in G$ so that $y = g.x$. So $f([g]) = g.x = y$ and thus f is surjective. Let $q : G \to G/Stab(x)$ be the quotient map induced by the relation $g \sim h$ if $h^{-1}g \in Stab(x)$. Then

$$f \circ q = \sigma_x$$

and since σ_x is continuous, then f is continuous by Theorem 22.2 (Munkres). Thus, f is a bijective continuous map from a compact space to a Hausdorff space, which means f is a homeomorphism. ☕

3. Let $\mathbb{Z}_2 = \{0, 1\}$ with the discrete topology. It is a group where 0 is the identity, $1 + 0 = 0 + 1 = 1$ and $1 + 1 = 0$.

   (a) Prove that $\mathbb{Z}_2$ is a compact Hausdorff topological group.

   Proof. Let $\{U_\alpha\}$ be an open cover of $\mathbb{Z}_2$. Take the sets U_{α_1} and U_{α_2} that contain 1 and 2, respectively, and there is your finite subcover of $\mathbb{Z}_2$, so that it is compact. Pick two distinct points in $\mathbb{Z}_2$, namely 0 and 1, the only two distinct points in $\mathbb{Z}_2$. Since we're in the discrete topology $\{0\}$ and $\{1\}$ are two disjoint open sets which contain 0 and 1, respectively, so that $\mathbb{Z}_2$ is Hausdorff. Under addition and inversion, the preimage of open sets (there are only 4 open sets in this topology) is certainly open. Thus we have a topological group. ☕

   (b) Let $G = \prod_{n \in \mathbb{N}} \mathbb{Z}_2$ be all sequences with values in $\mathbb{Z}_2$. Define the sum of two sequences to be the sequence whose values are the sums of the values of the two sequences. That is, if $(x_n), (y_n) \in G$, then $(x_n) + (y_n) = (x_n + y_n)$. Prove that G is a compact Hausdorff space and that G is a topological group.

   Proof. Since the product of compact spaces is compact by Tychonov's Theorem, G is compact. Similarly, since the product of Hausdorff spaces is Hausdorff (by a previous homework), G is Hausdorff. Since the product of groups is a group under componentwise addition, and the addition and inversion map for G is a composition of addition and inversion maps in each component, G is a topological group. ☕

(c) Let C be the standard Cantor set in the unit interval. It is the set of points in $[0,1]$ that can be written as $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ where $a_n \in \mathbb{Z}_2$. Prove that the map

$$S : \prod_{n\in\mathbb{N}} \mathbb{Z}_2 \to C$$

that sends each sequence a_n to $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ is a homeomorphism. Here we are assuming S is surjective. Prove that the map is injective and continuous, then use a theorem about bijective continuous maps from compact spaces to Hausdorff spaces.

Proof. **Solution from Alex Bates**
Suppose $S(a_n) = S(b_n)$. Then

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \implies \sum_{n=1}^{\infty} \frac{2(a_n - b_n)}{3^n} = 0 \implies a_n - b_n = 0\ \forall n \implies (a_n) = (b_n)$$

and so S is injective. Let $(x_n) \in G$ and V be a neighborhood of $S((x_n))$ in $\mathbf{C}$. Without loss of generality, let $\epsilon > 0$ and $V = (S((x_n)) - \epsilon, S((x_n)) + \epsilon) \cap \mathbf{C}$. By the Archimedean property, there exists $N \in \mathbb{N}$ so that $2/3^n < \epsilon$. Define $U = x_1 \times x_2 \times \ldots x_N \times x_{N+1} \times \prod_{n+2}^{\infty} \mathbb{Z}_2$. Then $(x_n) \in U$ and $S(U) \subseteq V$. Therefore, S is continuous.
Since $\mathbf{C}$ is a closed subset of the Hausdorff space $\mathbb{R}$, then $\mathbf{C}$ is Hausdorff. Since G is compact and S is bijective and continuous, then G and $\mathbf{C}$ are homeomorphic. ☕

§32, # 1 Show that a closed subspace of a normal space is normal.

Proof. Let A be a closed subspace of the normal space X. Let F and H be nonempty disjoint closed sets of A. Then there exists closed sets S, T of X so that

$$F = A \cap S \quad \text{and} \quad H = A \cap T.$$

Notice that since F and H are disjoint, so are S and T. Since S and T are closed in X and X is normal, there exists disjoint open sets U, V so that $S \subset U$ and $T \subset V$. Then $A \cap U, A \cap V$ are disjoint open sets in the subspace A for which

$$F \subset A \cap U \quad \text{and} \quad H \subset A \cap V.$$

§32, # 3 Show that every locally compact Hausdorff space is regular.

Proof. Let X be a locally compact Hausdorff space, $x \in X$ and U a neighborhood of x. By Theorem 29.2, there exists a neighborhood V of x so that $\overline{V} \subset U$. By Lemma 31.1, X is regular.

§33, # 1 Examine the proof of the Urysohnn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q,$$

p, q rational.

Proof. " $\subseteq$ " Suppose $x \in f^{-1}(r)$, that is $f(x) = r$. Recall that by construction we have

$$\ldots U_{r-2} \subset \overline{U}_{r-2} \subset U_{r-1} \subset \overline{U}_{r-1} \subset U_r \subset \overline{U}_r \subset U_{r+1} \subset \overline{U}_{r+1} \subset U_{r+2} \subset \overline{U}_{r+2} \ldots$$

By definition of f, $r = f(x) = \inf\{p \mid x \in U_p\}$, and so if $p > r$, then $x \in U_p$. Hence $x \in \bigcap_{p>r} U_p$. Also, if $q < r$, then $x \notin U_q$. Hence $x \in \bigcup_{q<r} U_q$.
" $\supseteq$ " Consider the following two facts from the proof of the Urysohn lemma:

(1) $x \in \overline{U_r} \implies f(x) \leq r$.

(2) $x \notin U_r \implies f(x) \geq r$.

For all $p > r$, since $x \in U_p$, then $x \in \overline{U}_p$. So by (1), $f(x) \leq p$ for all $p > r$, which implies $f(x) \leq r$. For all $q < r$ since $x \notin U_q$, then by (2) $f(x) \leq q$, which implies $f(x) \geq r$. Hence $f(x) = r$ so that $x \in f^{-1}(r)$.

§33, # 2 (a) Show that a connected normal space having more than one point is uncountable.

Proof. Let X be a connected normal space having more than one point. Suppose for contradiction that X is countable, (finite or infinite). Let $x, y \in X$. Then $\{x\}, \{y\}$ are closed sets. By Urysohn's lemma, there exists a continuous map $f : X \to [0,1]$ so that $f(x) = 0$ and $f(y) = 1$. Notice that since X is connected and $[0,1]$ is an ordered set in the order topology, then by the Intermediate Value Theorem, every point in $(0,1)$ is attained through f by some value in X. Hence, f is a surjective map from a countable space to an uncountable space, which is a contradiction. Therefore, X must be uncountable. ☕

(b) Show that a connected regular space having more than one point is uncountable. [*Hint:* Any countable space is Lindelöf.]

Proof. Let X be a connected regular space having more than one point. Suppose for contradiction that X is countable. Given any open cover of X, say $\{U_\alpha\}_{\alpha \in A}$, we can choose one element from the cover for each $x \in X$, and the collection $\{U_x\}_{x \in X}$ will be a countable cover of X. Therefore, X is Lindelöf. So, X is a regular space with a countable basis, and by Theorem 32.1, X is normal. Apply part (a) to arrive at a contradiction. ☕

§33, # 3 Give a direct proof of the Urysohn lemma for a metric space (X, d) by setting

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Proof. Let (X, d) be a metric space and A and B be disjoint closed subsets of X. Note that f is continuous since it is the composition of continuous functions. Namely, the metric function, the addition function in $\mathbb{R}$, and division by nonzero elements in $\mathbb{R}$. Notice that quantity $d(x, A) + d(x, B)$ will never equal 0. If so, then $d(x, A) = 0 = d(x, B)$ and by Exercise 2(a), §27 we have that $x \in \overline{A}$ and $x \in \overline{B}$. Since A and B are closed, $\overline{A} = A$ and $\overline{B} = B$. So, $x \in A \cap B$, which is a contradiction since A and B are disjoint.

If $x \in A$, then $d(x, A) = 0$ so that $f(x) = 0/(0 + d(x, B)) = 0$. Hence $f(A) = 0$. If $x \in B$, then $d(x, B) = 0$ so that $f(x) = d(x, A)/(d(x, A) + 0) = 1$. Hence $f(B) = 1$. ☕

§33, # 4 Recall that A is a "G_δ set" in X if A is the intersection of a countable collection of open sets of X.

Theorem. *Let X be normal. There exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$, if and only if A is a closed G_δ set in X.*

A function satisfying the requirements of this theorem is said to ***vanish precisely on A***.

Proof. " $\Rightarrow$ " First note that A is closed since $f^{-1}(\{0\}) = A$ and $\{0\}$ is closed in X. Notice that $\{0\} = \bigcap_{n\in\mathbb{Z}^+}[0, 1/n)$. Then,

$$A = f^{-1}(\{0\}) = f^{-1}\left(\bigcap_{n\in\mathbb{Z}^+}[0,1/n)\right) = \bigcap_{n\in\mathbb{Z}^+} f^{-1}([0,1/n).$$

For each n, since f is continuous and $[0, 1/n)$ is open in the subspace $[0, 1]$, then $f^{-1}([0, 1/n)$ is open. Therefore A is a closed G_δ set.
" $\Leftarrow$ " * From Online Solution Manual*
Since A is closed and there is a countable collection of open sets V_n whose intersection is A, let $U_1 = V_1$ and for every n, if $U_{1/n}$ is defined, let

$$A \subset U_{1/(n+1)} \subset \overline{U}_{1/(n+1)} \subset U_{1/n} \subset \overline{U}_{1/n} \cap V_{n+1}.$$

This is possible because the space is normal and A is closed. We need to slightly modify the construction in the proof of the Uryshon Lemma. Namely, we do no define U_0, we define a sequence of points in $\mathbb{Q} \cap (0, 1]$ starting from 1 such that no U_p is defined before $U_{1/n}$ is $p < 1/n$. So, first we define $U_1 = V_1$, then $U_{1/2}$, then some U_p for $1/2 < p < 1$, then $U_{1/3}$, then some U_p for $1/3 < p < 1$, etc. This way we define U_p for all rational points in $(0, 1]$ with the properties required by the proof. Moreover,

$$A \subseteq f^{-1}(0) \subseteq \bigcap_{p>0} U_p = A.$$

§35, # 3 Let X be metrizable. Show that the following are equivalent:

(i) X is bounded under every metric that gives the topology of X.

(ii) Every continuous function $\phi : X \to \mathbb{R}$ is bounded.

(iii) X is limit point compact.

[*Hint:* If $\phi : X \to \mathbb{R}$ is a continuous function, then $F(x) = x \times \phi(x)$ is an embedding of X in $X \times \mathbb{R}$. If A is an infinite subset of S having no limit point, let ϕ be a surjection of A onto $\mathbb{Z}_+$.]

Proof. $(iii) \implies (ii)$ Since X is metrizable and limit point compact, then X is compact by Theorem 28.2. Let $\phi : X \to \mathbb{R}$ be a continuous map. Then, the image of the compact space X under the continuous map ϕ is compact. In particular, $\phi(X)$ is bounded.
$(iii) \implies (i)$. Since X is metrizable and limit point compact, then X is compact by Theorem 28.2. Let d be a metric which gives the topology of X. Since X is compact, then X is bounded. Therefore, d is bounded.
$(ii) \implies (i)$ Let d be a metric which gives the topology of X. Since d is a continuous function $d : X \to \mathbb{R}$ then by (ii), d is bounded. Therefore, given any two points $x, y \in X$ we have $d(x, y)$ is bounded. Hence X is bounded.
$(ii) \implies (iii)$
$(i) \implies (ii)$ ☕

§35, # 4 Let Z be a topological space. If Y is a subspace of Z, we say that Y is a ***retract*** of Z if there is a continuous map $r : Z \to Y$ such that $r(y) = y$ for each $y \in Y$.

(a) Show that if Z is Hausdorff and Y is a retract of Z, then Y is closed in Z.

Proof. Let Y is a retract of the Hausdorff space Z by the continuous map $r : Z \to Y$. Let $x \in Z - Y$. We want to show that $x \notin \overline{Y}$, i.e., that there exists an open neighborhood of x which is disjoint from Y. Suppose $r(x) = y$. Then $x \neq y$ and since Z is Hausdorff, there exists disjoint neighborhoods U and V of x and y, respectively. Then, $V \cap Y$ is open in the subspace Y. Since r is continuous, then $r^{-1}(V \cap Y)$ is open in Z. Notice that $y \in V \cap Y$, so $r(x) = y \in V \cap Y$ and $r(y) = y \in V \cap Y$. Therefore, $x, y \in r^{-1}(V \cap Y)$. Let $W = r^{-1}(V \cap Y)$. We claim $W \cap U$ is a neighborhood of x which is disjoint from Y. Suppose $z \in W \cap U \cap Y$. Since $z \in Y$ then, $r(z) = z$. Since $z \in W$, then $z = r(z) \in V \cap Y$. So, $z \in V$ and $z \in U$, which cannot happen since U and V are disjoint. ☕

(b) Let A be a two-point set in $\mathbb{R}^2$. Show that A is not a retract of $\mathbb{R}^2$.

Proof. Let $A = \{p, q\}$ for some points $p, q \in \mathbb{R}^2$. By way of contradiction, suppose there was a map $f : \mathbb{R}^2 \to A$ which was continuous and $f(p) = p, f(q) = q$. Then f is surjective. Since $\mathbb{R}^2$ is connected and f is continuous, then by Theorem 23.5, the continuous image of a connected space is connected. But A is not connected. This is because A, being a subspace of a Hausdorff space, is Hausdorff. So that there exists disjoint open neighborhoods of p and q. The union of these neighborhoods equals A, which constitutes a separation of A. Therefore, A is not a retract of $\mathbb{R}^2$. ☕

(c) Let S^1 be the unit circle in $\mathbb{R}^2$; show that S^1 is a retract of $\mathbb{R}^2 - \{\mathbf{0}\}$, where $\mathbf{0}$ is the origin. Can you conjecture whether or not S^1 is a retract of $\mathbb{R}^2$?

Proof. Recall that $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Consider the map $f : \mathbb{R}^2 - \{\mathbf{0}\} \to S^1$ given by

$$f((x,y)) = \begin{cases} (x,y) & \text{if } (x,y) \in S^1 \\ \\ \dfrac{(x,y)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \notin S^1. \end{cases}$$

Then, f is continuous since it is the identity map on S^1 and is the composition of a continuous functions off of S^1. Off of S^1, we have a well defined function since $x^2 + y^2 \neq 0$ for any $(x, y) \in \mathbb{R}^2 - \{\mathbf{0}\}$.
And no, I cannot conjecture about whether S^1 is a retract of $\mathbb{R}$. Any map I try to come up with to meet the conditions of being a retract are not continuous. So, I don't really know! ☕

§51, # 1 Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. Since $h \simeq h'$ and $k \simeq k'$, there exists continuous maps

$$H : X \times I \to Y \quad \text{and} \quad K : Y \times I \to Y$$

for which

$$H(x,0) = h(x), H(x,1) = h'(x) \quad \text{and} \quad K(x,0) = k(x), K(x,1) = k'(x).$$

Consider the map $F(x,t) := K(H(x,t),t)$. This map is continuous because it is the composition of continuous functions. Notice that

$$F(x,0) = K(H(x,0),0) = k(H(x,0)) = k(h(x)) = (k \circ h)(x)$$

and

$$F(x,1) = K(H(x,1),1) = k'(H(x,1)) = k'(h'(x)) = (k' \circ h')(x).$$

Therefore, $k \circ h \simeq k' \circ h'$. ☕

§51, # 2 Given spaces X and Y, let $[X,Y]$ denote the set of homotopy classes of X into Y.

(a) Let $I = [0,1]$. Show that for any X, the set $[X,I]$ has a single element.

Proof. Let $f, f' : X \to I$ be maps where $f \equiv 0$ on X and $f'(x)$ be any continuous map. Then $F(x,t) = tf'(x)$ is a homotopy between f and f' because

$$F(x,0) = 0 \cdot f'(x) = 0 = f \quad \text{and} \quad F(x,1) = 1 \cdot f'(x) = f'(x)$$

Therefore any continuous map $f' : X \to I$ is homotopic to the zero map and hence $[X,I]$ has a single class. ☕

(b) Show that if Y is path connected, the set $[I,Y]$ has a single element.

Proof. Fix $y \in Y$, let $f \equiv y$ be a constant map on I, and let $h : I \to Y$ be a continuous map. We show that $f \simeq h$ so that all continuous maps from I to Y are homotopic to f and hence, $[I,Y]$ has a single element.

Suppose $h(0) = a$. Since Y is path connected, there exists a path $g : I \to Y$ from a to y, i.e., g is continuous and $g(0) = a, g(1) = y$. Consider the map

$$F(x,t) = \begin{cases} h((1-2t)x) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Then

$$F(x,0) = h((1-2\cdot 0)x) = h(x) \quad \text{and} \quad F(x,1) = g(1) = y = f(x).$$

F is continuous because both h and g are continuous and each of the pieces of F agree when $t = 1/2$:

$$h((1-2(1/2))x) = h(0) = a = g(0) = g(2(1/2)-1)$$

☕

§51, # 3 The space X is said to be ***contractible*** if the identity map $i_X : X \to X$ is nulhomotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

Proof. The identity map i_I is homotopic to the constant map $f \equiv 0$ on I by the homotopy

$$F(x,t) = x(1-t)$$

since $F(x,0) = x = i_I$ and $F(x,1) = 0 = f$. Similarly, the identity map $i_{\mathbb{R}}$ is homotopic to the constant map $f \equiv 0$ on $\mathbb{R}$ by the same homotopy. ☕

(b) Show that a contractible space is path connected.

Proof. Let X be a contractible space and $a, b \in X$. For some fixed $x_0 \in X$, there exists a homotopy $F(x,t)$ between i_X and a the constant map $f \equiv x_0$. Then $F(a,t)$ is a path in X from a to x_0 since F is continuous and

$$F(a,0) = a \quad \text{and} \quad F(a,1) = x_0.$$

Similarly, $\hat{F}(b,t) = F(b, 1-t)$ is a path in X between x_0 and b since F is continuous and

$$\hat{F}(b,0) = F(b,1) = x_0 \quad \text{and} \quad \hat{F}(b,1) = F(b,0) = b.$$

Consider the map

$$H(t) = \begin{cases} F(a,2t) & t \in [0, \frac{1}{2}] \\ \hat{F}(b, 2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Then $H(0) = F(a,0) = a$, $H(1) = \hat{F}(b,1) = b$, and

$$H(1/2) = F(a,1) = x_0 = F(b,1) = \hat{F}(b,0).$$

Therefore, H is a path between a and b and hence X is path connected. ☕

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

Proof. Let $h : X \to Y$ be any continuous map. Since Y is contractible, there exists a homotopy $F(y,t)$ between i_Y and a constant map $g \equiv y_0$ on Y for some fixed $y_0 \in Y$. Then the following map is a homotopy between h and f, so that all maps from X to Y are homotopic to the constant map y_0:

$$H(x,t) = F(h(x), t)$$

☕

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof. Since X is contractible, there exists a homotopy $F(x,t)$ between the identity map i_X and the constant map $f \equiv x_0$ for some fixed $x_0 \in X$. Let $h : X \to Y$ be any continuous map. Fix $y \in Y$. Since Y is path connected, there exists a path $p : I \to Y$ such that $p(0) = h(x_0)$ and $p(1) = y$. Then

$$G(x,t) = \begin{cases} h(F(x,2t)) & t \in [0, \frac{1}{2}] \\ p(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

is a homotopy between h and f, so that every continuous map from X to Y is homotopic to f. Lets check that:

$$G(x,0) = h(F(x,0)) = h(x) \quad \text{and} \quad G(x,1) = p(1) = y$$

and

$$G(x,1/2) = h(F(x,1)) = h(x_0) = p(0) = p(2(1/2)-1).$$

☕

§52, # 1 A subset A of $\mathbb{R}^n$ is said to be ***star convex*** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(a) Find a star convex that is not convex.

Proof. Consider the set $A = \{(x,0) \in \mathbb{R}^2 \mid |x| \leq 1\} \cup \{(0,y) \in \mathbb{R}^2 \mid |y| \leq 1\}$, i.e., the x-axis between -1 and 1, and the y-axis between -1 and 1. Let $a_0 = (0,0)$. If (x,y) is any point in A, then either x or y is zero. Without loss of generality, suppose $y = 0$ and $|x| \leq 1$. Then the line

$$L_1 = (t-1)(0,0) + t(x,0) = (0,0) + (tx, 0)$$

for $0 \leq t \leq 1$ lies completely in A since $|tx| \leq |x| \leq 1$. So A is star convex. However, the line

$$L_2 = (1-t)(0,1) + t(1,0)$$

for $0 \leq t \leq 1$ does not lie completely in A since the point $(1/2, 1/2)$ is a point in L_2, but this point is not in A. Thus, A is not convex. ☕

(b) Show that if A is star convex, A is simply connected.

Proof. Let $x, y \in A$. Then $p_1(t) = (1-t)x + ta_0$ is a path from x to a_0 and $p_2(t) = (1-t)a_0 + tx$ is a path from a_0 to y. So,

$$p_3(t) = \begin{cases} p_1(2t) & t \in [0, \frac{1}{2}] \\ p_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

is a path from x to y since $p_3(0) = x, p_3(1) = y$ and $p_1(2(\frac{1}{2})) = p_2(2(\frac{1}{2}) - 1) = a_0$. Thus A is path connected. Consider the fundamental group $\pi_1(A, a_0)$. Given any continuous map $f : I \to A$, the homotopy $H(x,t) = (1-t)a_0 + tf(x)$ shows that all continuous maps in A are homotopic to the constant map a_0, so that $\pi_1(A, a_0)$ is trivial. Thus A is simply connected. ☕

§52, # 2 Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. Since $\gamma = \alpha * \beta$, then γ is a path in X from x_0 to x_2. So we have the map $\hat{\gamma} : \pi_1(X, x_0) \to \pi_1(X, x_2)$, where if f is a loop in X based at x_0, we have

$$\hat{\gamma}([f]) = [\overline{\gamma}] * [f] * [\gamma].$$

Then $\hat{\beta} \circ \hat{\alpha} : \pi_1(X, x_0) \to (X, x_2)$ and

$$\begin{aligned}\hat{\beta}\left(\hat{\alpha}([f])\right) &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta] \\ &= [\overline{\gamma}] * [f] * [\gamma] \\ &= \hat{\gamma}([f])\end{aligned}$$

☕

§52, # 4 Let $A \subset X$; suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a ***retraction*** of X onto A.) If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Proof. If g is a loop in A based at a_0, then $r \circ g = g$ since $r|_A \equiv i_A$. So, r_* is surjective since $r_*([g]) = [r \circ g] = [g]$ and g was arbitrary. ☕

§52, # 7 Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

Proof. Identity: Let $e_{x_0} \equiv x_0$ be the constant loop in G based at x_0. If $f \in \Omega(G, x_0)$ then

$$(e_{x_0} \otimes f)(s) = x_0 \cdot f(s) = f(s) \cdot x_0 = (f \otimes e_{x_0})(s).$$

Inverses: Let $f \in \Omega(G, x_0)$ and define. Then

$$(f \otimes f^{-1})(s) = f(s) \cdot f^{-1}(s) = e_{x_0},$$

and

$$(f^{-1} \otimes f)(s) = f^{-1}(s) \cdot f(s) = e_{x_0}.$$

Associativity: This follows from the associativity of $\cdot$:

$$(f \otimes g)(s) \otimes h(s) = (f(s) \cdot g(s)) \cdot h(s) = f(s) \cdot (g(s) \cdot h(s)) = f(s) \otimes (g \otimes h)(s).$$

☕

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

Proof. Let $[f], [g] \in \pi_1(G, x_0)$. We show that the operation $[f] \otimes [g] = [f \otimes g]$ is well-defined. To that end, suppose $[f] = [f']$ and $[g] = [g']$. Since $f \simeq f'$ and $g \simeq g'$, there exists homotopies H, F so that

$$H_0(s) = f(s), H_1(s) = f'(s) \quad \text{and} \quad F_0(s) = g(s), F_1(s) = g'(s)$$

Define $J(s,t) = H(s,t) \cdot F(s,t)$. Then J is a homotopy between $f \otimes g$ and $f' \otimes g$ because

$$J_0(s) = H_0(s) \cdot F_0(s) = f(s) \cdot g(s) \quad \text{and} \quad J_1(s) = H_1(s) \cdot F_1(s) = f'(s) \cdot g'(s).$$

Therefore,

$$[f] \otimes [g] = [f \otimes g] = [f' \otimes g'] = [f'] \otimes [g'].$$

The identity element, inverses, and associativity of this operation on $\pi_1(G, x_0)$ follow in a very similar way as shown in part (a) so that $\pi_1(G, x_0)$ with the operation $\otimes$ forms a group. ☕

(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]

Proof. Let $[f], [g] \in \pi_1(G, x_0)$. We claim

$$[f] \otimes [g] = [f \otimes g] = [f * g] = [f] * [g]$$

To verify this, let f, g be representatives of $[f], [g]$, respectively. Let $[e_{x_0}]$ be the identity of $\pi_1(G, x_0)$. Then

$$\begin{aligned}((f * e_{x_0}) \otimes (e_{x_0} * g))(s) &= \begin{cases} (f \cdot e_{x_0})(2s) & \text{if } s \in [0, \frac{1}{2}] \\ (e_{x_0} \cdot g)(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases} \\ &= \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases} \\ &= f * g.\end{aligned}$$

Since $f * e_{x_0} \simeq f$ and $e_{x_0} * g \simeq g$ there exists homotopies H, F so that

$$H_0(s) = (f * e_{x_0})(s), H_1(s) = f(s) \quad \text{and} \quad F_0(s) = (e_{x_0} * g)(s), F_1(s) = g(s).$$

Define $J(s,t) = H(s,t) \cdot F(s,t)$. Then J is a homotopy between $(f * e_{x_0}) \otimes (e_{x_0} * g)$ and $f \otimes g$ because

$$\begin{aligned}J(s,0) &= H(s,0)F(s,0) = (f * e_{x_0})(s) \cdot (e_{x_0} * g)(s) = (f * e_{x_0}) \otimes (e_{x_0} * g) \\ J(s,1) &= H(s,1)F(s,1) = f(s) \cdot g(s) = (f \otimes g)(s)\end{aligned}$$

Therefore,

$$f * g = (f * e_{x_0}) \otimes (e_{x_0} * g) \simeq f \otimes g.$$

☕

(d) Show that $\pi_1(G, x_0)$ is abelian.

Proof. Let f, g be representatives of $[f], [g] \in \pi_(G, x_0)$, respectively. Notice that

$$(e_{x_0} * f) \otimes (g * e_{x_0}) = \begin{cases} (e_{x_0} \cdot g)(2t) & \text{if } t \in [0, \frac{1}{2}] \\ (f \cdot e_{x_0})(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} = g * f.$$

Recall that $f \simeq e_{x_0} * f$ and $g \simeq g * e_{x_0}$. Therefore,

$$[f] * [g] = [f] \otimes [g] = [e_{x_0} * f] \otimes [g * e_{x_0}] = [g * f] = [g] * [f].$$

☕

§53, # 3 Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a *k-fold covering* of B.

Proof. Define $C := \{b \in B \mid |p^{-1}(b)| = k\}$ and $D := B \setminus C$. Notice that $C \neq \emptyset$ since $b_0 \in C$. To get a contradiction, suppose $D \neq \emptyset$, i.e. there exists an element in B whose preimage under p noes not have k elements. Then $B = C \cup D$ and $C \cap D = \emptyset$. Therefore, if we can show that C and D are open, we have found a separation of B, contradicting the fact that B is connected.

Let $x \in C$. Since p is a covering map, there exists a neighborhood U of x so that $p^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha$ and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for all $\alpha \in A$. Since $x \in C$, then $|p^{-1}(x)| = k$, which means $p^{-1}(x)$ intersects $\{V_\alpha\}_{\alpha\in A}$ at exactly k points. Since the V_α are disjoint, then $|A| = k$. Thus, we can write $p^{-1}(U) = \bigsqcup_{i=1}^{k} V_i$. If $y \in U$, then $p^{-1}(y) \subseteq p^{-1}(U) = \bigsqcup_{i=1}^{k} V_i$. Since $p|_{V_i} : V_i \to U$ is a homeomorphism for each i, then each V_i must contain $p^{-1}(y)$, i.e., $|p^{-1}(y)| = k$. Thus $y \in C$. Therefore,

$$x \in U \subseteq C$$

i.e., C is open. Similarly, we get that D is open. ☕

§53, # 5 Show that the map of Example 3 is a covering map. Generalize to the map $p(z) = z^n$.

Proof. *** Adapted proof from online resource ***
That $p(z) = z^n$ is continuous and surjective is clear. Fix $z = e^{it} \in S^1$ and let

$$U = \left\{ e^{i\theta} \;\middle|\; t - \frac{\pi}{2} < \theta < t + \frac{\pi}{2} \right\}$$

be a neighborhood around z. Then $p^{-1}(U) = \bigsqcup_{k=1}^{n} V_k$ where

$$V_k = \left\{ e^{i\theta} \;\middle|\; (-1)^k t + (-1)^{k+1}\frac{\pi}{2n} < \theta < (-1)^k t + (-1)^k \frac{\pi}{2n} \right\}.$$

For an explicit example of the disjoint open sets V_k, consider $n = 2$. Then $p^{-1}(U) = V_1 \bigsqcup V_2$, where

$$V_1 = \left\{ e^{i\theta} \;\middle|\; -t + \frac{\pi}{4} < \theta < -t - \frac{\pi}{4} \right\} \quad \text{and} \quad V_2 = \left\{ e^{i\theta} \;\middle|\; t - \frac{\pi}{4} < \theta < t + \frac{\pi}{4} \right\}.$$

For all n, the open set U is a semicircle around z, and V_k is a "$1/2n$ circle" around z (for k even) and a "$1/2n$ circle" around$-z$ (for k odd). In the case $n = 2$, the open sets V_1 and V_2 are "$1/4$ circles" around z and $-z$, respectively. Moreover, the map $p|_{V_k} : V_k \to U$ is a homeomorphism for each k since the inverse map $p^{-1} : U \to V_k$ is well-defined and continuous. ☕

§53, # 6 Let $p : E \to B$ be a covering map.

(a) If B is Hausdorff, regular, completely regular, locally compact Hausdorff, then so is E. [*Hint:* Is $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices, and C is a closed set of B such that $C \subset U$, then $p^{-1}(C) \cap V_\alpha$ is a closed set of E.]

Proof. $\underline{B \text{ Hausdorff} \implies E \text{ Hausdorff}}$

Let $a, b \in E$ be distinct points. First suppose $p(a) = p(b) = c$. Then there is an admissible open set U in E containing c. Let $\{V_\alpha\}$ be the partition of $p^{-1}(U)$ into slices. Then there exists β, γ so that $a \in V_\beta$ and $b \in V_\gamma$. Since $p|_{V_\beta}$ and $p|_{V_\gamma}$ are injective, then $\beta \neq \gamma$. Therefore V_β and V_γ are disjoint open sets containing a and b, respectively.

Now suppose $p(a) \neq p(b)$. Then There exists disjoint open sets $U, V \subset B$ respectively containing $p(a), p(b)$. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint open sets respectively containing a, b.

$\underline{B \text{ regular} \implies E \text{ regular}}$

Let $x \in E$ and U a neighborhood of x. We will show that there is a neighborhood V of x for which $\overline{V} \subset U$.

Let W be the admissible neighborhood of $p(x)$ and let $\{V_\alpha\}$ be the partition of $p^{-1}(W)$ into slices. Let V_β be the slice containing x. Then $V_\beta \cap U$ is an open set in E containing x. Since $p|_{V_\beta}$ is homeomorphic to W, then $p(V_\beta \cap U)$ is homeomorphic to an open subset Y of W. Since B is regular and $p(x) \in Y$, there exists an open set Z containing $p(x)$ for which $\overline{Z} \subset Y$.

Now, $p^{-1}(Z) \cap V_\beta$ is an open set in E containing x which is contained in $V_\beta \cap U$. Using the hint, $p^{-1}(\overline{Z}) \cap V_\beta$ is closed in E. Then, setting $V = p^{-1}(Z) \cap V_\beta$, we get

$$x \in V \subset \overline{V} \subseteq p^{-1}(\overline{Z}) \cap V_\beta \subset V_\beta \cap U \subset U.$$

$\underline{B \text{ locally compact Hausdorff} \implies E \text{ locally compact Hausdorff}}$

We've already shown that since B is Hausdorff, then so is E. Let W be the admissible neighborhood of $p(x)$ and let $\{V_\alpha\}$ be the partition of $p^{-1}(W)$ into slices. Let V_β be the slice containing x. Since B is locally compact, then so is W. Since $p|_{V_\beta}$ is homeomorphic to W, and W is locally compact, then so is V_β. Then V_β contains a compact subspace C that contains a neighborhood of x. Since C is compact in V_β, then it is compact in E as well. Thus E is locally compact. (see definition at the beginning of §29).

☕

(b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

Proof. ☕

§54, # 3 Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

Proof. We need to show that $\alpha * \beta = p \circ (\tilde{\alpha} * \tilde{\beta})$. Since $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts of α and β, respectively, then $\alpha = p \circ \tilde{\alpha}$ and $\beta = p \circ \tilde{\beta}$. Since $\alpha(1) = \beta(0)$ and $\tilde{\alpha}(1) = \tilde{\beta}(0)$, then each piece of the piecewise maps below agree at the point $t = 1/2$, and thus are continuous by the pasting lemma:

$$\begin{aligned} p \circ (\tilde{\alpha} * \tilde{\beta}) &= \begin{cases} p \circ (\tilde{\alpha})(2t) & \text{if } t \in [0, \frac{1}{2}] \\ p \circ (\tilde{\beta})(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \\ &= \begin{cases} \alpha(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \beta(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \\ &= \alpha * \beta. \end{aligned}$$

☕

§54, # 5 Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 4 of §53. Consider the path

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$$

in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\tilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$, and sketch it.

Solution: Recall that $p(x) = (\cos 2\pi x, \sin 2\pi x)$ and hence $p \times p$ is defined by

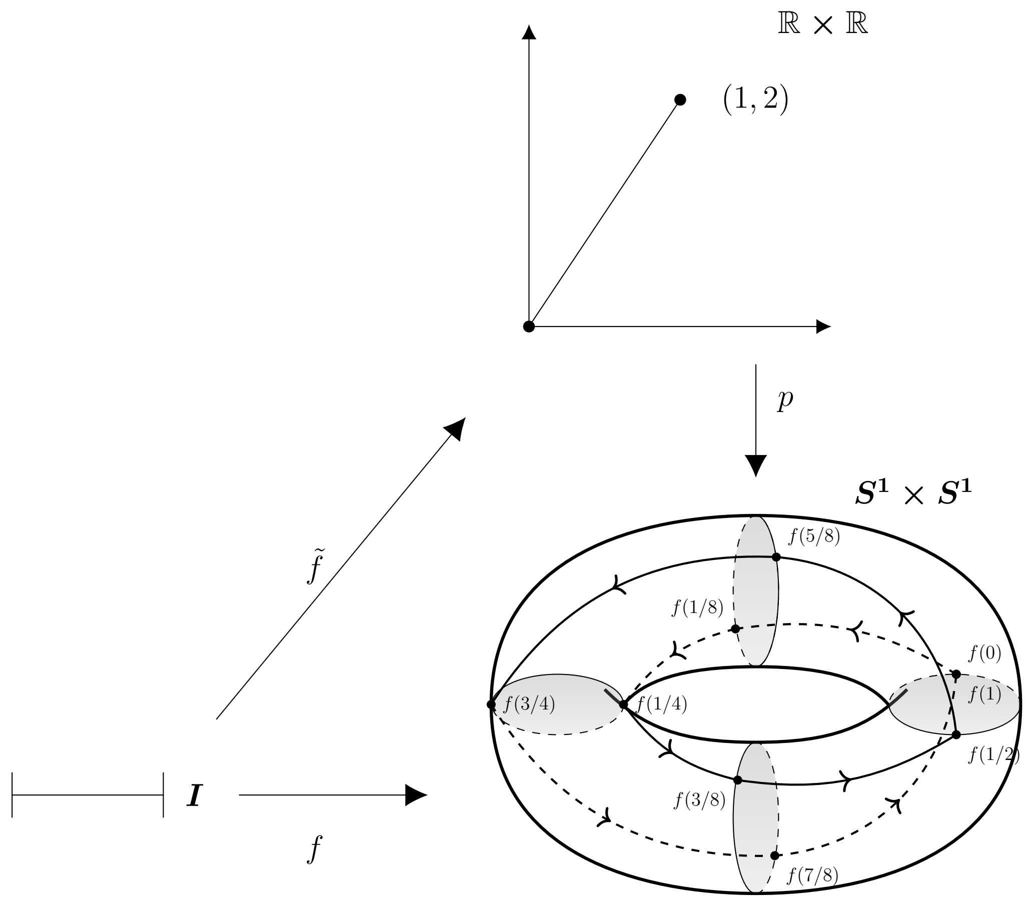
$$(x, y) \mapsto (\cos 2\pi x, \sin 2\pi x) \times (\cos 2\pi y, \sin 2\pi y).$$

Define $\tilde{f}(t) = (t, 2t)$. Then

$$p \times p\ (\tilde{f}(t)) = p \times p\ (t, 2t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$$

and thus $\tilde{f}$ is a lift of f. Notice that $\tilde{f}$ maps the interval $I = [0, 1]$ to the straight line between the origin and the point $(1, 2)$ in $\mathbb{R} \times \mathbb{R}$. We use the following points to sketch the path on $S^1 \times S^1$:

$$\begin{aligned} f(0) = f(1) = (1,0) \times (1,0), &\quad f(1/8) = (\sqrt{2}/2, \sqrt{2}/2) \times (0,1), \\ f(1/4) = (0,1) \times (-1,0), &\quad f(3/8) = (-\sqrt{2}/2, \sqrt{2}/2) \times (0,-1), \\ f(1/2) = (-1,0) \times (1,0), &\quad f(5/8) = (-\sqrt{2}/2, -\sqrt{2}/2) \times (0,1), \\ f(3/4) = (0,-1) \times (-1,0), &\quad f(7/8) = (\sqrt{2}/2, -\sqrt{2}/2) \times (0,-1), \end{aligned}$$



§54, # 6 Consider the maps $g, h : S^1 \to S^1$ given by $g(z) = z^n$ and $h(z) = 1/z^n$. (Here we represent S^1 as the set of complex numbers z of absolute value 1.) Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [*Hint:* Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.]

Proof. We know that the generator of $\pi_1(S^1, b_0)$ is $\alpha(t) = \cos 2\pi t + i\sin 2\pi t$, which is a loop that wraps around S^1 exactly once. Thus it suffices to determine where g_* and h_* send $[\alpha]$. Using the hint, we have

$$g \circ \alpha = (\alpha(t))^n = (\cos 2\pi t + i\sin 2\pi t)^n = (\cos 2\pi nt + i\sin 2\pi nt).$$

So, $g \circ \alpha$ is a loop that wraps around the circle n times in unit time, i.e., as "time" t goes from 0 to 1, $g \circ \alpha$ wraps around the circle n times while α wraps around the circle once in the same time frame. Thus

$$g_*([\alpha]) = [g \circ \alpha] = [\cos 2\pi nt + i\sin 2\pi nt] = [\alpha * \alpha * \cdots * \alpha] = [\alpha] * [\alpha] * \cdots * [\alpha].$$

We have a similar homomorphism for h_*, except $h \circ \alpha$ wraps around the circle n times in the opposite direction of α.

$$h \circ \alpha = (\alpha(t))^{-n} = (\cos 2\pi t + i\sin 2\pi t)^{-n} = (\cos 2\pi(-nt) + i\sin 2\pi(-nt)),$$

which gives

$$h_*([\alpha]) = [h \circ \alpha] = [\cos 2\pi(-nt) + i\sin 2\pi(-nt)] = [\overline{\alpha} * \overline{\alpha} * \cdots * \overline{\alpha}] = [\overline{\alpha}] * [\overline{\alpha}] * \cdots * [\overline{\alpha}].$$

where $\overline{\alpha}$ is the inverse map of α. ☕

§55, # 3 Show that if A is a nonsingular 3 by 3 matrix having nonnegative entries, then A has a positive real eigenvalue.

Proof. [1]

Let $T : \mathbb{R}^3 \to R^3$ be the linear transformation whose matrix is A. Let $B = S^2 \cap O^1$ where O^1 is the octant in $\mathbb{R}^3$ consisting of those points whose coordinates are all nonnegative. Then B is homeomorphic to the ball B^2, so that the fixed point theorem holds for continuous maps of B into itself.

Now if $x = (x_1, x_2, x_3)$ is in B, then all of the components of x are nonnegative and at least one is positive. because all entries of A are nonnegative, the vector $T(x)$ is a vector all of whose components are nonnegative. Moreover, $T(x) \neq 0$ since A is nonsingular so that $T(x)$ has at least one positive component. Therefore, the map $x \to T(x)/||T(x)||$ is a well-defined continuous map of B onto itself, which therefore has a fixed point x_0. So $x_0 = T(x_0)/||T(x_0)||$, which gives

$$T(x_0) = x_0||T(x_0)||,$$

i.e., A has eigenvalue $||T(x_0)|| > 0$ with eigenvector x_0. ☕

[1]This is essentially the same proof as the one given for Corollary 55.7, except we use the fact that A is nonsingular to note that $T(x) \neq 0$ so that the map $x \to T(x)/||T(x)||$ is well-defined.

§55, # 4 Suppose that you are given the fact that for each n, there is no retraction $r : B^{n+1} \to S^n$. (This result can be proved using more advanced techniques of algebraic topology.)

Proof. We first prove a generalization of (1) $\iff$ (2) in Lemma 55.3. That is, if $h : S^n \to X$ is a continuous map, then

(1) h is nullhomotopic. $\iff$ (2) h extends to a continuous map $k : B^{n+1} \to X$.

($\Rightarrow$) Let $H : S^n \times I \to X$ be a homotopy between h and a constant map. Define $\pi : S^n \times I \to B^{n+1}$ by

$$\pi(x,t) = (1-t)x.$$

Then π is continuous, surjective, and closed so that it is a quotient map. Notice that π sends $S^n \times 1$ to $\mathbf{0}$ and is injective elsewhere. Being injective for $t \neq 1$, H is constant on $\pi^{-1}(x,t)$ for all x and $t \neq 1$. Moreover, H is constant on $\pi^{-1}(\mathbf{0}) = S^n \times 1$ because $H(\pi^{-1}(\mathbf{0})) = H(S^1, 1)$ is a constant map.

Therefore, H induces (by Theorem 22.2, via the quotient map π), a continuous map $k : B^{n+1} \to X$. Since $H = k \circ \pi$, then $h(S^n) = H(S^n, \mathbf{0}) = k(\pi(S^n, \mathbf{0})) = k(S^n)$, i.e., h and k agree on the domain S^n. Hence, k is an extension of h to B^{n+1}.

($\Leftarrow$) If k is an extension of h, then

$$k \circ \pi : S^n \times I \to B^{n+1} \to X$$

is a homotopy between h and a constant map because

$$k(\pi(S^n, 0)) = k(S^n) = h(S^n) \quad \text{and} \quad k(\pi(S^n, 1)) = k(\mathbf{0}) = H(\pi^{-1}(\mathbf{0})) = H(S^1, 1),$$

and the latter is a constant map. ☕

(a) The identity map $i : S^n \to S^n$ is not nullhomotopic.

Proof. If i were nullhomotopic, then the continuous extension $k : B^{n+1} \to S^n$ is a retraction of B^{n+1} onto S^n, a contradiction. ☕

(b) The inclusion map $j : S^n \to R^{n+1} - \mathbf{0}$ is not nullhomotopic.

Proof. Suppose j were nullhomotopic. Then it extends to a continuous map $k : B^{n+1} \to R^{n+1} - \mathbf{0}$. There is a retraction $r : R^{n+1} - \mathbf{0} \to S^n$ given by $r(x) = x/||x||$. But then $r \circ k$ is a retraction of B^{n+1} onto S^n, a contradiction. ☕

(c) Every nonvanishing vector field on B^{n+1} points directly outward at some point of S^n, and directly inward at some point of S^n.

Proof. Let $v : B^{n+1} \to \mathbb{R}^{n+1} - \mathbf{0}$ be a continuous map so that $(x, v(x))$ is a nonvanishing vector field on B^{n+1}. Suppose that $v(x)$ does not point directly inward at any point x of S^n. Let w be the restriction of v to S^n. Since w extends to a map of B^{n+1} into $\mathbb{R}^{n+1} - \mathbf{0}$, it is nullhomotopic.

We claim that w is homotopic to to the inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$. Consider the homotopy

$$F : S^n \times I \to \mathbb{R}^{n+1} \quad \text{where} \quad F(x,t) = tx + (1-t)w(x).$$

It is clear that $F(x,1) = x = j(x)$ on S^n and $F(x,0) = w(x)$. However, we need to show that $F(x,t) \neq \mathbf{0}$ so that F maps $S^n \times I$ into $\mathbb{R}^{n+1} - \mathbf{0}$. When $t = 0$, then $F(x,t) = w(x) \neq \mathbf{0}$ since w maps into $\mathbb{R}^{n+1} - \mathbf{0}$. When $t = 1$, then $F(x,t) = j(x) \neq \mathbf{0}$ since $x \neq \mathbf{0}$ for all $x \in S^n$. Now, if $F(x,t) = \mathbf{0}$ for any $t \in (0,1)$, then

$$tx + (1-t)w(x) = \mathbf{0} \implies w(x) = -\frac{t}{(1-t)}x$$

i.e., $w(x)$ equals a negative scalar multiple of x so that $w(x)$ points directly inward at x, which we're assuming cannot happen. So F indeed maps $S^n \times I$ into $\mathbb{R}^{n+1} - \mathbf{0}$. Therefore, j is nullhomotopic, a contradiction to part (b). Therefore, v points directly inward at some point of S^n. We apply the result just proved to the nonvanishing vector field $(x, -v(x))$ to show that v points directly outward at some point of S^n. ☕

(d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.

Proof. Suppose that $f(x) \neq x$ for every $x \in B^{n+1}$. Then defining $v(x) = f(x) - x$ gives a nonvanishing vector field $(x, v(x))$ on B^{n+1}. But the vector field v cannot point directly outward at any point x of S^n, for that would mean

$$f(x) - x = ax$$

for some *positive* real number a, so that $f(x) = (1+a)x$ would lie outside the unit ball B^{n+1}, a contradiction to (c). ☕

(e) Every $n+1$ by $n+1$ matrix with positive real entries has a positive eignenvalue.

Proof. Let A be a $n+1$ by $n+1$ matrix of positive real numbers and let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be its linear transformation. Let B be the intersection of the n-sphere, S^n, with the collection of points $\{(x_1, x_2, \ldots, x_{n+1}) \mid x_i \geq 0 \ \forall i\} \subset \mathbb{R}^{n+1}$. Then B is homeomorphic to B^{n+1}, so that part (d) holds for continuous maps of B onto itself.

Now if $x = (x_1, x_2, \ldots, x_{n+1})$ is in B, then all components of x are nonnegative and at least one is positive. Because all entries of A are positive, the vector $T(x)$ is a vector all of whose components are positive. As a result, the map $x \mapsto T(x)/||T(x)||$ is a continuous map of B to itself, which therefore has a fixed point x_0. Then

$$T(x_0) = ||T(x_0)||x_0,$$

so that T (and therefore the matrix A) has a positive real eigenvalue $||T(x_0)||$. ☕

(f) If $h : S^n \to S^n$ is nullhomotopic, then h has a fixed point and h maps some point x to is antipode $-x$.

Proof. Since h is nullhomotopic, then it extends to a map $k : B^{n+1} \to S^n$. If $j : S^n \to B^{n+1}$ is the inclusion map then $j \circ k : B^{n+1} \to B^{n+1}$ is continuous map from B^{n+1} into B^{n+1} and thus has a fixed point x_0 by part (d). Then

$$x_0 = j(k(x_0)) = j(h(x_0)) = h(x_0),$$

and thus h has a fixed point. Now, let $f : S^n \to S^n$ be given by $f(x) = -x$. Then $f \circ h$ is nullhomotopic so that it extends to a continuous map $k' : B^{n+1} \to S^n$. Then $j \circ k'$ has a fixed point x_1 so that

$$x_1 = j(k'(x_1)) = j(f(h(x_1))) = j(-h(x_1)) = -h(x_1),$$

i.e., $h(x_1) = -x_1$ so that h maps x_1 to its antipode. ☕

§56, # 1 Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with real or complex coefficients. Show that if $|a_{n-1}| + \ldots |a_1| + |a_0| < 1$, then *all* the roots of the equation lie interior to the unit ball B^2. [*Hint:* Let $g(x) = 1 + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n$, and show that $g(x) \neq 0$ for $x \in B^2$.]

Proof. Let $f(x)$ be the given polynomial equation and suppose $g(x) = 0$ for some $x \in B^2$. Then

$$1 = -a_{n-1} - \cdots - a_1 - a_0 \implies 1 \leq |a_{n-1}| + \cdots + |a_1| + |a_0|,$$

which is a contradiction. Notice that $f(1/x) = g(x) \cdot x^n$. Thus if x is a root of g, then $1/x$ is a root of f. Since all the roots of g lie outside B^2, then if x is a root of g, $|x| > 1$, which implies $\frac{1}{|x|} < 1$, i.e., $1/x \in B^2$. Therefore all of the roots of f lie inside B^2. ☕

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Chapter 5 Exercises

2. Prove that a space X is locally path connected if and only if each path component of each open subset of X is open.

Proof. ($\Rightarrow$) Let U be open in X and C a path component of U with $x \in C$. Since X is locally path connected, there exists $V \subset U$ open, a neighborhood of x which is path connected. Since C is a path component of U, it is maximal with respect to path connectedness, i.e., C is not a proper subset of any path connected subset of U. In particular, C is not a proper subset of V and so $x \in V \subset C$, hence C is open.

($\Leftarrow$) Let $x \in X$ and $\mathcal{O}$ a neighborhood of x. Let C be the path component of $\mathcal{O}$ containing x. By hypothesis, C is open, and so C is a path connected neighborhood of x contained in $\mathcal{O}$. Therefore, X is locally path connected. ☕

5. **Definition.** A function $f : X \to Y$ is a *local homeomorphism* provided that each point x in X has an open neighborhood U such that f maps U homeomorphically onto $f(U)$.

   (a) Prove that every covering projection is a local homeomorphism.

   Proof. Let $p : X \to Y$ be a covering projection and $x \in X$. Then $p(x)$ has an admissible neighborhood U. Let V be the path component of $p^{-1}(U)$ containing x. Then V is a neighborhood of x in X that is homeomorphic to $p(V) = U$ in Y. Therefore p is a local homeomorphism. ☕

   (b) Give an example to show that a local homeomorphism may fail to be a covering projection.

   Proof. Consider the map $p : \mathbb{R}^+ \to S^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Given $x \in \mathbb{R}^+$, we can choose $\epsilon > 0$ small enough so that $V = (x - \epsilon, x + \epsilon) \subset \mathbb{R}^+$.

   However, if we consider the point $(0, 1) \in S^1$, we can choose any neighborhood U of $(0, 1)$ in so that

$$p^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}^+} (n - \delta, n + \delta) \sqcup (0, \delta).$$

   Then for $n \in \mathbb{Z}^+$, p maps $(n - \delta, n + \delta)$ homeomorphically to U, but $p((0, \delta)) \subsetneq U$. Thus $(0, 1)$ does not have an admissible neighborhood and hence p is not a covering projection. ☕

6. Let (E, p) be a covering space of B. Show that the family of admissible neighborhoods is a basis for the topology of B.

Proof. If $x \in B$, then by definition of a covering space, x is contained in an admissible neighborhood. If $x \in U_1 \cap U_2$ for two admissible neighborhood U_1, U_2 then there exists $U_3 \subset U_1 \cap U_2$ which is a path component of $U_1 \cap U_2$. Since U_3 is a path component of an admissible set, it is admissible. Therefore the collection of admissible neighborhoods is a basis for a topology on B. ☕

8. **Theorem 5.2** (The Covering Homotopy Property) *Let (E, p) be a covering space of B and $F : I \times I \to B$ a homotopy such that $F(0, 0) = b_0$. If e_0 is a point in E with $p(e_0) = b_0$, then there is a unique covering homotopy $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0, 0) = e_0$.*

   Proof. Given F, we first define $\tilde{F}(0, 0) = e_0$. Then we use to path lifting homotopy property to extend $\tilde{F}$ to $0 \times I$ and $I \times 0$. Now we extend $\tilde{F}$ to all of $I \times I$ as follows:

   Using the Lebesgue Number Lemma, we can choose subdivisions of I

$$s_0 < s_1 < \cdots < s_m$$
$$t_0 < t_1 < \cdots < t_n$$

   so that each rectangle

$$I_i \times I_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$$

   is mapped by F into an open set of B that is evenly covered by p. We now define $\tilde{F}$ inductively on $I \times I$, beginning with $I_1 \times J_1$ and continuing with the rectangles $I_i \times J - 1$ in the "bottom row," then with the rectangles $I_i \times J_2$ in the next row and so on.

   In general, given i_0 and j_0, define A to be the union of $0 \times I$, $I \times 0$ and all the rectangles "previous" to $I_{i_0} \times I_{j_0}$. That is,

$$A := (0 \times I) \cup (I \times 0) \cup \bigcup_{\substack{1 \leq i \leq n,\\ j < j_0}} I_i \times I_j \cup \bigcup_{i < i_0} I_i \times I_{j_0}.$$

   We assume $\tilde{F}$ is already defined on A and that $\tilde{F}$ is a continuous lifting of $F|_A$.

   We now define $\tilde{F}$ on $I_{i_0} \times I_{j_0}$. Choose and open set U of B that is evenly covered by p and contains the set $F(I_{i_0} \times I_{j_0})$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Now $\tilde{F}$ is already defined on the set $C = A \cap (I_{i_0} \times I_{j_0})$, which is the set of all rectangles previous to $I_{i_0} \times I_{j_0}$ and the bottom and left edges of $I_{i_0} \times I_{j_0}$ and thus C is connected. Therefore, $\tilde{F}(C)$ is connected and must lie entirely in some V_α. Suppose it lies in V_0. Then $p|_{v_0} : V_0 \to U$ is a homeomorphism so that we can extend $\tilde{F}$ on $I_{i_0} \times I_{j_0}$ by defining

$$\tilde{F}(x) = (p|_{V_0})^{-1} F(x) \text{ for all } x \in I_{i_0} \times I_{j_0}.$$

   This extension of $\tilde{F}$ will be continuous since it agrees on the overlap in C with where $\tilde{F}$ was previously defined. ☕

12. Prove that a homomorphism of covering spaces is a covering projection.

    Proof. Let (E_1, p_1) and (E_2, p_2) be covering spaces of B and h a homomorphism between them. Let $x \in E_2$ and $p_2(x) = b$. Let U be an admissible neighborhood of b so that each path component of $p_1^{-1}(U)$ is homeomorphic to U via p_1. Similarly, let V be an admissible neighborhood of b so that each path component of $p_2^{-1}(V)$ is homeomorphic to V via p_2. Now, let W be the path component of $U \cap V$ containing b. Then W is admissible since is is a path connected subset of an admissible set.

Let $\bigsqcup_{\alpha\in A} W_\alpha$ be a partition of $p_1^{-1}(W)$ into slices and let Y be the path component of $p_2^{-1}(W)$ containing e. Since p_1^{-1} is defined on W, then we have the following:

$$h^{-1}(Y) = p_1^{-1}(p_2(Y)) = p_1^{-1}(W) = \bigsqcup_{\alpha\in A} W_\alpha.$$

Moreover, since $p_1|_{W_\alpha} : W_\alpha \to W$ is a homeomorphism for each W_α and $(p_2|_Y)^{-1} : W \to Y$ is a homeomorphism, then so is their composition. Therefore,

$$h|_{W_\alpha} = (p_2|_Y)^{-1} \circ p_1|_{W_\alpha}.$$

is a homeomorphism for each W_α. Hence h is a covering map. ☕

13. Show that isomorphism of covering spaces is an equivalence relation.

Proof. Let $(E_1, p_1), (E_2, p_2), (E_2, p_3)$ be covering spaces of B. Reflexivity is clear since $E_1 \cong E_1$ via the identity map.

If $E_1 \cong_h E_2$, then $p_1 = p_2 h$. So $h^{-1} : E_2 \to E_1$ is a homeomorphism for which $p_2 = p_1 h^{-1}$ and thus $E_2 \cong_{h^{-1}} E_1$ so that $\cong$ is symmetric.

Finally, if $E_1 \cong_{h_1} E_2$ and $E_2 \cong_{h_2} E_3$, then $p_1 = p_2 h_1$ and $p_2 = p_3 h_2$. Then $h_2 h_1 : E_1 \to E_3$ is a homeomorphism for which $p_1 = h_2 h_1 p_3$ and hence $E_1 \cong_{h_2 h_1} E_3$. Therefore $\cong$ is an equivalence relation. ☕

16. Determine all covering spaces of the torus and exhibit a representative covering space from each isomorphism class.

Proof. The map covering space $(\mathbb{R}^2, p)$ where $p(s,t) = (\cos 2\pi s \sin 2\pi t)$ is the universal covering space of $S^1 \times S^1$ since $\pi_1(\mathbb{R}^2) = \{0\}$. Since the conjugacy classes $\pi_1(S^1 \times S^1)$ are in bijection with the covering spaces of $S^1 \times S^1$, we look at the conjugacy classes of the subgroups of $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z}$ is abelian, the conjugacy classes are simply the subgroups of $\mathbb{Z} \times \mathbb{Z}$. The subgroups of $\mathbb{Z} \times \mathbb{Z}$ fall into one of three classes of subgroups: $\{0\}, \{\langle m, n\rangle\}, \{\langle (a,b), (c,d)\rangle\}$. That is, the trivial subgroup, the subgroups generated by one element, and the subgroups generated by two elements. ☕

19. (a) Prove that the set $A(E,p)$ of all automorphisms of a covering space (E,p) is a group.

Proof. The identity in $A(E,p)$ is the identity map $\mathbb{1}_E$ since $p = \mathbb{1}p = p$. The inverse of $f \in A(E,p)$ is f^{-1} since f^{-1} is an isomorphism from E into itself for which $pf^{-1} = p$ and $ff^{-1} = \mathbb{1}_E = f^{-1}f$. Finally, if $f, g, h \in A(E,p)$, then $(fg)h = f(gh)$ since map composition is associative and $p(fg)h = (pf)gh = (pg)h = ph = p$.

☕

(b) Prove that members f, g of $A(E,p)$ must be identical or must agree at no point of E.

Proof. Since E is path connected then it is connected. Moreover $f, g : E \to E$ are homeomorphisms for which $pf = p = pg$. By Theorem 5.2, the set of points at which f and g agree is clopen, and since E is connected, this set must either be all of E or the empty set; this means f and g are identical or that they agree at no point of E. ☕

(c) Prove that the identity map is the only member of $A(E, p)$ that has a fixed point.

Proof. If $f \in A(E, p)$ for which there exists $x_0 \in E$ so that $f(x_0) = x_0$ then the set $\{x \in E \mid f(x) = \mathbb{1}_E\} \neq \emptyset$ and so by part (a), we have $f = \mathbb{1}_E$. ☕

20. Prove that if B is simply connected, then (B, i) is the universal covering space of B. (Here i denotes the identity map.)

Proof. Since B is simply connected, then $\pi_(B) = \{0\}$ so that $i_* : \{0\} \to \{0\}$ and thus (B, i) is a universal covering space of B. If (U, q) is a universal covering space, then

$$q_*\pi_1(U) = \{0\} = i_*\pi(B)$$

and so the conjugacy class determined by $q_*\pi_1(U)$ and $i_*\pi(B)$ is the trivial subgroup of B and hence $(B, i) \cong (U, q)$ by Theorem 5.9. ☕

21. Prove that the fundamental group $\pi_1(P^n)$ of the projective n-space P^n is isomorphic to the group of integers module 2 for $n \geq 2$. What about $n = 1$?

Proof. Since $\pi_1(S^n) = \{0\}$ for all $n \geq 2$, then (S^n, p) is the universal cover of P^n for all $n \geq 2$, where $p : S^2 \to P^n$ identifies antipodal points. Moreover, since p identifies pairs of antipodal points, then (S^n, p) has two automorphisms: the identity and the antipodal map. Thus $\pi_1(P^n) \cong A(S^n, p) \cong \mathbb{Z}/2\mathbb{Z}$. If $n = 1$, then P^1 is homeomorphic to S^1 so that $\pi(P^1) \cong \mathbb{Z}$. ☕