Homework for
Real Analysis

Nicholas Camacho
Department of Mathematics
University of Iowa
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Most exercises are from
Real Analysis (4th Edition) by Royden & Fitzpatrick.
For example, "6.2.19" means exercise 19 from
section 2 of chapter 6 in Royden & Fitzpatrick.
Beware: Some solutions may be incorrect!

1.1.2 Verify the following:

(i) For each real number $a \neq 0, a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.

Proof. Let $a \in \mathbb{R}, a \neq 0$. Notice that $a \cdot 0 = 0$ because

$$\begin{aligned} a0 &= a(0+0) \\ a0 &= a0 + a0 \\ a0 + (-a0) &= a0 + a0 + (-a0) \\ 0 &= a0 \end{aligned}$$

If $a > 0$, then $a^2 = aa > 0$ by the positivity axioms. If $a < 0$, then $(-a) > 0$ and so $(-a)^2 = (-a)(-a) > 0$. Then,

$$\begin{aligned} (-a)^2 &= (-a)(-a) \\ &= (-a)(-a) + 0 \\ &= (-a)(-a) + (a)0 \\ &= (-a)(-a) + (a)(-a+a) \\ &= (-a)(-a) + (a)(-a) + (a)(a) \\ &= (-a)(-a+a) + (a)(a) \\ &= (-a)0 + (a)(a) \\ &= 0 + (a)(a) \\ &= (a)(a) = (a)^2 \end{aligned}$$

and so $0 < (-a)^2 = a^2$. ■

(ii) For each positive number a, its multiplicative inverse, a^{-1} is also positive.

Proof. Let $a > 0$. Then, its multiplicative inverse, $a^{-1} \neq 0$. If $a^{-1} < 0$, then $-a^{-1} > 0$ and so

$$\begin{aligned} aa^{-1} &= 1 \\ (-1)aa^{-1} &= (-1)1 \\ a(-1)a^{-1} &= -1 \\ a(-a^{-1}) &= -1 \end{aligned}$$

and thus, the product of two positive real numbers is not positive, a contradiction to the positivity axioms. Thus, $a^{-1} > 0$. ■

(iii) If $a > b$, then

$$ac > bc \quad \text{if} \quad c > 0 \quad \text{and} \quad ac < bc \quad \text{if} \quad c < 0$$

Proof. Let $a > b$, i.e., $a - b > 0$ and so $a - b$ is positive, and suppose $c > 0$. Then

$$ac - bc = (a - b)c > 0$$

by the positivity axioms, and thus $ac > bc$. Similarly, suppose now that $c < 0$. Then, $-c > 0$ and so

$$bc - ac = (-b + a)(-c) = (a + (-b))(-c) = (a - b)(-c) > 0$$

by the positivity axioms, and thus $ac < bc$. ■

1.2.8 Use an induction argument to show that for each natural number n, the interval $(n, n + 1)$ fails to contain any natural number.

Proof. We will first show by induction that for any natural number, n, we must have $n - 1$ is a natural number for $n \neq 1$. First notice that the set $I = \{1\} \cup \{2\} \cup \{x \in \mathbb{R} | x > 2\}$ is an inductive set. Since I contains no numbers between 1 and 2, then neither do the natural numbers. Additionally, since all inductive sets contain 1, and $1 + 1 = 2$, then 2 is also in every inductive set and so 2 is a natural number. Thus, 2 is the next natural number after 1. So for the base case, $n = 2$, we have $2 - 1 = 1$ is a natural number. Now, suppose that for some natural number $k \geq 2$, $k - 1$ is a natural number. Then, $(k + 1) - 1 = k$ is a natural number by our inductive hypothesis. Thus, $n - 1$ is a natural number for all $n \geq 2$.
Now, we prove by induction that for each natural number n, the interval $(n, n+1)$ does not contain a natural number. For the base case, $n = 1$ we consider the interval $(1, 2)$. Suppose there were a natural number a so that $1 < a < 2$. This implies that $0 < a - 1 < 1$, i.e., there exists a natural number less than 1, a contradiction. Now, suppose that the claim holds for some k, a natural number greater than 1. That is, suppose the interval $(k, k + 1)$ fails to contain a natural number. Now, consider the interval $(k+1, k+2)$ and suppose there were a natural number b so that $k+1 < b < k+2$. This implies $k < b-1 < k+1$, i.e., a natural number lies in $(k, k + 1)$, a contradiction to our induction hypothesis. Thus, our claim holds for all natural numbers n. ■

1.3.25 Show that any two nondegenerate intervals of real numbers are equipotent.

Proof. Let $\sim$ be the equivalence relation that two sets are equipotent (equipotency is certainly reflexive, symmetric, and transitive). We claim that all nondegenerate intervals of real numbers create an equivalence class under $\sim$. We will first show that for $a, b \in \mathbb{R}, a < b$:

$$
\begin{aligned}
&(1): \\
&\quad (a,b) \sim (0,1) && \text{(a)} \\
&\quad (a,b] \sim (0,1] && \text{(b)} \\
&\quad [a,b) \sim [0,1) && \text{(c)} \\
&\quad [a,b] \sim [0,1] && \text{(d)} \\
&(2): \\
&\quad (0,1] \sim (0,1) && \text{(a)} \\
&\quad [0,1) \sim (0,1) && \text{(b)} \\
&\quad [0,1] \sim (0,1) && \text{(c)}
\end{aligned}
$$

and then show

$$
\begin{aligned}
&(3): \\
&\quad (b,\infty) \sim (0,1) && \text{(a)} \\
&\quad [b,\infty) \sim (0,1) && \text{(b)} \\
&\quad (-\infty,b) \sim (0,1) && \text{(c)} \\
&\quad (-\infty,b] \sim (0,1) && \text{(d)} \\
&\quad (-\infty,\infty) \sim (0,1) && \text{(e)}
\end{aligned}
$$

(1) *(a)* Define a function $g_1 : (a,b) \to (0,1)$ by $g_1(x) = (x-a)/(b-a)$. If $(x_1 - a)/(b-a) = (x_2 - a)/(b-a)$, then simple cancellation shows $x_1 = x_2$, and thus g_1 is injective. For any $c \in (0,1)$, let $x = c(b-a)+a$ and then $g_1(x) = c$, and thus g_1 is surjective. Thus, $(0,1) \sim (a,b)$.

(b) Define a function $g_2 : [a,b) \to [0,1)$ by $g_2(x) = (x-a)/(b-a)$. If $(x_1 - a)/(b-a) = (x_2 - a)/(b-a)$, then simple cancellation shows $x_1 = x_2$, and thus g_2 is injective. For any $c \in [0,1)$, let $x = c(b-a)+a$ and then $g_2(x) = c$, and thus g_2 is surjective. Thus, $[0,1) \sim [a,b)$.

(c) Define a function $g_3 : [a,b] \to [0,1]$ by $g_3(x) = (x-a)/(b-a)$. If $(x_1 - a)/(b-a) = (x_2 - a)/(b-a)$, then simple cancellation shows $x_1 = x_2$, and thus g_3 is injective. For any $c \in [0,1]$, let $x = c(b-a)+a$ and then $g_3(x) = c$, and thus g_3 is surjective. Thus, $[0,1] \sim [a,b]$.

(d) Define a function $g_4 : (a,b] \to (0,1]$ by $g_4(x) = (x-a)/(b-a)$. If $(x_1 - a)/(b-a) = (x_2 - a)/(b-a)$, then simple cancellation shows $x_1 = x_2$, and thus g_4 is injective. For any $c \in (0,1]$, let $x = c(b-a)+a$ and then $g_4(x) = c$, and thus g_4 is surjective. Thus, $(0,1] \sim (a,b]$.

(2) (a) Let $A = \{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{1}{n+1}$. Define $f_1 : [0,1) \to (0,1)$ by

$$f_1(x) = \begin{cases} x & \text{if } x \notin A \text{ or } x \neq 0 \\ a_1 & \text{if } x = 0 \\ a_2 & \text{if } x = a_1 \\ & \vdots \\ a_{n+1} & \text{if } x = a_n \\ & \vdots \end{cases}$$

Then, f_1 is injective since each point in $[0,1)$ is mapped to it's own unique point in $(0,1)$, and surjective because every point in $(0,1)$ has been accounted for: If $y \in (0,1)\backslash A$, then $f_1(y) = y$. If $y = a_1$, then $f(0) = a_1$. If $y \in A\backslash\{a_1\}$, then $f(a_{n-1}) = y$.

(b) For the case $(0,1] \sim (0,1)$, we let $f_2 : (0,1] \to (0,1)$ be defined by

$$f_2(x) = \begin{cases} x & \text{if } x \notin A \text{ or } x \neq 1 \\ a_1 & \text{if } x = 1 \\ a_2 & \text{if } x = a_1 \\ & \vdots \\ a_{n+1} & \text{if } x = a_n \\ & \vdots \end{cases}$$

Then, f_2 is injective since each point in $(0,1]$ is mapped to it's own unique point in $(0,1)$, and surjective because every point in $(0,1)$ has been accounted for: If $y \in (0,1)\backslash A$, then $f_2(y) = y$. If $y = a_1$, then $f(1) = a_1$. If $y \in A\backslash\{a_1\}$, then $f(a_{n-1}) = y$.

(c) And finally, for the case $[0,1] \sim (0,1)$, the construction of the function $f_3 : [0,1] \to (0,1)$ is similar:

$$f_3(x) = \begin{cases} x & \text{if } x \notin A, x \neq 0, \text{ or } x \neq 1 \\ a_1 & \text{if } x = 0 \\ a_2 & \text{if } x = 1 \\ a_3 & \text{if } x = a_1 \\ & \vdots \\ a_{n+2} & \text{if } x = a_n \\ & \vdots \end{cases}$$

Then, f_3 is injective since each point in $[0,1]$ is mapped to it's own unique point in $(0,1)$, and surjective because every point in $(0,1)$ has been accounted for: If $y \in (0,1) \setminus A$, then $f_3(y) = y$. If $y = a_1$, then $f(0) = a_1$. If $y = a_2$, then $f(1) = a_2$. If $y \in (A \setminus (\{a_1\} \cup \{a_2\}))$, then $f(a_{n-2}) = y$.

(3) (a) We now show that for $b > 0$, $(b,\infty) \sim (0,1)$. Define a function $h_1 : (0,1) \to (b,\infty)$ by $h_1(x) = b/x$. If $b/x_1 = b/x_2$ this implies $x_1 = x_2$ and so h_1 is injective. For any $y \in (b,\infty)$, let $x = b/y$. Then $h_1(x) = h_1(b/y) = b/(b/y) = y$, and thus h_1 is surjective. Thus, $(b,\infty) \sim (0,1)$.

Suppose now that $b \in \mathbb{R}$. By the Archimedean Property, there exists $n \in \mathbb{N}$ so that $n > b$. So,

$$(b,\infty) = [(b,n] \cup (n,\infty)] \sim [(0,1) \cup (0,1)] = (0,1)$$

and so $(b,\infty) \sim (0,1)$ for all $b \in \mathbb{R}$.

(b) We now show that $[b,\infty) \sim (0,1)$ for all $b \in \mathbb{R}$. Similar to the previous case, we have

$$[b,\infty) = [[b,n] \cup (n,\infty)] \sim [(0,1) \cup (0,1)] = (0,1)$$

and so $[b,\infty) \sim (0,1)$ for all $b \in \mathbb{R}$.

(c) Assume that $b \in \mathbb{R}$ and consider the function $h_2 : (\infty, b) \to (b,\infty)$ defined by $h_2(x) = -x$. If $y_1 = y_2$, then $-y_2 = -y_2$ and so h_2 is injective. Clearly, if $y \in (b,\infty)$, let $x = -y$ and then $h_2(x) = h_2(-y) = y$ and so h_2 is surjective. Thus, $(-\infty, b) \sim (b,\infty)$. Then, by part 3$(a)$, $(-\infty,b) \sim (0,1)$.

(d) Assume that $b \in \mathbb{R}$ and consider the function $h_3 : (\infty, b] \to [b,\infty)$ defined by $h_3(x) = -x$. If $y_1 = y_2$, then $-y_2 = -y_2$ and so h_3 is injective. Clearly, if $y \in [b,\infty)$, let $x = -y$ and then $h_3(x) = h_3(-y) = y$ and so h_3 is surjective. Thus, $(-\infty, b] \sim [b,\infty)$. Then, by part 3$(b)$, $(-\infty,b] \sim (0,1)$.

(e) Notice that

$$(-\infty,\infty) = [(\infty,0) \cup [0,\infty]] \sim (0,1) \cup (0,1) = (0,1)$$

and so $(-\infty,\infty) \sim (0,1)$

■

Problem 1 — Let X be a non-empty set and denote $P(X) = \{Y | Y \text{ is a subset of } X\}$ its power set. Show that there are not bijections between X and $P(X)$.

Proof. To get a contradiction, assume such a bijection exists, say $f : X \to P(X)$. Now consider the set

$$A = \{x \in X | x \notin f(x)\}$$

Notice that $A \neq \emptyset$ because, since f is surjective and $\emptyset \in P(X)$, then there is a $a \in X$ so that $f(a) = \emptyset$. Certainly $a \notin \emptyset$, and so $a \in A$. Also notice that $A \subset X$. As f is surjective, there exists $x \in X$ so that $f(x) = A$. Now, either $x \in A$ or $x \notin A$. If $x \in A$, then by definition of A, we have $x \notin f(x) = A$, a contradiction. On the other hand, if $x \notin A$, then $x \in f(x) = A$, a contradiction. So, no such bijection f exists. ■

Problem 2 — Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that $f(2) = 2$, f is strictly increasing, and $f(nm) = f(n)f(m)$ for all n, m such that $\gcd(n, m) = 1$.

Proof. First note that for odd k, 2 and k are relatively prime, and so

$$f(2k) = f(2)f(k) = 2f(k)$$

We now proceed by induction on n to show that $f(n) = n$ is the only function with such properties. We consider $n = 1$, $n = 2$, and $n = 3$ for the base case. By the Pigeonhole Principle, $f(1) = 1$, and $f(2) = 2$ by definition. To see that $f(3) = 3$, assume for contradiction that $f(3) > 3$. Then, since f is stricly increasing, $f(21) < f(22)$, and so

$$\begin{aligned} f(14) &= 2f(7) \\ &< 3f(7) \\ &< f(3)f(7) \\ &= f(21) \\ &< f(22) \\ &= 2f(11) \\ &< f(11) \end{aligned}$$

a contradiction. So, $f(3) = 3$. Now, assume that $f(k) = k$ for $k < n$. If $n + 1$ is even, then n is odd, then

$$f(2n) = 2f(n) = 2n.$$

This means that there are exactly $2n - n = n$ natural numbers to assign to the n points $f(n + 1), f(n + 2) \dots, f(2n)$. Since f is strictly increasing, then $f(x) = x$ for all $x \in \{n + 1, n + 2, \dots, 2n\}$ by the Pigeonhole Principle. Thus, $f(n + 1) = n + 1$. This completes the case when $n + 1$ is even. If $n + 1$ is odd, then $n - 1$ is odd, which means

$$f(2(n - 1)) = 2f(n - 1) = 2(n - 1) = 2n - 2$$

This means that there are exactly $(2n-2)-(n-1)=n-1$ natural numbers to assign to the $n-1$ points $f(n), f(n+1), \dots f(2n-2)$. Since f is strictly increasing, then $f(x)=x$ for all $x \in \{n, n+1 \dots, 2n-1\}$ by the Pigeonhole Principle. Thus, $f(n+1)=n+1$. This completes the case when $n+1$ is odd. Therefore, by mathematical induction $f(n)=n$ for all $n \in \mathbb{N}$. ∎

1.4.30 A point x is called an *accumulation point* of a set E provided it is a point of closure of $E \sim \{x\}$.

(i) Show that the set E' of the set of accumulation points of E is a closed set.

Proof. Let $x \in \overline{E'}$. Then, any open interval containing x also contains a point in E'. Pick an open interval $\mathcal{O}$ containing x. Then, there exists $x' \in E'$ in $\mathcal{O}$. Choose $\epsilon > 0$ small enough so that $(x' - \epsilon, x' + \epsilon) \subseteq \mathcal{O}$ and $x \notin (x' - \epsilon, x' + \epsilon)$. Since $x' \in E'$, there exists a point $e \in E \sim \{x'\}$ that is also in $(x' - \epsilon, x' + \epsilon)$. Thus, $e \in E \sim \{x\}$, which implies $x \in E'$. Thus, E' is closed. ■

(ii) Show that $\overline{E} = E \cup E'$.

Proof. Let $x \in \overline{E}$. Suppose $\mathcal{O}$ is an open interval containing x. Since $x \in \overline{E}$, then $\mathcal{O}$ contains a point $e \in E$. If $e = x$, then $x \in E$. If $e \neq x$, then $x \in E'$ since there exists an $e \in E$ so that $e \in E \sim \{x\}$. Thus, $\overline{E} \subseteq E \cup E'$. Now, suppose $x \in E \cup E'$. If $x \in E$, then $x \in \overline{E}$ trivially. If $x \in E'$, then any open interval containing x also contains a point in $E \setminus \{x\}$, which certainly implies that any open interval containing x also contains a point in E. Thus, $x \in \overline{E}$. So, $E \cup E' \subseteq \overline{E}$, and we conclude $\overline{E} = E \cup E'$. ■

1.4.32 A point x is called and *interior point* of a set E if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in E. The set of interior points of E is called the *interior* of E denoted by int$E$. Show that

(i) E is open if and only if $E = \text{int}E$.

Proof. Suppose E is open and let $x \in E$. Since E is open, there exists $r > 0$ so that $(x - r, x + r) \subseteq E$. In other words, x is an interior point of E. Thus, $x \in \text{int}E$. Now, suppose $x \in \text{int}E$. So, there exists an $r > 0$ so that $(x-r, x+r) \subseteq E$, i.e., $x \in E$ and thus $E = \text{int}E$.
Now, suppose $E = \text{int}E$ and let $x \in E$. Since $E = \text{int}E$, then $x \in \text{int}E$. So, there exists an $r > 0$ so that $(x - r, x + r) \subseteq E$. Thus, E is open. ■

(ii) E is dense if and only if $\text{int}(\mathbb{R} \sim E) = \emptyset$.

Proof. Assume E is dense in $\mathbb{R}$. To get a contradiction, suppose $\text{int}(\mathbb{R} \sim E) \neq \emptyset$. So, there exists $x \in \text{int}(\mathbb{R} \sim E)$ which means there exists $r > 0$ so that $(x - r, x + r) \subseteq (\mathbb{R} \sim E)$ Since E is dense in $\mathbb{R}$, we should find a point $e \in E$ so that $e \in (x - r, x + r)$, a contradiction. Thus, $\text{int}(\mathbb{R} \sim E) = \emptyset$.
Assume that $\text{int}(\mathbb{R} \sim E) = \emptyset$. To get a contradiction, assume E is not dense in $\mathbb{R}$. So, there exists two points in $x, y \in \mathbb{R}, x < y$, so that there does not exist a point in E between them. So, $(x, y) \subseteq (R \sim E)$, i.e., $(y - x)/2 \in \text{int}(R \sim E)$, a contradiction. Thus, E is dense in $\mathbb{R}$. ■

1.4.35 Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

Proof. Since, by definition, the collection of Borel sets is the smallest σ-algebra that contains all of the open sets of $\mathbb{R}$, and all closed sets are in this σ-algebra by closure under complements (i.e., all closed sets are Borel), then the collection of Borel sets is the smallest σ-algebra that contains all closed sets. To see this more clearly, suppose $\mathcal{A}$ is a σ-algebra that contains all closed sets. Then, $\mathcal{A}$ necessarily contains all open sets by closure under complements. Since the collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra containing all open sets, then $\mathcal{B} \subseteq \mathcal{A}$, and thus $\mathcal{B}$ is the smallest σ-algebra containing closed sets. ∎

1.5.40 Show that a sequence (a_n) is convergent to an extended real number if and only if there is exactly one extended real number that is a cluster point of the sequence.

Proof. ($\Rightarrow$) Suppose $\{a_n\} \to a \in \mathbb{R} \cup \pm\infty$. By Proposition 19 part (iv),

$$\liminf\{a_n\} = \limsup\{a_n\} = a$$

By Exercise 38, $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$. So, a is the largest and smallest cluster point of the sequence so that a is the unique cluster point of $\{a_n\}$.

($\Leftarrow$) Suppose there is exactly one cluster point of $\{a_n\}$, say a. So, a is the largest and smallest cluster point of $\{a_n\}$. This means $a = \liminf\{a_n\} = \limsup\{a_n\}$ by Exercise 38. Then, by Proposition 19 part (iv), $\{a_n\}$ converges to a. ∎

1.5.42 Prove that if, for all $n, a_n \geq 0$ and $b_n \geq 0$, then

$$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n)$$

provided the product on the right is not of the form $0 \cdot \infty$.

Proof. First note that if either (a_n) or (b_n) is unbounded, then the inequality holds trivially. So, suppose both sequences are bounded above and let $n \geq 1$ and $k \geq n$. Then,

$$0 \leq a_k \leq \sup\{a_\ell \mid \ell \geq n\} \quad \text{and} \quad 0 \leq b_k \leq \sup\{b_\ell \mid \ell \geq n\}$$

In other words, any term, a_k, will be less than or equal to the supremum of any (countably) infinite collection of terms of the sequence (a_n). Likewise for the sequence (b_n). Combining these inequalities, we have that for all $k \geq n$,

$$a_k b_k \leq \sup\{a_\ell \mid \ell \geq n\} \cdot \sup\{b_\ell \mid \ell \geq n\}.$$

So, we have found an upper bound for the sequence $(a_k b_k)$. So,

$$\sup_{k \geq n}\{a_k b_k\} \leq \sup\{a_\ell \mid \ell \geq n\} \cdot \sup\{b_\ell \mid \ell \geq n\}$$

Taking limits as $n \to \infty$, we obtain the desired result:

$$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n)$$

$\blacksquare$

1.6.48 Define the real-valued function f on $\mathbb{R}$ by setting

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ p \sin \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous?

Proof. f is continuous at 0 and all irrational points. Note that $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$.
f is continuous at 0:
Let $\epsilon > 0$ and $x = 0$. Then, for $\delta = \epsilon$, we have

$$|f(x) - f(0)| = |f(x)| \leq |x| < \epsilon$$

when $|x - 0| = |x| < \delta = \epsilon$.

f is continuous at all irrationals:
Let x be irrational. First we show that for any M, there exists $\delta > 0$ such that $q \geq M$ for any rational $p/q \in (x - \delta, x + \delta)$. Otherwise, there exists M such that for any n, there exists a rational $p_n/q_n \in (x - 1/n, x + 1/n)$ with $q_n < M$. Then

$$|p_n| \leq q_n \cdot \max\{|x - 1|, |x + 1|\} < M \cdot \max\{|x - 1|, |x + 1|\}$$

for all N. Thus there are only finitely many choices of p_n and q_n for each n. This implies that there exists a rational p/q in $(x - 1/n, x + 1/n)$ for infinitely many n, a contradiction.
Given $\epsilon > 0$, choose M such that

$$M^2 > \frac{2 \cdot \max\{|x + 1|, |x - 1|\}}{6\epsilon}.$$

This implies

$$\frac{\max\{|x + 1|, |x - 1|\}}{6M^2} < \frac{\epsilon}{2}.$$

Then choose $\delta > 0$ such that $\delta < \min\{1, \epsilon/2\}$ and $q \geq M$ for any rational $p/q \in (x - \delta, x + \delta)$. Suppose $|x - y| < \delta$. If y is irrational, then

$$|f(y) - f(x)| = |x - y| < \delta < \epsilon.$$

Note that $|x - \sin x| < x^3/6$ for all $x \neq 0$ by Taylor's Theorem. If $y = p/q$ is rational. Then,

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - y| + |y - x| \\ &= |p| \cdot |1/q - \sin(1/q)| + |y - x| \\ &< \frac{|p|}{6q^3} + \delta \\ &< \frac{\max(|x+1|, |x-1|)}{6q^2} + \delta \\ &\leq \frac{\max(|x+1|, |x-1|)}{6M^2} + \delta \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

f does not converge at any nonzero rational:
Let q be a nonzero rational. Let $\epsilon = |f(q) - q| > 0$. If $q > 0$, given any $\delta > 0$, pick an irrational $x \in (q, q + \delta)$. Then,

$$|f(x) - f(q)| = x - f(q) > q - f(q) = \epsilon$$

If $q < 0$, given any $\delta > 0$ pick any irrational $x \in (q - \delta, q)$. Then

$$|f(x) - f(q)| = f(q) - x > f(q) - q = \epsilon$$

■

1.6.56 Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.

Proof. Let $\mathcal{C}$ be the set of all continuous points of f. For each $n \in \mathbb{N}$ and $x \in \mathcal{C}$, let $\delta_{x,n}$ be such that $|f(x) - f(y)| < 1/n$ for all $y \in B(x, \delta_{x,n})$. Then let

$$\mathcal{O}_n = \bigcup_{x \in \mathcal{C}} B(x, \delta_{x,n}).$$

Note that $\mathcal{O}_n$ is open for all n, each being the union of open intervals. Then $G := \cap_{n \in \mathbb{N}} \mathcal{O}_n$ is a G_δ set and $\mathcal{C} \subseteq G$. Let $x_0 \in G$ and $\epsilon > 0$. Since $x_0 \in \mathcal{O}_n$ for all n, there exists $n_0 \in \mathbb{N}$ and $x \in \mathcal{C}$ such that $1/n_0 < \epsilon/2$ and $x_0 \in B(x, \delta_{x,n_0})$. Hence, $|f(x) - f(x_0)| < 1/n_0 < \epsilon/2$. Let $\delta > 0$ be such that $B(x_0, \delta) \subseteq B(x, \delta_{x,n_0})$. Then for any $y \in B(x_0, \delta)$,

$$|f(x_0) - f(y)| \leq |f(x_0) - f(x)| + |f(x) - f(y)| < \epsilon/2 + 1/n_0 < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $x_0 \in \mathcal{C}$ and so $\mathcal{C} = G$.

■

2.2.9 Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Proof. Since $B \subseteq A \cup B$, then $m^*(B) \leq m^*(A \cup B)$ by monotonicity of outer measure. Since outer measure in subadditive, then $m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$. ■

Problem 1 Let X be a set and $\mathcal{A} \subset P(X)$ be a countable σ-algebra. Show that $\mathcal{A}$ is finite.

Proof. Suppose $\mathcal{A} = \{A_i\}_i^\infty$ is a countable σ-algebra. For each $x \in X$, define

$$B_x := \bigcap_{x \in A_i} A_i$$

Note that $B_x \in A$ since this is a countable intersection. For $x, y \in X$, we claim that if $B_x \cap B_y \neq \emptyset$, then $B_x = B_y$. Let $z \in B_x \cap B_y$. Then $B_z \subseteq B_x \cap B_y$. If $x \notin B_z$, then $B_x \setminus B_z$ is a set in $\mathcal{A}$ containing x and is strictly contained in B_x, which contradicts the definition of B_x. Hence, $B_z = B_x$ and similarly, $B_z = B_y$, and thus $B_x = B_y$. Now consider $\{B_x\}_{x \in X}$. If there is a finite set of the form B_x then $\mathcal{A}$ is a union of a finite number of disjoint sets and hence, $\mathcal{A}$ is finite. ■

Problem 2 Let $x_1 \in (0, \pi)$ and for every $n \in \mathbb{N}$ let $x_{n+1} = \sin(x_n)$. Compute

$$\lim_{n \to \infty} x_n \sqrt{n}.$$

Justify your answer!

Proof. Let $y_n = n$ for all n in the sequence $(y_n)_{n=1}^\infty$. We claim that

$$\lim_{n \to \infty} \frac{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}{y_{n+1} - y_n} = \frac{1}{3}.$$

If we can show this, then by the Stolz–Cesàro theorem,

$$\lim_{n \to \infty} \frac{1}{x_n^2 n} = \frac{1}{3}$$

which implies

$$\lim_{n \to \infty} x_n \sqrt{n} = \lim_{n \to \infty} \sqrt{x_n^2 n} = \sqrt{3}$$

To prove the claim, first notice that since $\sin(x) < x$ for all $x > 0$ then

$$0 < x_{n+1} = \sin x_n < x_n$$

for all n and so (x_n) is strictly decreasing. Thus,

$$\lim_{n \to \infty} x_n = \lim_{x \to 0} \sin(x) = 0$$

Then,

$$\lim_{n\to\infty} \frac{1}{x_{n+1}^2} - \frac{1}{x_n^2} = \lim_{x\to 0} \frac{1}{\sin^2 x} - \frac{1}{x^2} = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x\to 0} \frac{\frac{x^2}{\sin^2 x} - 1}{x^2}$$

Notice that all terms in the ratio of the rightmost limit above are differentiable, so we can compute:

$$\left(\frac{x^2}{\sin^2 x} - 1\right)' = 2x(\csc^2 x)(1 - x\cot x) \quad \text{and} \quad \left(x^2\right)' = 2x$$

Notice that

$$\frac{2x(\csc^2 x)(1 - x\cot x)}{2x} = \frac{-(x\cos x - \sin x)}{\sin^3 x}.$$

Since each term in the above ratio is differentiable, we can compute

$$-(x\cos x - \sin x)' = x\sin x \quad \text{and} \quad (\sin^3 x)' = 3\cos x \sin x$$

Notice that

$$\lim_{x\to 0} \frac{x\sin x}{3\cos x \sin x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1}{3\cos x} = \frac{1}{3}$$

because

$$\lim_{x\to 0} \frac{x}{\sin x} = 1.$$

So, by L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{x\sin x}{3\cos x \sin x} = \frac{1}{3} \implies \lim_{x\to 0} \frac{\frac{x^2}{\sin^2 x} - 1}{x^2} = \frac{1}{3}$$

Finally, notice that $y_{n+1} - y_n = 1$ and so

$$\frac{1}{3} = \lim_{x\to 0} \frac{\frac{x^2}{\sin^2 x} - 1}{x^2} = \lim_{n\to\infty} \frac{1}{x_{n+1}^2} - \frac{1}{x_n^2} = \lim_{n\to\infty} \frac{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}{y_{n+1} - y_n}$$

which proves the claim. ■

Problem 3 Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Assume that for every $x \geq 0$ we have $\lim_{n\to\infty} f(nx) = 0$. Show that $\lim_{x\to\infty} f(x)$ exists and compute it.

Proof. We first state and prove the following lemma:

Lemma. *Let X be a nonempty complete metric space. If*

$$X = \bigcup_{n=1}^{\infty} A_n \tag{1}$$

where each A_n is closed, then at least one A_n contains a non-empty open subset.

Proof. Assume none of the sets A_n contain a nonempty open subset. So, A_1 does not contain a nonempty open subset, and therefore its open complement A_1^c must. In other words, there must exists $x_1 \in X$ and $0 < \epsilon_1 < 1$ such that

$$B_{x_1}(\epsilon_1) \subset A_1^c$$

Similarly, A_2 does not contain a non-empty set, and therefore there must be a point x_2 in the open set $A_2^c \cap (B_{x_1}(\epsilon_1)$. hence we can find $\epsilon_2 < \frac{1}{2}$ such that

$$B_{x_2}(\epsilon_2) \subset (A_2^c \cap B_{x_1}(\epsilon_1))$$

Continuing this iterative process, we can construct a sequence of points $(x_n)_{n=1}^{\infty}$ and positive reals $(\epsilon_n)_{n=1}^{\infty}$ such that

$$B_{x_{n+1}}(\epsilon_{n+1}) \subset (A_n^c \cap B_{x_n}(\epsilon_n))$$

and $\epsilon_n < \frac{1}{2^n}$. Notice by our construction

$$B_{x_1}(\epsilon_1) \supset B_{x_2}(\epsilon_2) \supset \cdots \supset B_{x_n}(\epsilon_n) \supset B_{x_{n+1}}(\epsilon_{n+1}) \supset \ldots$$

Here we have a sequence of nested balls (open intervals in $\mathbb{R}$). The sequence (x_n) is Cauchy since $n, m \geq N$ implies that $x_n, x_m \in B_{x_N}(\epsilon_N)$ and $\epsilon_N < \frac{1}{2^N}$. Since X is a complete space, there exists a point $x \in X$ such that $x_n \to x$. In particular, $x \in B_{x_n}(\epsilon_n)$ and therefore $x \notin A_n$ for all $n \in \mathbb{N}$, a contradiction to our statement in (1). ■

Let $\epsilon > 0$. Define $E_n := \{x \in [0, \infty) \mid |f(kx)| \leq \epsilon \ \forall k \geq n\}$. We claim

$$[0, \infty) = \bigcup_{n\in\mathbb{N}} E_n.$$

If $x \in \bigcup_{n\in\mathbb{N}} E_n$, then $x \in [0, \infty)$ by the definition of E_n. If $x \in [0, \infty)$, then $\lim\limits_{n\to\infty} f(nx) = 0$. This implies that there exists an $N \in \mathbb{N}$ so that $|f(nx)| < \epsilon$ whenever $n \geq N$. Thus, $x \in E_N$ so that $x \in \bigcup_{n\in\mathbb{N}} E_n$.

Also, notice that each E_n is closed. By the Lemma, there exists $n_0 \in \mathbb{N}$ so that $\text{int}(E_{n_0}) \neq \emptyset$. Let $x_0 \in \text{int}(E_{n_0})$. Then there exists $r > 0$ so that $(x_0 - r, x + r_0) \subseteq E_{n_0}$. Let $t_0 := n_1 x_0$ where $n_1 \in \mathbb{N}$ and $n_1 \geq n_0$ and $\frac{x_0}{n_1} < r$ (we can choose such an n_1 by the Archimedean Property). Now, let $x \in (0, \infty)$ so that $x \geq t_0$. This means

$$x \geq n_1 x_0 > x_0$$

Then, $x \geq n_x x_0$ for some $n_x \geq n_1$. Then, $x = n_x x_0 + \alpha$ where $0 \leq \alpha \leq x_0$. Now, notice

$$|f(x)| = |f(n_x x_0 + \alpha)| = \left| f\left(n_x \left(x_0 + \frac{\alpha}{n_x} \right) \right) \right| \leq \epsilon$$

because $n_x \geq n_1$ and

$$\left| \frac{\alpha}{n_x} \right| \leq \left| \frac{x_0}{n_x} \right| \leq \left| \frac{x_0}{n_1} \right| < r.$$

Hence, $\lim_{n \to \infty} f(x) = 0$. ■

2.3.15 Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

Proof. Let $\epsilon > 0$ and define $B_n := [-n, n] \cap E$. Notice that $\cup_{n\in\mathbb{N}} B_n = E$ and that $\{B_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets. By the continuity of measure,

$$\lim_{n\to\infty} m(B_n) = m\left(\bigcup_{n=1}^{\infty} B_n\right).$$

So, there exists $N \in \mathbb{N}$ so that

$$\epsilon > m\left(\bigcup_{n=1}^{\infty} B_n\right) - m(B_N) = m(E) - m(B_N) = m(E \setminus B_N)$$

by the excision property. Let $I_0 = E \setminus B_N$ and so $m(I_0) < \epsilon$. Then, notice that $B_N = [-N, N] \cap E$ is bounded and contained in E. Let

$$x_1 = -N < x_2 < \cdots < x_k = N$$

so that $|x_{i+1} - x_i| = \epsilon/2$ for each $0 \leq i \leq k-1$. Now, let $I_i = [x_i, x_{i+1})$ for all i. Then, $\ell(I_i) < \epsilon$ for all i and

$$E = \bigsqcup_{i=0}^{k} I_i.$$

☕

2.4.18 Let E have finite outer measure. Show that there is a G_δ set G for which $E \subseteq G$ and $m(G) = m^*(E)$. Show that E is measurable if and only if there is an F_σ set F for which $F \subseteq E$ and $m(F) = m^*(E)$.

Proof. Since $m^*(E) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \middle| E \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$, then for each $n \in \mathbb{N}$ we can find a collection of open sets $\{I_{k,n}\}_{k=1}^{\infty}$ which covers E so that

$$\sum_{k=1}^{\infty} \ell(I_{k,n}) \leq m^*(E) + \frac{1}{n}.$$

Notice that each $\sum_{k=1}^{\infty} \ell(I_{k,n})$ is open since each is a countable union of open sets. Define

$$G = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=1}^{\infty} I_{k,n}\right).$$

Since G is a countable intersection of open sets, then G is a G_δ set. Since $E \subseteq \sum_{k=1}^{\infty} \ell(I_{k,n})$ for each n, then $E \subseteq G$. So, $m^*(E) \leq m^*(G)$.
Since $G \subseteq \bigcup_{k=1}^{\infty} I_{k,n}$ for each n, then

$$m^*(G) \leq m^*\left(\bigcup_{k=1}^{\infty} I_{k,n}\right) \leq m^*(E) + \frac{1}{n}$$

for all n, which means $m^*(G) \le m^*(E)$, and so $m^*(G) = m^*(E)$.
By Theorem 11 part (iv)[1], E is measurable if and only if there exists an F_σ set F so that $F \subseteq E$ and

$$m^*(E \setminus F) = 0.$$

By excision property of outer measure,

$$0 = m^*(E \setminus F) = m^*(E) - m^*(F) \implies m^*(E) = m^*(F).$$

2.4.20 (Lebesgue) Let E have finite outer measure. Show that E is measureable if and only if for each open, bounded interval (a,b),

$$b - a = m^*((a,b) \cap E) + m^*((a,b) \setminus E). \tag{1}$$

Proof. ($\Rightarrow$) The forward direction is clear: If (a,b) is any open, bounded interval, then

$$E \text{ measurable } \implies b - a = m^*((a,b)) = m^*((a,b) \cap E) + m^*((a,b \cap E^c))$$

($\Leftarrow$) Let A be a set and suppose (1) holds. If $m^*(A) = \infty$, then $m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c)$ and thus E is measurable. Suppose $m^*(A) < \infty$. Let $\epsilon > 0$ and $\{I_k\}_{k=1}^\infty$ be a collection of open sets so that

$$\bigcup_{k=1}^\infty I_k \supseteq A \quad \text{and} \quad m^*(A) + \epsilon \ge \sum_{k=1}^\infty \ell(I_k).$$

Since each I_k is an open set, then each I_k is a countable union of disjoint open intervals. Thus, let $\{\mathcal{O}_n\}_{n=1}^\infty$ be a collection of open intervals so that

$$\bigcup_{k=1}^\infty I_k = \bigcup_{n=1}^\infty \mathcal{O}_n.$$

Then

$$\bigcup_{n=1}^\infty \mathcal{O}_n \supseteq A \quad \text{and} \quad m^*(A) + \epsilon \ge \sum_{n=1}^\infty \ell(\mathcal{O}_n).$$

Notice that since each $\mathcal{O}_n$ is an interval, then $\ell(\mathcal{O}_n) = m^*(\mathcal{O}_n)$ for each $\mathcal{O}_n$. So,

$$\begin{aligned} m^*(A) + \epsilon &\ge \sum_{n=1}^\infty \ell(\mathcal{O}_n) \\ &= \sum_{n=1}^\infty m^*(\mathcal{O}_n) \\ &= \sum_{n=1}^\infty [m^*(\mathcal{O}_n \cap E) + m^*(\mathcal{O}_n \cap E^c)] && \text{(by (1))} \end{aligned}$$

[1]Part (iv) is easily proven using part (ii) of the same theorem: Since E is measurable, then so is E^c. By part (ii), there is a G_δ set G containing E^c for with $0 = m^*(G \setminus E^c) = m^*(G \cap E)$. By DeMorgan's Laws, $G^c = F$ for some F_σ set F, and since G contains E^c, then F is contained in E. Notice: $G \cap E = (G^c)^c \cap E = E \setminus G^c = E \setminus F$, and so $m^*(E \setminus F) = 0$.

$$= \sum_{n=1}^{\infty}[m^*(\mathcal{O}_n \cap E)] + \sum_{n=1}^{\infty}[m^*(\mathcal{O}_n \cap E^c)]$$

$$\geq m^*\left(\bigcup_{n=1}^{\infty}(\mathcal{O}_n \cap E)\right) + m^*\left(\bigcup_{n=1}^{\infty}(\mathcal{O}_n \cap E^c)\right) \qquad \text{(subadditivity of } m^*\text{)}$$

$$= m^*\left(\bigcup_{n=1}^{\infty}\mathcal{O}_n \cap \bigcup_{n=1}^{\infty} E\right) + m^*\left(\bigcup_{n=1}^{\infty}\mathcal{O}_n \cap \bigcup_{n=1}^{\infty} E^c\right)$$

$$= m^*\left(\bigcup_{n=1}^{\infty}\mathcal{O}_n \cap E\right) + m^*\left(\bigcup_{n=1}^{\infty}\mathcal{O}_n \cap E^c\right)$$

$$\geq m^*(A \cap E) + m^*(A \cap E^c) \qquad \text{(monotonicity of } m^*\text{)}$$

This inequality holds for all $\epsilon > 0$ and so E is measurable.

2.5.28 Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

Proof. Let $\{E_k\}_{k=1}^{\infty}$ be a set of disjoint, measurable sets. By finite additivity,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k) \text{ for each } n.$$

Let $F_n = \bigcup_{k=1}^{n} E_k$ for each $n \in \mathbb{N}$. Then, $\{F_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets. Notice that

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{n=1}^{\infty} F_n.$$

By continuity of measure,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} m(F_n)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} E_k\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} m(E_k)\right) = \sum_{k=1}^{\infty} m(E_k)$$

2.6.30 Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountable infinite.

Proof. Let E be a set with positive outer measure. By way of contradiction, suppose there exists a countable choice set C_E for the rational equivalence relation on E. Since each equivalence class is a translate of $\mathbb{Q}$, then each equivalence class is countable. Thus, E is the union of a countable collection of countable sets, which is countable and so $m(E) = 0$, a contradiction. Thus, any choice set for the rational equivalence relation on E must be uncountable.

2.7.39 Let F be the subset of $[0,1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0,1] \setminus F$ is dense in $[0,1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.

Proof. Let $0 < \alpha < 1$. Consider the interval $[0,1]$. To construct F, we begin by removing the open interval

$$\mathcal{O}_1 = \left(\frac{3-\alpha}{6}, \frac{3+\alpha}{6}\right)$$

from $[0,1]$ to obtain the set

$$F_1 = [0,1] \setminus \mathcal{O}_1 = \left[0, \frac{3-\alpha}{6}\right] \cup \left[\frac{3+\alpha}{6}, 1\right].$$

Notice that F_1 is the disjoint union of 2^1 closed intervals, each of length $(3-\alpha)/(2 \cdot 3^1)$. Continuing in a manner similar to the construction of the Cantor set $\boldsymbol{C}$, we construct the collection $\{F_k\}_{k=1}^{\infty}$ of descending closed sets, where F_{k+1} is obtained by removing open intervals of length $\alpha/3^{k+1}$ from the middle of each of the 2^k closed intervals of F_k. Then, let

$$\boldsymbol{F} = \bigcap_{k=1}^{\infty} F_k.$$

Since each F_k is the disjoint union of 2^k closed intervals, then each F_k is closed. Then F is a closed set, as it is the intersection of closed sets. Let $\boldsymbol{F}^c = [0,1] \setminus F$.
Let $\{\mathcal{O}_k\}_{k=1}^{\infty}$ be the ascending collection of open intervals where $\mathcal{O}_1$ is defined as above and $\mathcal{O}_{k+1} = [0,1] \setminus F_{k+1}$. Then,

$$\boldsymbol{\mathcal{O}} = \bigcup_{k=1}^{\infty} \mathcal{O}_k = \boldsymbol{F}^c.$$

Since we removed 2^{n-1} intervals of length $\alpha/3^n$ at the n-th stage, then by countable additivity of Lebesgue measure,

$$m\left(\boldsymbol{\mathcal{O}}\right) = \sum_{k=1}^{\infty} \alpha \cdot \frac{2^{k-1}}{3} = \frac{\alpha}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{\alpha}{2} \cdot \frac{2/3}{1-2/3} = \frac{\alpha}{2} \cdot 2 = \alpha.$$

Then by the excision property of measurable sets,

$$m(\boldsymbol{F}) = m([0,1]) - m(\boldsymbol{\mathcal{O}}) = 1 - \alpha.$$

Finally, to see that $\boldsymbol{\mathcal{O}}$ is dense in $[0,1]$ we show that $\operatorname{int}(\boldsymbol{F})$ is empty. To get a contradiction, suppose $a \in \operatorname{int}(\boldsymbol{F})$. Then, there exists $\epsilon > 0$ so that $I = (a-\epsilon, a+\epsilon) \subseteq \boldsymbol{F}$. Then, $I \subseteq F_k$ for all k. Let f_k be the length of one of the closed intervals of F_k. Since F_k is a collection of disjoint intervals, I must be contained in one of the 2^k closed intervals. In other words,

$$\ell(I) = 2\epsilon < f_k \quad \forall k. \tag{1}$$

We claim that $f_k < 1/2^k$ for all k. We proceed by induction. Clearly, we have

$$f_1 = \frac{\ell([0,1]) - \ell(\mathcal{O}_1)}{2} = \frac{1 - \frac{\alpha}{3^1}}{2} < \frac{1}{2^1}$$

and

$$f_2 = \frac{f_1 - \frac{\alpha}{3^2}}{2} < \frac{1/2}{2} < \frac{1}{2^2}.$$

If $f_{n-1} < 1/2^{n-1}$, then

$$f_n = \frac{f_{n-1} - \frac{\alpha}{3^n}}{2} < \frac{1/2^{n-1}}{2} = \frac{1}{2^n}.$$

Thus by induction, $f_k \leq 1/2^k$ for all k. Now, we can choose $N \in \mathbb{N}$ large enough so that

$$\frac{1}{2^N} < 2\epsilon.$$

Thus,

$$f_N = \frac{1}{2^N} < 2\epsilon = \ell(I)$$

which is a contradiction of (1). Thus, $a \notin \text{int}(\boldsymbol{F})$ and so $\text{int}(\boldsymbol{F})$ is empty. ☕

2.7.40 Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: Consider the complement of the generalized Cantor set of the preceding problem.)

Proof. Note that for any set of real numbers $\partial(A) = \overline{A} \setminus \text{int}(A)$. Consider the open set $\boldsymbol{\mathcal{O}}$ from the previous exercise. Since $\boldsymbol{\mathcal{O}}$ is open, then $\text{int}(\boldsymbol{\mathcal{O}}) = \boldsymbol{\mathcal{O}}$. Also, as $\boldsymbol{\mathcal{O}}$ is dense in $[0,1]$, then $\overline{\boldsymbol{\mathcal{O}}} = [0,1]$. So,

$$\partial(\boldsymbol{\mathcal{O}}) = \overline{\boldsymbol{\mathcal{O}}} \setminus \text{int}(\boldsymbol{\mathcal{O}}) = [0,1] \setminus \boldsymbol{\mathcal{O}} = \boldsymbol{F}.$$

By the previous exercise, $m(\boldsymbol{F}) = 1 - \alpha$ and so,

$$m(\partial(\boldsymbol{\mathcal{O}})) = m(\boldsymbol{F}) = 1 - \alpha,$$

which is positive. ☕

2.7.45 Show that a strictly increasing function that is defined on an interval has a continuous inverse.

Proof. Let f be a strictly increasing function defined on some interval I. Since f is increasing, it is injective. So, the inverse function $f^{-1} : f(I) \to I$ is well defined. To get a contradiction, suppose f^{-1} is not continuous at the point $y \in f(I)$. There exists a sequence $\{y_n\} \subset f(I)$ that converges to y but the corresponding image sequence $\{f^{-1}(y_n)\}$ does not converge to $f^{-1}(y)$. Let $f^{-1}(y_n) = x_n$ for all n and $f^{-1}(y) = x$. Thus, we can find $\epsilon_0 > 0$ so that for all $N \in \mathbb{N}$, there exists $n_0 > N$ so that

$$x_{n_0} < x - \epsilon_0 < x \quad \text{or} \quad x < x + \epsilon_0 < x_{n_0}.$$

Since f is increasing, then

$$f(x_{n_0}) = y_{n_0} < y = f(x) \quad \text{or} \quad f(x) = y < y_{n_0} = f(x_{n_0}).$$

So, there exists $\delta_0 > 0$ so that

$$y_{n_0} < y - \delta_0 < y \quad \text{or} \quad y < y + \delta_0 < y_{n_0}.$$

This implies that for all $N \in \mathbb{N}$, there exists $n_0 > N$ such that

$$|y - y_{n_0}| > \delta_0$$

or in other words, $\{y_n\}$ does not converge to y, which is a contradiction. So, f^{-1} is continuous. ☕

2.7.46 Let f be a continuous function and B be a Borel set. Show that $f^{-1}(B)$ is a borel set. (Hint: The collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets).

Proof. Let $\mathcal{A}$ be the collection of sets E for which $f^{-1}(E)$ is Borel. Since f is continuous, $f^{-1}(\mathcal{O})$ is open for any open set $\mathcal{O}$. Thus, $\mathcal{A}$ contains the open sets. Since preimages of a continuous function preserve complements, unions ,and intersections, then $\mathcal{A}$ is a σ-algebra containing the open sets. Since the collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra containing the open sets, then $\mathcal{B} \subseteq \mathcal{A}$. So, $\mathcal{B}$ is the smallest collection of sets B for which $f^{-1}(B)$ is Borel. Thus, $f^{-1}(B)$ is Borel for any Borel set B. ☕

2.7.47 Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

Proof. Let f be a continuous strictly increasing function that is defined on an interval. By Exercise 45, f^{-1} is continuous. Let B be any Borel set. By Exercise 46, $(f^{-1})^{-1}(B) = f(B)$ is Borel. Thus, f maps Borel sets to Borel sets. ☕

3.1.9 Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points $x \in E$ at which $\{f_n\}$ converges. Is the set E_0 measurable?

Proof. Notice that

$$\begin{aligned} E_0 &= \{x \in E \mid \{f_n(x)\} \text{ converges}\} \\ &= \{x \in E \mid \{f_n(x)\} \text{ is Cauchy}\} \\ &= \{x \in E \mid \forall\, \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall\, n, m \geq N, |f_n(x) - f_m(x)| < \epsilon\}. \end{aligned}$$

By the Archimedian Property, for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ so that $1/k < \epsilon$, and so

$$\begin{aligned} E_0 &= \{x \in E \mid \forall\, k \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \forall\, n, m \geq N, |f_n(x) - f_m(x)| < 1/k\} \\ &= \bigcap_{k\in\mathbb{N}} \{x \in E \mid \exists N \in \mathbb{N} \text{ such that } \forall\, n, m \geq N, |f_n(x) - f_m(x)| < 1/k\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \{x \in E \mid \forall\, n, m \geq N, |f_n(x) - f_m(x)| < 1/k\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} \{x \in E \mid |f_n(x) - f_m(x)| < 1/k\}. \end{aligned}$$

Let h be the absolute value function and define $g_{mn} := h \circ (f_n - f_m)$. Since f_n and f_m are measurable, then $f_n - f_m$ is measurable. Since h is continuous and $f_n - f_m$ is measurable, then g_{mn} is measurable by Proposition 7 (§3.1, Royden). So,

$$\begin{aligned} E_0 &= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} \{x \in E \mid |f_n(x) - f_m(x)| < 1/k\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} \{x \in E \mid g_{mn} < 1/k\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} \{x \in E \mid x \in g_{mn}^{-1}([0,1/k))\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} g_{mn}^{-1}([0,1/k)). \end{aligned}$$

Since g_{mn} is a measurable function, and $[0, 1/k)$ is a measurable set, then $g_{mn}^{-1}([0,1/k))$ is a measurable set for all k. Since the countable intersection of the countable union of the countable intersection of measurable sets is a measurable set, then E_0 is measurable. ☕

3.2.22 (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$.

Proof. Let $\epsilon > 0$. Using the hint, we define $E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}$ for each $n \in \mathbb{N}$. We show that $\cup_n E_n$ is an open cover of $[a, b]$. First, define $g_n := f - f_n$ for all n. Then, as f and f_n are continuous, then so is g_n. Thus, $g_n^{-1}((-1, \epsilon)) = E_n$ is open.
Since $\{f_n\}$ converges pointwise to f, then if $x \in [a, b]$, there exists $N \in \mathbb{N}$ so that $f(x) - f_N(x) < \epsilon$, which means $x \in E_N$. So, $\{E_n\}$ is an open cover for $[a, b]$, i.e.,

$$[a, b] \subseteq \bigcup_{n\in\mathbb{N}} E_n.$$

Since $[a, b]$ is closed and bounded, it is compact, and so there exists a finite subcover $\{E_{n_1}, \ldots, E_{n_m}\}$ of $[a, b]$ by the Heine-Borel Theorem.
Let $N_0 = \max_{1\leq i\leq m}\{n_i\}$. Since $\{f_n\}$ is increasing, then $f_{N_0} \leq f_{n_i} \leq f$ for all $n_i \geq N_0$, which means $f - f_{n_i} \leq f - f_{N_0}$ for all $n_i \geq N_0$. So, $E_{n_i} \subseteq E_{N_0}$ for all $1 \leq i \leq m$. Thus

$$\bigcup_{i=1}^{m} E_{n_i} = E_{N_0}$$

which implies $[a, b] \supseteq E_{N_0}$, and by definition of E_{N_0}, we have $E_{N_0} \supseteq [a, b]$. Hence, $E_{N_0} = [a, b]$. Therefore, for $\epsilon > 0$ and $x \in [a, b]$, we have $N_0 \in \mathbb{N}_0$ so that $m \geq N$ implies $f(x) - f_m(x) < \epsilon$. Thus, $\{f_n\}$ converges to f uniformly. ☕

3.3.25 Suppose f is a function that is continuous on a closed set G of real numbers. Show that f has a continuous extension to all of $\mathbb{R}$.

Proof. Since F is closed, $\mathbb{R} \sim F$ is open, and thus we can write $\mathbb{R} \sim F$ as a countable collection of disjoint intervals;

$$\mathbb{R} \sim F = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

The closure of each bounded interval (a_k, b_k) is $[a_k, b_k]$. For all such intervals, $a_k, b_k \in F$ and so $f(a_k)$ and $f(b_k)$ are already defined. So, for all $x \in [a_k, b_k]$ we define

$$f(x) = \frac{f(b_k) - f(a_k)}{b_k - a_k} \cdot (x - a_k) + f(a_k).$$

Notice that this definition of f on all the bounded intervals $[a_k, b_k]$ agrees with the definition of f at the points a_k, b_k. Thus, f is continuous on these intervals since it is linear. Now, suppose (a_k, b_k) is bounded from below and unbounded above so that $(a_k, b_k) = (a_k, \infty)$. The closure of this unbounded interval is $[a_k, \infty)$. As before, $a_k \in F$ and so $f(a_k)$ is defined. Then, for all $x \in [a_k, \infty)$ define $f(x) = f(a_k)$, which is clearly continuous since it is constant. Similarly, for intervals which are unbounded below, $(-\infty, b_k)$ define $f(x) = f(b_k)$ for all $x \in (-\infty, b_k]$. In the stupid case where $F = \emptyset$, then $\mathbb{R} - F = (-\infty, \infty)$ and so we just define $f(x) = 1$ for all $x \in \mathbb{R}$. Thus, f is defined and continuous on all of $\mathbb{R}$. ☕

3.3.31 Let $\{f_n\}$ be a sequence of measurable functions on E that converge pointwise to f on E. Show that $E = \bigcup_{k=1}^{\infty} E_k$ where for each index k, E_k is measurable and $\{f_n\}$ converges uniformly to f on each E_k if $k > 1$ and $m(E_1) = 0$.

Proof. First suppose that $m(E) < \infty$. By Egoroff's Theorem, for all $k > 2, k \in \mathbb{N}$ there exists closed subsets $E_k \subseteq E$ so that

$$\{f_n\} \to f \text{ uniformly on } E_k \text{ and } m(E \sim E_k) < \frac{1}{k}.$$

Then $\{E_k\}_{k=2}^{\infty}$ is an ascending collection of measurable sets. Let $E_1 = E - \bigcup_{k=2}^{\infty} E_k$, and so by DeMorgan's identities,

$$E_1 = \bigcap_{k=2}^{\infty} [E \sim E_k].$$

Then $\{E \sim E_k\}_{k=2}^{\infty}$ is a descending collection of measurable sets and since $m(E - E_2) < 1/2 < \infty$, then by the continuity of measure

$$m(E_1) = \lim_{k\to\infty} [E \sim E_k] = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Then $E = \bigcup_{k=1}^{\infty} E_k$ and each E_k is measurable. This completes the case when E has finite measure.
Now suppose $m(E) = \infty$. For all $i \in \mathbb{Z}$, define $G_i = [i, i+1) \cap E$. Since $m([i, i+1)) = 1$ then $m(G_i) \leq 1 < \infty$. By the previous case

$$G_i = \bigcup_{k=1}^{\infty} A_{i,k} \text{ where } m(A_{i,1}) = 0 \quad \forall\, i \in \mathbb{Z}$$

and $\{f_n\} \to f$ uniformly on $A_{i,k}$ for all $k > 1$. Let $E_1 = \bigcup_{i\in\mathbb{Z}} A_{i,1}$. Relabel $\{A_i, 1\}_{i\in\mathbb{Z}}$ as $\{A_{\ell,1}\}_{\ell\in\mathbb{N}}$. Since $\{A_\ell, 1\}_{\ell\in\mathbb{N}}$ is a collection of pairwise disjoint measurable sets, then by countable additivity of measure we get

$$m(E_1) = m\left(\bigcup_{\ell\in\mathbb{N}} A_{\ell,1}\right) = \sum_{\ell\in\mathbb{N}} m(A_{\ell,1}) = 0.$$

Relabel $\{A_{i,k} \mid i \in \mathbb{Z}, k \in \mathbb{N} \setminus \{1\}\}$ as $\{E_j \mid j \in \mathbb{N} \setminus \{1\}\}$. So, $\{f_n\} \to f$ uniformly on E_j for all $j > 1$ and

$$\begin{aligned} E = \bigcup_{i\in Z} G_i = \bigcup_{i\in\mathbb{Z}}\left(\bigcup_{k=1}^{\infty} A_{i,k}\right) = \bigcup_{i\in\mathbb{Z}}\left(A_{i,1} \cup \bigcup_{k=2}^{\infty} A_{i,k}\right) &= \left(\bigcup_{i\in Z} A_{i,1}\right) \cup \left(\bigcup_{i\in\mathbb{Z}}\bigcup_{k=2}^{\infty} A_{i,k}\right) \\ &= \left(\bigcup_{\ell=1}^{\infty} A_{\ell,1}\right) \cup \left(\bigcup_{j=2}^{\infty} E_j\right) \\ &= E_1 \cup \bigcup_{j=2}^{\infty} E_j \\ &= \bigcup_{j=1}^{\infty} E_j \end{aligned}$$

4.2.16 Let f be a nonnegative bounded measurable function on a set of finite measure E. Assume $\int_E f = 0$. Show that $f = 0$ almost everywhere on E.

Proof. For all $n \in \mathbb{N}$ define

$$F_n := \{x \in E \mid f(x) \geq 1/n\}.$$

Then, let

$$F = \bigcup_{n=1}^{\infty} F_n = \{x \in E \mid \exists n \in \mathbb{N} \ s.t.\ f(x) \geq 1/n\} = \{x \in E \mid f(x) > 0\}.$$

Notice that for each fixed $n \in \mathbb{N}$, we have

$$f \geq \chi_{F_n} \cdot f \text{ on } E \quad \text{and} \quad f \geq \frac{1}{n} \text{ on } F_n.$$

So, by monotonicity of the integral

$$0 = \int_E f \geq \int_E \chi_{F_n} \cdot f = \int_{F_n} f \geq \int_{F_n} \frac{1}{n} = \frac{1}{n} \cdot m(F_n).$$

Therefore for each $n \in \mathbb{N}$ we have $0 \geq m(F_n)/n$ which means $m(F_n) = 0$. Then by countable subadditivity of measure, we get

$$m(F) = m\left(\bigcup_{n=1}^{\infty} F_n\right) \leq \sum_{n=1}^{\infty} m(F_n) = 0.$$

and so the collection points of E at which f exceeds 0 has measure zero, which means $f = 0$ almost everywhere on E.

4.3.22 Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f and f be integrable over $\mathbb{R}$. Show that

$$\text{if } \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

Proof. Let E be any measurable set. Since $\{f_n\} \to f$ pointwise on $\mathbb{R}$, then $\{f_n\} \to f$ pointwise on E and $\mathbb{R} \sim E$. By Fatou's Lemma,

$$\int_E f \leq \liminf \int_E f_n. \tag{1}$$

Again by Fatou's Lemma,

$$\int_{\mathbb{R}\sim E} f \leq \liminf \int_{\mathbb{R}\sim E} f_n.$$

By Additivity over Domains, we get

$$\int_{\mathbb{R}\sim E} f_n = \int_{\mathbb{R}} f_n - \int_E f_n = \int_{\mathbb{R}} f_n + \int_E -f_n$$

for each n. So, we get

$$\begin{aligned} \liminf \int_{\mathbb{R}\sim E} f_n &= \liminf \left(\int_{\mathbb{R}} f_n + \int_E -f_n \right) \\ &= \int_{\mathbb{R}} f + \liminf \int_E -f_n \\ &= \int_{\mathbb{R}} f - \limsup \int_E f_n \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}\sim E} f &\leq \int_{\mathbb{R}} f - \limsup \int_E f_n \\ \limsup \int_E f_n &\leq \int_{\mathbb{R}} f - \int_{\mathbb{R}\sim E} f \\ \limsup \int_E f_n &\leq \int_E f. \end{aligned} \tag{2}$$

Then by (1) and (2), the desired result follows. ☕

4.3.27 Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

Proof. Define $g_n := \inf\{f_k \mid k \geq n\}$. Then $\{g_n\}$ is an increasing sequence and

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} [\inf_k \{f_k \mid k \geq n\}] = \liminf f_n.$$

Then by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E g_n = \int_E \liminf f_n.$$

Since $g_n \leq f_n$ for all n, then my monotonicity of the integral

$$\int_E g_n \leq \int_E f_n$$

for all n. Then by the Order Limit Theorem,

$$\int_E \liminf f_n = \lim_{n\to\infty} \int_E g_n \leq \lim_{n\to\infty} \int_E f_n. \tag{3}$$

Since it is always the case that

$$\lim_{n\to\infty} \int_E f_n \leq \liminf \int_E f_n, \tag{4}$$

then we conclude

$$\int_E \liminf f_n \leq \lim_{n\to\infty} \int_E f_n \leq \liminf \int_E f_n.$$

by (3) and (4) ☕

4.4.32 Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.

Proof. Since $|f_n| \leq g_n$ on E and $\{g_n\} \to g$ pointwise, then $|g_n| < g$ on E. Also, since $|f| < g$ almost everywhere on E and g is integrable over E, then by the Integral Comparison Test, f and each f_n also are integrable over E. We infer from Proposition 15 that, by possibly excising from E a countable collection of sets of measure zero and using the countable additivity of Lebesgue measure, we may assume that f and each f_n is finite almost everywhere on E. The function $g - f$ and for each n, the function $g_n - f_n$, are properly defined, nonnegative, and measurable. Moreover, the sequence $\{g_n - f_n\}$ converges pointwise almost everywhere to $g - f$. Fatou's Lemma tells us that

$$\int_E (g - f) \leq \liminf \int_E (g_n - f_n).$$

Thus by Linearity of integration for integrable functions,

$$\begin{aligned}\int_E g - \int_E f &\leq \liminf \left(\int_E (g_n + \int_E -f_n) \right) \\ &= \liminf \int_E g_n + \liminf \int_E -f_n \\ &= \int_E g - \limsup \int_E f_n\end{aligned}$$

and so

$$\limsup \int_E f_n \leq f.$$

Similarly, considering the sequence $\{g_n + f_n\}$, we obtain

$$\int_E f \leq \liminf \int_E f_n.$$

☕

4.4.33 let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ almost everywhere on E and f is integrable over E. Show that

$$\int_E |f - f_n| \to 0 \text{ if and only if } \lim_{n\to\infty} \int_E |f_n| = \int_E |f|$$

Proof. " $\Rightarrow$ " In order to show that $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$, we need to show that

$$\left| \int_E |f_n| - \int_E |f| \right| \to 0. \tag{5}$$

Observe that by the Triangle Inequality

$$\begin{aligned}\big||f_n| - |f|\big| &= \big||f_n| - f + f + f_n - f_n - |f|\big| \\ &\leq |f - f_n| + \big||f_n| - f\big| + \big|f_n - |f|\big| \\ &\leq |f - f_n|.\end{aligned}$$

Then by Linearity of the Integral, the Integral Comparison Test, and Monotonicity of the Integral, we get

$$\begin{aligned}0 \leq \left| \int_E |f_n| - \int_E |f| \right| &= \left| \int_E |f_n| - |f| \right| \\ &\leq \int_E \big||f_n| - |f|\big| \\ &\leq \int_E |f - f_n|.\end{aligned}$$

for each n. Since $\int_E |f - f_n| \to 0$, then (5) holds.
" $\Leftarrow$ " Since $f_n \to f$ then $\{|f_n - f|\} \to 0$. Define $g_n := |f_n| + |f|$ and $g := 2|f|$. Notice that for each n

$$|f_n - f| \leq |f_n| + |f| = g_n.$$

Since

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|, \quad \text{then} \quad \lim_{n\to\infty} \int_E |f_n| + |f| = \int_E 2|f|.$$

Since f is integrable over E, then $\int_E 2|f|$ is integrable as well. So,

$$\{|f_n - f|\} \to 0, \quad \{g_n\} \to g, \quad |f_n - f| \leq g_n \ \forall n, \quad \text{and} \quad \lim_{n\to\infty} \int_E g_n = \int_E g < \infty.$$

Therefore by the General Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0.$$

☕

4.4.34 Let f be a nonnegative measurable function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f$$

Proof. Define $E_n := [-n, n]$. Notice that

$$\int_{-n}^{n} f = \int_{\mathbb{R}} f \cdot \chi_{E_n}.$$

By Linearity of the Integral and the Integral Comparison Test,

$$\left| \int_{\mathbb{R}} f \cdot \chi_{E_n} - \int_{\mathbb{R}} f \right| = \left| \int_{\mathbb{R}} f \cdot \chi_{E_n} - f \right| \leq \int_{\mathbb{R}} |f \cdot \chi_{E_n} - f| \, .$$

For each fixed n, $|f \cdot \chi_{E_n} - f| = 0$. So,

$$\int_{\mathbb{R}} |f \cdot \chi_{E_n} - f| \to 0.$$

Therefore, by the previous exercise, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f \cdot \chi_{E_n}| = \int_{\mathbb{R}} |f|$$

☕

4.5.37 Let f be an integrable function on E. Show that for each $\epsilon > 0$ there exists a natural number N for which if $n \geq N$, then $\left|\int_{E_n} f\right| < \epsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$.

Proof. Notice that

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty}\{x \in E \mid |x| \geq n\} = \{x \in E \mid |x| \geq n \ \forall n \in \mathbb{N}\} = \emptyset$$

because there is no real number which exceeds every natural number by the Archimedean property. Therefore, $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$. Since $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable sets, then by the Continuity of Integration

$$\lim_{n\to\infty}\int_{E_n} f = \int_{\bigcap_{n=1}^{\infty} E_n} f = 0.$$

4.6.44 Let f be integrable over $\mathbb{R}$ and $\epsilon > 0$. Establish the following three approximation properties.

(i) There is a simple function η on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \epsilon$.

Proof. Suppose that f is nonnegative. Then by definition of the integral of f over $\mathbb{R}$, there exists a bounded, measurable function h of finite support and $0 \leq h \leq f$ on $\mathbb{R}$ for which

$$\int_{\mathbb{R}} f - h < \frac{\epsilon}{4} \quad \text{and since } f - h = |f - h|, \quad \int_{\mathbb{R}} |f - h| < \frac{\epsilon}{4}$$

Since h is integrable, then by definition of the integral for bounded functions, there exists a simple function φ for which $\varphi \leq h$ on $\mathbb{R}$ and

$$\int_{\mathbb{R}} h - \varphi < \frac{\epsilon}{4} \text{ and since } h - \varphi = |h - \varphi|, \quad \int_{\mathbb{R}} |h - \varphi| < \frac{\epsilon}{4}.$$

Since $\varphi \leq h$ on $\mathbb{R}$ and h has finite support on $\mathbb{R}$, then so does φ. Therefore

$$\int_{\mathbb{R}} |f - \varphi| = \int_{\mathbb{R}} |f - h| + \int_{\mathbb{R}} |h - \varphi| < \frac{\epsilon}{2}.$$

Now, if f is any measurable function, we can apply the preceding argument to f^+ and f^- to obtain simple functions φ and ψ of finite support on $\mathbb{R}$ for which

$$\int_{\mathbb{R}} |f^+ - \varphi| < \frac{\epsilon}{2} \quad \text{and} \quad \int_{\mathbb{R}} |f^- - \psi| < \frac{\epsilon}{2}.$$

Then $\eta = \varphi - \psi$ is a simple function of finite support on $\mathbb{R}$ and

$$\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} |f^+ - \varphi| + \int_{\mathbb{R}} |\psi - f^-| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(ii) There is a step function s on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \epsilon$.

Proof. By part (i), there exists a simple function η for which $\int_{\mathbb{R}} |f-\eta| < \epsilon/2$. Let E be a closed and bounded interval and $\mathcal{O}$ an open set containing E for which $\mu(\mathcal{O} \sim E) < \epsilon/2$. Let $\mathcal{O} = \bigsqcup_{k=1}^{\infty} I_k$ for open intervals I_k. Then there exists N so that $\mu\left(\bigcup_{k=N+1}^{\infty} I_k\right) < \epsilon/2$. Define the step function

$$s = \sum_{k=1}^{N} \chi_{I_k}.$$

Suppose $\sum_{k=1}$ ☕

(iii) There is a continuous function g on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \epsilon$.

Proof. ☕

4.6.46 (Riemann-Lebesgue) Let f be integrable over $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)dx = 0$$

Proof. By Exercise 44 (ii), since f is integrable, then there is a step function φ which vanishes outside of a closed bounded interval, say $[a, b]$, and

$$\int_{\mathbb{R}} |f = \varphi| < \frac{\epsilon}{2}.$$

We first show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi(x)\cos(nx)dx = 0.$$

To that end, suppose $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ is a partition of $[a, b]$, and for each $0 \le i \le m-1$, let $a_i = \varphi(x)$ for all $x \in (x_i, x_{i+1})$. Then for each natural number n,

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x)\cos(nx)dx &= \int_{[a,b]} \varphi(x)\cos(nx)dx \\
&= \sum_{i=1}^{m} a_i \int_{[x_i,x_{i+1}]} \cos(nx)dx \\
&= \sum_{i=1}^{m} a_i \left.\frac{\sin(nx)}{n}\right|_{x_i}^{x_{i+1}} \\
&= \sum_{i=1}^{m} a_i \frac{(\sin(nx_{i+1}) - \sin(nx_i))}{n}
\end{aligned}$$

Notice that for each i, we have $|(\sin(nx_{i+1}) - \sin(nx_i))| \leq 2$. Thus,

$$\lim_{n\to\infty} \frac{(\sin(nx_{i+1}) - \sin(nx_i))}{n} \leq \lim_{n\to\infty} \frac{2}{n} = 0.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(x)\cos(nx)dx &= \lim_{n\to\infty} \sum_{i=1}^{m} a_i \frac{(\sin(nx_{i+1}) - \sin(nx_i))}{n} \\ &= \sum_{i=1}^{m} a_i \lim_{n\to\infty} \frac{(\sin(nx_{i+1}) - \sin(nx_i))}{n} = 0\end{aligned}$$

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ so that for all $n \geq N$ we have $\left|\int_{\mathbb{R}} \varphi(x)\cos(nx)\right| < \epsilon/2$. Therefore, for all $n \geq N$, we get

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x)\cos(nx)dx\right| &= \left|\int_{\mathbb{R}} (f(x) - \varphi(x))\cos(nx)\right| + \left|\int_{\mathbb{R}} \varphi(x)\cos(nx)\right| \\ &\leq \int_{\mathbb{R}} |(f(x) - \varphi(x)) \cdot 1| + \left|\int_{\mathbb{R}} \varphi(x)\cos(nx)\right| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

4.6.49 Let f be integrable over $\mathbb{R}$. Show that the following four assertions are equivalent:

(i) $f = 0$ a.e. on $\mathbb{R}$.

(ii) $\int_{\mathbb{R}} fg = 0$ for every bounded measurable function g on $\mathbb{R}$.

(iii) $\int_A f = 0$ for every measurable set A.

(iv) $\int_{\mathcal{O}} f = 0$ for every open set $\mathcal{O}$.

Proof. $(i) \implies (ii)$ We first show that in fact fg is integrable. Since f is integrable, then $\int_{\mathbb{R}} |f| < \infty$. Since g is bounded on $\mathbb{R}$, there exists $M > 0$ such that $|g| \leq M$ on $\mathbb{R}$. So,

$$\int_{\mathbb{R}} |fg| \leq \int_{\mathbb{R}} |f \cdot M| = M \int_{\mathbb{R}} |f| < \infty.$$

Let $E_0 = \{x \in \mathbb{R} \mid f(x) \neq 0\}$. Then $m(E_0) = 0$ since $f = 0$ a.e. on $\mathbb{R}$. Then $fg = 0$ on $\mathbb{R} \sim E_0$. So,

$$\int_{\mathbb{R}} fg = \int_{\mathbb{R}\sim E_0} fg + \int_{E_0} fg = 0.$$

$(ii) \implies (iii)$ Define $g = \chi_A$. Then g is measurable and bounded so that

$$0 = \int_{\mathbb{R}} fg = \int_{\mathbb{R}} f\chi_A = \int_A f$$

$(iii) \implies (iv)$ Since every open set is measurable, then by (iii) we have $\int_{\mathcal{O}} f = 0$.

$(iv) \implies (i)$ We show that $f^+, f^- \equiv 0$ a.e. on $\mathbb{R}$, so that $f^+ - f^- = f \equiv 0$ a.e. on $\mathbb{R}$. To do this, we show that

$$\int_{\mathbb{R}} f^+ = 0 \quad \text{and} \quad \int_{\mathbb{R}} f^- = 0$$

and then apply Proposition 9 of Chapter 4. Notice that since f is measurable, then the sets

$$E^+ = \{x \in \mathbb{R} \mid f(x) \geq 0\} \quad \text{and} \quad E^- = \{x \in \mathbb{R} \mid f(x) \leq 0\}$$

are measurable. Moreover,

$$\int_{\mathbb{R}} f^+ = \int_{E^+} f^+ \quad \text{and} \quad \int_{\mathbb{R}} f^- = \int_{E^-} f^-$$

Since E^+ is measurable there exist a G_δ set $G^+ = \bigcap_{k=1}^{\infty} U_k$ for which

$$E^+ \subseteq G^+ \quad \text{and} \quad m(G^+ \sim E^+) = 0.$$

Define $\mathcal{O}_n = \bigcap_{k=1}^{n} U_k$. Then $G^+ = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ and $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is a descending collection of open sets. Then by additivity over domains and continuity of integration

$$\int_{E^+} f^+ = \int_{G^+} f - \int_{G^+ \sim E^+} f = \int_{G^+} f = \int_{\bigcap_{n=1}^{\infty} \mathcal{O}_n} f = \lim_{n \to \infty} \int_{\mathcal{O}_n} f = 0.$$

Similarly, we can approximate the set E^- by a G_δ set G^- and use a similar argument to show that $\int_{E^-} f = 0$. ☕

5.2.7 Let E have finite, $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Prove that $\{f_n \cdot g\} \to f \cdot g$ in measure, and use this to show that $\{f_n^2\} \to f^2$ in measure. Infer from this that if $\{g_n\} \to g$ in measure, then $\{f_n \cdot g_n\} \to f \cdot g$ in measure.

Proof. Suppose, for contradiction, that $\{f_n \cdot g\} \not\to f \cdot g$ in measure. So, there exists an $\eta > 0, \epsilon_\eta > 0$ and an $N \in \mathbb{N}$ such that for all $n \geq N$

$$m(\{x \in E \mid |f_n g(x) - fg(x)| > \eta\}) \geq \epsilon_\eta.$$

Choose a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $n_k \geq N$ for all k. Since $\{f_n\} \to f$ in measure, then $f_{n_k} \to f$ in measure. By Riesz's Theorem, there exists a subsequence $\{f_{n_{k_\ell}}\}$ which converges pointwise a.e. to f on E. So, $\{f_{n_{k_\ell}} g\}$ converges pointwise a.e. to fg on E. This implies (by Proposition 3) that $\{f_{n_{k_\ell}} g\} \to fg$ in measure, a contradiction.

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5.3.17 Let f be a function on $[0, 1]$ that is continuous on $(0, 1]$. Show that it is possible for the sequence $\{\int_{[1/n,1]} f\}$ to converge and yet f is not Lebesgue integrable over $[0, 1]$. Can this happen if f is nonnegative?

Proof. ☕

6.2.9 Show that a set E of real numbers has measure zero if and only if there is a countable collection of open intervals $I_{k\,k=1}^{\infty}$ for which each point in E belongs to infinitely many of the I_k's and $\sum_{k=1}^{\infty} \ell(I_k) < \infty$.

Proof. ($\Rightarrow$) By definition of outer measure, for each natural number n, there exists a collection $\{I_{j,n}\}_{j=1}^{\infty}$ of open intervals for which $E \subseteq \bigcup_{j=1}^{\infty} I_{j,n}$ and

$$\sum_{j=1}^{\infty} \ell(I_{j,n}) \leq \mu(E) + \frac{1}{n^2} = \frac{1}{n^2}.$$

Then,

$$E \subseteq \bigcup_{n=1}^{\infty} \left(\bigcup_{j=1}^{\infty} I_{j,n} \right) \quad \text{and} \quad \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \ell(I_{j,n}) \right) \leq \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty.$$

Let $\{I_k\}_{k=1}^{\infty}$ be the countable collection of all the open intervals $I_{j,n}$ for all j and all n. Fix $x \in E$. Since $E \subseteq \bigcup_{j=1}^{\infty} I_{j,n}$ for each n, then there exists $I_{j_x,n} \in \{I_{j,n}\}_{j=1}^{\infty}$ containing x for each n. Thus, x belongs to infinitely many I_k's.
($\Leftarrow$) Since $\sum_{k=1}^{\infty} \ell(I_k) < \infty$, then by the Borel-Cantelli Lemma, almost all $x \in \mathbb{R}$ belong to at most finitely many of the I_k's. Thus, if we define E to be the set consisting of all $x \in \mathbb{R}$ which belong to infinitely many of the I_k's, then $\mu(E) = 0$. ☕

6.2.10 (Riesz-Nagy) Let E be a set of measure zero contained in the open interval (a, b). According to the proceeding problem, there is a countable collection of open intervals contained in (a, b), $\{(c_k, d_k)\}_{k=1}^{\infty}$, for which each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty}(d_k - c_k) < \infty$. Define

$$f(x) = \sum_{k=1}^{\infty} \ell((c_k, d_k) \cap (-\infty, x)) \text{ for all } x \in (a, b).$$

Show that f is increasing and fails to be differentiable at each point in E.

Proof. Suppose $x_1 \leq x_2$ for some $x_1, x_2 \in (a, b)$. Then for each k,

$$(c_k, d_k) \cap (-\infty, x_1) \subseteq (c_k, d_k) \cap (-\infty, x_2)$$

which implies

$$\ell((c_k, d_k) \cap (-\infty, x_1)) \leq \ell((c_k, d_k) \cap (-\infty, x_2)).$$

Since this is true for each k, $f(x_1) \leq f(x_2)$, i.e., f is increasing. We now show that f is not differentiable for all $x \in E$. Pick $x \in E$ and since x belongs to infinitely many of the open intervals (c_k, d_k), we can choose finitely many which x belongs to, say $\{(c_{k_1}, d_{k_1}), \ldots, (c_{k_n}, d_{k_n})\}$. For each $i \in \{1, \ldots, n\}$, let t_i be such that $x + t_i \in (c_{k_i}, d_{k_i})$, and $t = \min_i\{t_i\}$. Then for each i, we have

$$\ell((c_{k_i}, d_{k_i}) \cap (-\infty, x + t)) - \ell((c_{k_i}, d_{k_i}) \cap (-\infty, x)) = \ell((x, x + t)) = t.$$

Since f is increasing, then $f(x + t) - f(x) \geq nt$, which implies $\overline{D}f(x) \geq n$ for each natural number n, i.e., $\overline{D}f(x) = \infty$. Thus f is not differentiable at any point in E. ☕

6.2.18 Show that if f is defined on (a,b) and $c \in (a,b)$ is a local minimizer for f, then $\underline{D}f(c) \leq 0 \leq \overline{D}f(c)$.

Proof. Since c is a local minimum, there exists a $\delta > 0$ such that $f(c) \leq f(x)$ for all $x \in (c-\delta, c+\delta)$. Let $h \in (0,\delta)$. If $0 < t < h$, then

$$0 \leq \frac{f(c+t)-f(c)}{t},$$

which gives

$$0 \leq \sup_{0<t<h} \frac{f(c+t)-f(c)}{t} \leq \sup_{0<|t|<h} \frac{f(c+t)-f(c)}{t}.$$

Similarly, if $-h < t < 0$, then

$$\frac{f(c+t)-f(c)}{t} \leq 0.$$

and so

$$\inf_{0<|t|<h} \frac{f(c+t)-f(c)}{t} \leq \inf_{-h<t<0} \frac{f(c+t)-f(c)}{t} \leq 0.$$

Therefore,

$$\underline{D}f(c) = \lim_{h\to 0}\left[\inf_{0<|t|<h} \frac{f(c+t)-f(c)}{t}\right] \leq 0 \leq \lim_{h\to 0}\left[\sup_{0<|t|<h} \frac{f(c+t)-f(c)}{t}\right] = \overline{D}f(c)$$

6.2.19 Let f be continuous on $[a,b]$ with $\underline{D}f \geq 0$ on (a,b). Show that f is increasing on $[a,b]$. (Hint: First show this for a function g for which $\underline{D}g \geq \epsilon > 0$ on (a,b). Apply this to the function $g(x) = f(x) + \epsilon x$.)

Proof. Using the hint, if $\underline{D}g(x) > 0$ on (a,b), then there exists a $\delta > 0$ so that for all[1] $|h-0| < \delta$,

$$\inf_{0<|t|\leq h} \frac{g(x+t)-g(x)}{t} > 0.$$

Fix $x \in (a,b)$. Then for all $0 < |t| < h < \delta$, we get

$$\frac{g(x+t)-g(x)}{t} > 0.$$

So if $0 < t < h$, then the above implies $g(x+t) - g(x) > 0$ i.e., $g(x+t) > g(x)$. Similarly if $-h < t < 0$, then $g(x+t) - g(x) < 0$, i.e., $g(x+t) < g(x)$. Therefore,

$$g(x-t) < g(x) < g(x+t) \text{ for all } 0 < t < h < \delta. \qquad (*)$$

[1](i.e., $-\delta < h < \delta$, but since h is positive when we take the infimum, we have $0 < h < \delta$)

Fix $c < d$ in (a, b). We apply the above argument to each $x \in [c, d]$ to obtain a collection $\{(x - \delta_x, x + \delta_x)\}_{x\in[c,d]}$. Notice that is collection is an open cover of $[c, d]$. Since $[c, d]$ is compact, there exists $x_1, \ldots, x_{n-1}$ in $[c, d]$ so that $\{(x_i - \delta_{x_i}, x + \delta_{x_i})\}_{i=1}^{n-1}$ covers $[c, d]$. Add the intervals $(c - \delta_c, c + \delta_c)$ and $(d - \delta_d, d + \delta_d)$ to this finite subcover, where $c = x_0$ and $d = x_n$. Each consecutive pair of intervals must have nonempty intersection, or else a point in $[c, d]$ will not be contained in an element of the finite subcover. Therefore, there exists $y_1, \ldots, y_n$ so that

$$c = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < x_{n-1} < y_n < x_n = d$$

and y_i is in the delta neighborhood of both x_{i-1} and x_i for all $1 \leq i \leq n$. Therefore, by $(*)$

$$g(c) = g(x_0) < g(y_1) < g(x_1) < g(y_2) \cdots < g(x_{n-1}) < g(y_n) < g(x_n) = g(d).$$

Therefore, g is increasing on $[c, d]$. It remains to show that g is increasing on $[a, b]$. Define $a_n = a + 1/n$ for all n. Then $\{a_n\}$ is a decreasing sequence converging to a. Since g is continuous, the sequence $\{g(a_n)\}$ converges to $g(a)$. For sufficiently large N, $g(a_{n+1}) < g(a_n) < g(c)$ for all $n > N$. Therefore, $g(a) \leq g(c)$. So, g is increasing on $[a, c]$. By a similar argument applied to b, we get that $g(d) \leq g(b)$, so that g is increasing on all of $[a, b]$.
Let $g(x) = f(x) = \epsilon x$. Then $\underline{D}f(x) \geq \epsilon > 0$ on (a, b). Therefore by the above argument, $f(x) + \epsilon x$ is increasing on $[a, b]$. Let $x < y$ be in $[a, b]$. Then $f(x) + \epsilon x \leq g(x) + \epsilon y$. Suppose for contradiction that $f(x) > f(y)$. Then $0 < f(x) - f(y) \leq \epsilon(y - x)$. If $\epsilon = (f(x) - f(y))/(2(y - x))$, then $f(x) - f(y) \leq (f(x) - f(y))/2$, a contradiction. Thus, f is increasing on $[a, b]$.

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6.3.33 Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f. Show that

$$TV(f) \leq \liminf TV(f_n).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_k\}$ be a partition of $[a, b]$ and let n be a natural number. Then $V(f_n, P) \leq TV(f_n)$, which implies

$$\liminf V(f_n, P) \leq \liminf TV(f_n)$$

Notice that since $\{f_n\} \to f$ pointwise, then

$$\begin{aligned}
\liminf V(f_n, P) &= \liminf \sum_{i=1}^{k} |f_n(x_i) - f_n(x_{i-1})| \\
&= \lim \sum_{i=1}^{k} |f_n(x_i) - f_n(x_{i-1})| \\
&= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\
&= V(f, P).
\end{aligned}$$

Therefore,

$$V(f, P) = \liminf V(f_n, P) \leq \liminf TV(f_n).$$

Since this is true for all partitions P, we get

$$TV(f) = \sup\{V(f, P)\} \leq \liminf TV(f_n).$$

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6.3.35 For α and β positive numbers, define the function f on $[0, 1]$ by

$$f(x) = \begin{cases} x^{\alpha} \sin(1/x^{\beta}) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0. \end{cases}$$

Show that if $\alpha > \beta$, then f is of bounded variation on $[0, 1]$. Then show that if $\alpha \leq \beta$, then f is not of bounded variation on $[0, 1]$.

We first prove the following Lemma:

Lemma. *Let $\{x_n\}$ be a strictly decreasing sequence contained in $[0, 1]$ converging to 0 and define $f_n := f\chi_{[x_n,1]}$ for all n. Then $\lim\limits_{n\to\infty} TV(f_n) = TV(f)$.*

Proof. Notice that $\{f_n\} \to f$ pointwise on $[0,1]$ and so by the previous exercise, $TV(f) \leq \liminf TV(f_n)$. Hence it suffices to show that $\limsup TV(f_n) \leq TV(f)$. Fix $n \in \mathbb{N}$. Let $P_n = \{x_n = a_0, a_1, \ldots, a_k = 1\}$ be a partition of $[x_n, 1]$. Then,

$$\begin{aligned} V(f_n, P_n) = V(f\chi_{[x_n,1]}, P_n) &\leq V(f\chi_{[x_n,1]}, P_n \cup \{0\}) \\ &\leq V(f, P_n \cup \{0\}) \\ &\leq TV(f). \end{aligned}$$

Hence $TV(f_n) \leq TV(f)$. Since this is true for all n, then

$$\limsup TV(f_n) \leq TV(f).$$

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Proof. Define a decreasing sequence $(x_n)_{n=1}^{\infty}$ by

$$x_n = \left(\frac{1}{\pi/2 + \pi n}\right)^{1/\beta} \qquad \forall\ n.$$

Then for any points x_i and x_{i-1}, notice that

$$\begin{aligned} |f(x_i) - f(x_{i-1})| &= \left|(x_i)^\alpha \sin(1/(x_i)^\beta) - (x_{i-1})^\alpha \sin(1/(x_{i-1})^\beta)\right| \\ &= |(x_i)^\alpha \cdot (\pm 1) - (x_{i-1})^\alpha \cdot (\pm 1)| \\ &= x_i^\alpha + x_{i-1}^\alpha. \end{aligned}$$

Fix $n \in \mathbb{N}$ and let $P_n = \{1 = x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[x_n, 1]$. Then

$$\begin{aligned} V(f, P_n) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{n} (x_i)^\alpha + (x_{i-1})^\alpha \\ &= \sum_{i=1}^{n} \left(\frac{1}{\pi/2 + \pi i}\right)^{\alpha/\beta} + \sum_{i=1}^{n} \left(\frac{1}{\pi/2 + \pi(i-1)}\right)^{\alpha/\beta}. \end{aligned}$$

Define $f_n := f\chi_{[x_n,1]}$. Notice that for each pair of consecutive terms x_i, x_{i+1} in $\{x_n\}$, the variation of f on the interval $[x_i, x_{i+1}]$ is maximal because f is monotone on each of these intervals. Therefore, the total variation of f on the interval $[x_n, 1]$ is equal to the variation of f with respect to the partition P_n. In other words,

$$V(f, P_n) = TV(f\chi_{[x_n,1]}) = TV(f_n).$$

Note that $\{f_n\} \to f$ so by the lemma, $\lim TV(f_n) = TV(f)$. So,

$$\begin{aligned} TV(f) = \lim_{n\to\infty} TV(f_n) &= \lim_{n\to\infty} V(f, P_n) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{\pi/2 + \pi i}\right)^{\alpha/\beta} + \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{\pi/2 + \pi(i-1)}\right)^{\alpha/\beta} \\ &= \sum_{i=1}^{\infty} \left(\frac{1}{\pi/2 + \pi i}\right)^{\alpha/\beta} + \lim_{n\to\infty} \sum_{i=1}^{\infty} \left(\frac{1}{\pi/2 + \pi(i-1)}\right)^{\alpha/\beta}. \end{aligned}$$

Hence the total variation of f is equal to a sum of two p-series. If $\alpha > \beta$ then $\alpha/\beta > 1$ which means the p-series converges and thus $TV(f)$ is bounded, i.e., f has bounded variation. If $\alpha \leq \beta$, then $\alpha/\beta \leq 1$, which means the series diverges, i.e. f does not have bounded variation. ☕

6.4.38 Show that f is absolutely continuous on $[a, b]$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for every countable disjoint collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals in (a, b),

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon \text{ if } \sum_{k=1}^{\infty} (b_k - a_k) < \delta.$$

Proof. ($\Rightarrow$) Let $\epsilon > 0$. Since f is absolutely continuous on $[a, b]$ then there exists a $\delta > 0$ such that for every finite disjoint collection of intervals $\{(a_k, b_k)\}_{k=1}^{n}$ contained in (a, b) for which $\sum_{k=1}^{n}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon/2$.

Let $\{(c_k, d_k)\}_{k=1}^{\infty}$ be a countable collection of disjoint intervals contained in (a, b) for which $\sum_{k=1}^{\infty}(d_k - c_k) < \delta$. Hence, for each n, we have

$$\sum_{k=1}^{n}(d_k - c_k) < \delta \implies \sum_{k=1}^{n} |f(d_k) - f(c_k)| < \frac{\epsilon}{2}.$$

Since this holds for each n,

$$\lim_{n\to\infty} \sum_{k=1}^{n} |f(d_k) - f(c_k)| = \sum_{k=1}^{\infty} |f(d_k) - f(c_k)| \leq \frac{\epsilon}{2}.$$

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6.4.39 Use the preceding problem to show that if f is continuous and increasing on $[a, b]$, then f is absolutely continuous on $[a, b]$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for a measurable subset E of $[a, b]$,

$$m^*(f(E)) < \epsilon \text{ if } m(E) < \delta.$$

Proof. ($\Rightarrow$) Suppose f is absolutely continuous. Let $\epsilon > 0$ and $\delta > 0$ be such that for every countable disjoint collection of intervals $\{(a_k, b_k)\}_{k=1}^{\infty}$ contained in (a, b) for which $\sum_{k=1}^{\infty}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon$. Let E be a measurable subset of $[a, b]$ with $m(E) < \delta$. Then by definition of outer measure, there exists a disjoint collection of open intervals $\{(c_k, d_k)\}_{k=1}^{\infty}$ which cover E and for which

$$\sum_{k=1}^{\infty}(d_k - c_k) < m(E) + \delta/2 < \delta.$$

We can assume these intervals are disjoint because if not, their union is an open set, which can be written as the disjoint union of open intervals. Also, we can assume these disjoint intervals are contained in $[a, b]$ because if not, we can intersect each of them with (a, b) to get a collection of disjoint intervals which cover E and are contained in (a, b). Then

$$m(f(E)) \leq m\left(\bigcup_{k=1}^{\infty} f((c_k, d_k))\right) = \sum_{k=1}^{\infty} m(f(c_k, d_k)) \leq \sum_{k=1}^{\infty} f(d_k) - f(c_k) < \epsilon.$$

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6.4.40 Use the preceding problem to show that an increasing absolutely continuous function f on $[a, b]$ maps sets of measure zero onto sets of measure zero.

Proof. Let $\epsilon > 0$. If $E \subset [a, b]$ has measure 0, then in fact for all $\delta > 0$, we have $m(E) = 0 < \delta$, which implies by the previous problem that $m(f(E)) < \epsilon$. Since this is true for every $\epsilon > 0$, $m(f(E)) = 0$. ☕

6.4.41 Let f be an increasing absolutely continuous function on $[a, b]$. Use (i) and (ii) below to conclude that f maps measurable sets to measurable sets.

(i) Infer from the continuity of f and the compactness of $[a, b]$ that f maps closed sets to closed sets and therefore maps F_σ sets to F_σ sets.

(ii) The preceding problem tells us that f maps sets of measure zero to sets of measure zero.

Proof. Let E be a closed set in $[a, b]$. Since $[a, b]$ is compact, and closed subsets of compact sets are compact, then E is compact. Since f is continuous, and the continuous image of a compact set is compact, then $f(E)$ is compact. Since $\mathbb{R}$ is Hausdorff, and compact subsets of Hausdorff spaces are closed, then $f(E)$ is closed. Thus, f maps closed sets to closed sets. In particular, if $F = \bigcup_{k=1}^{\infty} E$ is an F_σ set, then

$$f\left(\bigcup_{k=1}^{\infty} E\right) = \bigcup_{k=1}^{\infty} f(E),$$

and since the RHS is a countable union of closed sets, then f sends F_σ sets to F_σ sets.

Now, let E be any measurable set in $[a, b]$. Then we can approximate E by an F_σ set F in the sense that $F \subseteq E$ and $m(E \sim F) = 0$. Then

$$f(E) = f((E \sim F) \cup F) = f(E \sim F) \cup f(F)$$

and since f sends measure zero sets to measure zero sets, $m(f(E \sim F)) = 0$. Moreover, $f(F)$ is an F_σ set so that $f(E)$ is measurable, since it is the union of two measurable sets. ☕

6.4.45 Let f be absolutely continuous on $\mathbb{R}$ and g be continuous and strictly monotone on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Proof. Let $\epsilon > 0$. Since f is absolutely continuous on $\mathbb{R}$ then there exists a $\delta_1 > 0$ such that for every finite disjoint collection of intervals $\{(c_k, d_k)\}_{k=1}^n$ in $\mathbb{R}$,

$$\text{if } \sum_{k=1}^n (d_k - c_k) < \delta_1 \text{ then } \sum_{k=1}^n |f(d_k) - f(c_k)| < \epsilon. \tag{$*$}$$

Without loss of generality, suppose g is increasing. Since g is absolutely continuous on $[a, b]$, then there exists $\delta_2 > 0$ such that for every finite disjoint collection of intervals $\{(a_k, b_k)\}_{k=1}^n$ contained in $[a, b]$,

$$\text{if } \sum_{k=1}^n (b_k - a_k) < \delta_2 \text{ then } \sum_{k=1}^n g(b_k) - g(a_k) < \delta_1.$$

Notice that since g is strictly increasing and $\{(a_k, b_k)\}_{k=1}^n$ is a disjoint collection, then $\{(g(a_k), g(b_k))\}_{k=1}^n$ is a disjoint collection of open intervals. In particular, $\{(g(a_k), g(b_k))\}_{k=1}^n$ is a collection of disjoint open intervals in $\mathbb{R}$ which satisfy the hypothesis in $(*)$. Therefore, for every finite disjoint collection of intervals $\{(a_k, b_k)\}_{k=1}^n$ contained in $[a, b]$

$$\text{if } \sum_{k=1}^n (b_k - a_k) < \delta_2 \text{ then } \sum_{k=1}^n |f(g(a_k) - f(g(b_k))| < \epsilon.$$

Therefore, $f \circ g$ is absolutely continuous on $[a, b]$. ☕

6.5.55 Let f be of bounded variation on $[a, b]$, and define $v(x) = TV(f_{[a,x]})$ for all $x \in [a, b]$.

(i) Show that $|f'| \leq v'$ a.e. on $[a, b]$, and infer from this that

$$\int_a^b |f'| \leq TV(f).$$

Proof. Let $c \in [a, b)$ and fix $t > 0$ so that $c + t \in [a, b]$ let $P = \{c, c + t\}$ be a partition of $[c, c + t]$. Then

$$|f(c + t) - f(c)| = V(f_{[c,c+t]}, P) \leq TV(f_{[c,c+t]}) = TV(f_{[a,c+t]}) - TV(f_{[a,c]}).$$

Since $TV(f_{[c,c+t]})$ is the difference of two increasing functions, then it is of bounded variation. Since f is of bounded variation, then both f and $TV(f_{[c,c+t]})$ are differentiable a.e. on (a, b) by Corollary 6, Chapter 6. So, the following limits exist a.e. on (a, b)

$$|f'(c)| = \lim_{t \to 0} \frac{|f(c + t) - f(c)|}{t} \leq \lim_{t \to 0} \frac{TV(f_{[a,c+t]}) - TV(f_{[a,c]})}{t} = v'(c).$$

Since c was arbitrary, we conclude that $|f'| \leq v'$ a.e. on $[a, b]$. By the same corollary, f' and v' are integrable, so by the monotonicity of integration, we have

$$\int_a^b |f'| \leq \int_a^b v'.$$

Since v is increasing, then $\int_a^b v' \leq v(b) - v(a)$, which gives

$$\int_a^b |f'| \leq \int_a^b v' \leq v(b) - v(a) = TV(f_{[a,b]}) - TV(f_{[a,a]}) = TV(f_{[a,b]}) = TV(f).$$

☕

(ii) Show that the above is an equality if and only if f is absolutely continuous on $[a, b]$.

Proof. ($\Rightarrow$) Suppose $\int_a^b |f'| = TV(f)$. Hence

$$\int_a^b |f'| = TV(f) = v(b) - v(a).$$

Let $\epsilon > 0$. Since v is the difference of two increasing functions then v is of bounded variation on $[a, b]$. Thus, there exists $\delta > 0$ so that for every disjoint collection of open intervals $\{(a_k, b_k)\}_{k=1}^n$ contained in $[a, b]$ for which $\sum_{k=1}^n (b_k - a_k) < \delta$, we have

$$\sum_{k=1}^n |TV(f_{[a,b_k]}) - TV(f_{[a,a_k]})| = \sum_{k=1}^n |v(b_k) - v(a_k)| < \epsilon.$$

Notice that

$$|f(b_k) - f(a_k)| \leq |TV(f_{[a_k,b_k]})| = |TV(f_{[a,b_k]}) - TV(f_{[a,a_k]})|$$

for all k and so

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq \sum_{k=1}^n |TV(f_{[a_k,b_k]})| < \epsilon.$$

($\Leftarrow$) Suppose f is absolutely continuous on $[a, b]$. Then by Theorem 10, Chapter 6, for any open interval (c, d) in $[a, b]$, we have

$$\int_c^d f' = f(d) - f(c).$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Then

$$V(f, P) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})| = \sum_{k=1}^n \left| \int_{x_{k-1}}^{x_k} f' \right| \leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|.$$

Hence,

$$TV(f) \leq \int_a^b |f'|$$

and together with (i) we have $TV(f) = \int_a^b |f'|$. ☕

(iii) Compare parts (i) and (ii) with Corollaries 4 and 12, respectively.
Solution:
Since $TV(f) = f(b) - f(a)$ for an increasing function f, then Corollary 4 is equivalent to (i) for increasing functions. If f is increasing and of bounded variation, then Corollary 12 is equivalent to (ii).

9.1.5 *The Nikodym Metric.* Let E be a Lebesgue measurable set of real numbers of finite measure, X the set of Lebesgue measurable subsets of E, and m Lebesgue measure. For $A, B \in X$, define $\rho(A, B) = m(A \triangle B)$, where $A \triangle B = [A \sim B] \cup [B \sim A]$, the symmetric difference of A and B.

(i) Show that this is a pseudometric on X.

Proof. The fact that $\rho(A, B) \geq 0$ follows from the fact that Lebesgue measure is nonnegative. Symmetry follows from the fact that

$$\rho(A, B) = m(A \sim B) + m(B \sim A) = m(B \sim A) + m(A \sim B) = \rho(B, A).$$

To show that the triangle inequality holds for this metric, we first show that

$$A \triangle B \subseteq (A \triangle C) \triangle (C \triangle B). \tag{1}$$

To that end, suppose $x \in A \triangle B$ where $x \in A$ but $x \notin B$. If $x \in C$, then $x \in A \cap C$ so that $x \notin A \triangle C$. But since $x \in C \sim B$, then $x \in C \triangle B$. Therefore, $x \in (A \triangle C) \triangle (C \triangle B)$. If $x \notin C$, then $x \notin C \triangle B$ and $x \in A \sim C$. Then $x \in A \triangle C$ but $x \notin C \triangle B$, which means $x \in (A \triangle C) \triangle (C \triangle B)$.

Similarly, if $x \notin A$ but $x \in B$, then $x \in (A \triangle C) \triangle (C \triangle B)$. Therefore, (1) holds. Notice that

$$(A \triangle C) \triangle (C \triangle B) \subseteq (A \triangle C) \cup (C \triangle B). \tag{2}$$

Therefore by (1), (2), and monotonicity of measure,

$$m(A \triangle B) \leq m((A \triangle C) \triangle (C \triangle B)) \leq m((A \triangle C) \cup (C \triangle B)) \leq m(A \triangle C) + m(C \triangle B),$$

that is, $\rho(A, B) \leq \rho(A, C) + \rho(C, B)$.

Although $A = B \implies \rho(A, B) = 0$, the converse may not be true. For if $A, B \in X$ are disjoint countable sets, then $\rho(A, B) = 0$ but $A \neq B$. Therefore, ρ cannot be a metric on X, but instead a pseudometric on X. ☕

(ii) Define two measurable sets to be equivalent provided their symmetric difference has measure zero. Show that ρ induces a metric on the collection of equivalence classes.

Proof. We need to show that $\tilde{\rho}([A], [B]) = \rho(A, B)$ defines a metric on $X/\cong$. Based on the work done in part (i), it remains to show that $\tilde{\rho}([A], [B]) = 0$ implies $[A] = [B]$. But this is clear, since

$$0 = \tilde{\rho}([A], [B]) = \rho(A, B) \implies A \cong B \implies [A] = [B].$$

☕

(iii) Finally, show that for $A, B \in X$,

$$\rho(A, B) = \int_E |\chi_A - \chi_B|,$$

where χ_A and χ_B are the characteristic functions of A and B, respectively.

Proof. Since

$$\int_E \chi_{A \triangle B} = m(A \triangle B) = \rho(A, B),$$

it suffices to show that $\chi_{A \triangle B} = |\chi_A - \chi_B|$ on E. Let $x \in E$.

If $x \in A \cap B$, then $\chi_{A \triangle B}(x) = 0$ and $X_A(x) = 1 = \chi_B(x)$ so that

$$\chi_{A \triangle B}(x) = 0 = |\chi_A(x) - \chi_B(x)|.$$

If $x \in A \sim B$, then $\chi_{A \triangle B}(x) = 1, \chi_A(x) = 0$, and $\chi_B(x) = 0$ so that

$$\chi_{A \triangle B}(x) = 1 = |\chi_A(x) - \chi_B(x)|.$$

If $x \in B \sim A$, then $\chi_{A\triangle B}(x) = 1, \chi_A(x) = 1$, and $\chi_B(x) = 1$ so that

$$\chi_{A\triangle B}(x) = 1 = |\chi_A(x) - \chi_B(x)|.$$

If $x \notin A \cup B$, then $\chi_{A\triangle B}(x) = 0$, $\chi_A(x) = 0$, and $\chi_B(x) = 0$ so that

$$\chi_{A\triangle B}(x) = 0 = |\chi_A(x) - \chi_B(x)|.$$

☕

9.1.11 Let (X, ρ) be a metric space and A any set for which there is a one-to-one mapping f of A onto the set X. Show that there is a unique metric on A for which f is an isometry of metric spaces. (This is the sense in which an isometry amounts merely to a relabeling of the points in a space.)

Proof. Let $a_1, a_2 \in A$ and define a metric σ on A by $\sigma(a_1, a_2) = \rho(f(a_1), f(a_2))$. Then the nonnegativity, symmetry, and triangle inequality of σ follow from that of ρ. Since $fA \to X$ is injective then if $f(a_1) = f(a_2)$, then $a_1 = a_2$ so that $\sigma(a_1, a_2) = 0 \iff a_1 = a_2$. Thus σ is indeed a metric on A.

If τ is another metric on A for which $\tau(a_1, a_2) = \rho(f(a_1), f(a_2))$, then $\tau(a_1, a_2) = \sigma(a_1, a_2)$ so that σ is unique. ☕

9.2.15 Let X be a metric space, x belong to X and $r > 0$.

(i) Show that $\overline{B}(x, r)$ is closed and contains $B(x, r)$.

Proof. Let $y \in X \sim \overline{B}(x, r)$ and define $r' = \rho(x, y) - r$. Then $B(y, r')$ is a neighborhood of y disjoint from $\overline{B}(x, r)$. Otherwise, if $z \in B(y, r') \cap \overline{B}(x, r)$, then

$$\rho(x, z) \leq r \quad \text{and} \quad \rho(z, y) < r' = \rho(x, y) - r.$$

But by the triangle inequality, this would imply

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y) < r + r' = \rho(x, y),$$

which is a contradiction.

To see that $\overline{B}(x, r)$ contains $B(x, r)$, notice that if $y \in B(x, r)$, then $\rho(x, y) < r$ and hence $y \in \overline{B}(x, r)$. ☕

(ii) Show that in a normed linear space X, the closed ball $\overline{B}(x, r)$ is the closure of the open ball $B(x, r)$, but this is not so in a general metric space.

Proof. Since $\overline{B}(x, r)$ is closed and contains $B(x, r)$, and $\overline{B(x, y)}$ is the smallest closed set containing $B(x, r)$, then $\overline{B(x, r)} \subseteq \overline{B}(x, r)$. Conversely, let $y \in \overline{B}(x, r)$ with $\rho(x, y) = r$ (if $\rho(x, y) < r$ then $y \in B(x, r)$ and so $y \in \overline{B(x, r)}$ and we are done). Then if we define $y_n := y(1 - 1/n)$ for all n, then $\{y_n\} \to y$ which means $y \in \overline{B(x, r)}$. Thus $\overline{B(x, r)} = \overline{B}(x, r)$. Notice that our definition of y_n is only valid in a normed linear space.

This is not true in a general metric space because if we consider any space with the discrete metric, the open unit ball of radius 1 around any point is just the singleton set. Its closure is also a singleton set. However, the closed unit ball of radius 1 is everything. ☕

9.2.22 For a subset E of a metric space X, a point $x \in X$ is called a boundary point of E provided every open ball centered at x contains points in E and points in $X \sim E$: the collection of boundary points of E is called the boundary of E and denoted by bd E. Show

(i) Show that bd E is always closed.

Proof. Let $x \in X \sim \text{bd}\, E$. Then there exists $r > 0$ so that $B(x, r) \cap E = \emptyset$ *or* $B(x, r) \cap (X \sim E) = \emptyset$. If the former is true, then $B(x, r)$ is a neighborhood of x disjoint from $\text{bd}\, E$ since every open ball of a point in $\text{bd}\, E$ must intersect E. Similarly, if the latter is true then $B(x, y)$ is a neighborhood of x disjoint from $\text{bd}\, E$ since every open ball of a point in $\text{bd}\, E$ must intersect $X \sim E$. Therefore, $\text{bd}\, E$ is closed. ☕

(ii) Show that E is open if and only if $E \cap \operatorname{bd} E = \emptyset$.

Proof. ($\Rightarrow$) If $x \in E$ then there exists $r > 0$ so that $B(x, r) \subseteq E$. Since $B(x, r)$ is an open ball about X contained in E, then $B(x, r) \cap (X \sim E) = \emptyset$, and hence $x \notin \operatorname{bd} E$. Therefore, $E \cap \operatorname{bd} E = \emptyset$.

($\Leftarrow$) If $x \in E$ then $x \notin \operatorname{bd} E$. Then there exists $r > 0$ so that $B(x, r) \cap E = \emptyset$ *or* $B(x, r) \cap (X \sim E) = \emptyset$. However, the former cannot be true since $x \in E$. So

$$\emptyset = B(x, r) \cap (X \sim E) = B(x, r) \cap (X \cap E^c) = B(x, r) \cap E^c \implies B(x, r) \subseteq E.$$

☕

(iii) Show that E is closed if and only if $\operatorname{bd} E \subseteq E$.

Proof. ($\Rightarrow$) If $x \in \operatorname{bd} E$, then for every neighborhood $\mathcal{O}$ of x, $\mathcal{O} \cap E \neq \emptyset$. Hence $x \in \overline{E} = E$.

($\Leftarrow$) If $x \in X \sim E$, then $x \notin \operatorname{bd} E$ since $\operatorname{bd} E \subseteq E$. So there exists $r > 0$ so that $B(x, r) \cap E = \emptyset$ *or* $B(x, r) \cap (X \sim E) = \emptyset$. But since $x \in X \sim E$, the former must be true. Then $B(x, r)$ is a neighborhood of x disjoint from E, and hence E is closed. ☕

9.3.32 For a nonempty subset E of the metric space (X, ρ) and a point $x \in X$, define the distance from x to E, $\operatorname{dist}(x, E)$, as follows:

$$\operatorname{dist}(x, E) = \inf\{\rho(x, y) \mid y \in E\}.$$

(i) Show that the distance function $f : X \to \mathbb{R}$ defined by $f(x) = \operatorname{dist}(x, E)$, for $x \in X$, is continuous.

Proof. Let $\epsilon > 0$ and $x \in X$. Choose any point $y \in X$ for which $\rho(x, y) < \epsilon$ (in the $\epsilon - \delta$ criterion for continuity, we are choosing $\delta = \epsilon$). We claim

$$|\operatorname{dist}(x, E) - \operatorname{dist}(y, E)| \leq \rho(x, y) < \epsilon \tag{3}$$

Notice that by the triangle inequality

$$\operatorname{dist}(x, E) \leq \rho(x, e) \leq \rho(x, y) + \rho(y, e) \text{ for all } e \in E,$$

from which we get

$$\operatorname{dist}(x, E) - \rho(x, y) \leq \rho(y, e) \text{ for all } e \in E.$$

Then

$$\operatorname{dist}(x, E) - \rho(x, y) \leq \inf\{\rho(y, e) \mid e \in E\} = \operatorname{dist}(y, E),$$

which implies

$$\operatorname{dist}(x, E) - \operatorname{dist}(y, E) \leq \rho(x, y).$$

Switching the roles of x and y shows $\operatorname{dist}(y, E) - \operatorname{dist}(x, E) \leq \rho(x, y)$ as well so that (3) holds. ☕

(ii) Show that $\{x \in X \mid \operatorname{dist}(x, E) = 0\} = \overline{E}$.

Proof. If $\operatorname{dist}(x, E) = 0$ then there exists $y \in E$ so that $\rho(x, y) = 0$, i.e., $x = y$. Thus $x \in E \subseteq \overline{E}$. If $x \in \overline{E}$ then for each $n \in \mathbb{N}$, $B(x, 1/n) \cap E \neq \emptyset$. So there exists $x_n \in B(x, 1/n) \cap E$ for each n. Thus

$$\inf\{\rho(x, x_n) \mid x_n \in E\} \to 0 \implies \operatorname{dist}(x, E) = 0.$$

☕

9.3.34 Show that a subset E of a metric space X is closed if and only if there is a continuous real-valued function f on X for which $E = f^{-1}(0)$.

Proof. ($\Rightarrow$) Consider the function given in Exercise 32. Then

$$f^{-1}(0) = \{x \in X \mid \operatorname{dist}(x, E) = 0\} = \overline{E} = E.$$

($\Leftarrow$) Since $\mathbb{R} \sim \{0\} = (-\infty, 0) \cup (0, \infty)$ is open, then $\{0\}$ is closed. Since f is continuous then $f^{-1}(0) = E$ is closed. ☕

9.4.42 Prove that the product of two complete metric spaces is complete.

Proof. Let (X, ρ_1) and (Y, ρ_2) be two complete metric spaces and $\{(x_n, y_n)\}$ be a Cauchy sequence in the product space $(X \times Y, \tau)$. Let $\epsilon > 0$. Since $\{(x_n, y_n)\}$ is a Cauchy sequence there exists $N \in \mathbb{N}$ so that for all $n, m \geq N$

$$\tau((x_n, y_n), (x_m, y_m)) = \{(\rho_1(x_n, x_m))^2 + (\rho_2(y_n, y_m))^2\}^{1/2} < \sqrt{\epsilon}.$$

So for all $n, m \geq N$

$$\rho_1(x_n, x_m), \rho_2(y_n, y_m) < \sqrt{\epsilon} < \epsilon.$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y, respectively, and since X and Y are complete metric spaces, $\{x_n\} \to x \in X$ and $\{y_n\} \to y \in Y$. It follows that $\{(x_n, y_n)\} \to (x, y) \in X \times Y$ and so $X \times Y$ is complete. ☕

9.4.49 For a metric space (X, ρ), complete the following outline of a proof of Theorem 13:

(i) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, show that $\{\rho(x_n, y_n)\}$ is a Cauchy sequence of real numbers and therefore converges.

Proof. Let $\epsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy in X, let $N \in \mathbb{N}$ be large enough so that for all $n, m \geq N$,

$$\rho(x_n, x_m) < \epsilon/2 \quad \text{and} \quad \rho(y_n, y_m) < \epsilon/2 \tag{4}$$

By the reverse triangle inequality, it follows that

$$|\rho(x_n, y_n) - \rho(x_m, y_n)| \leq \rho(x_n, x_m) \quad \text{and} \quad |\rho(x_m, y_n) - \rho(x_m, y_m)| \leq \rho(y_n, y_m). \tag{5}$$

Then (4) and (5) give

$$\begin{aligned}|\rho(x_n, y_n) - \rho(x_m, y_m)| &\leq |\rho(x_n, y_n) - \rho(x_m, y_n)| + |\rho(x_m, y_n) - \rho(x_m, y_m)| \\ &\leq \rho(x_n, x_m) + \rho(y_n, y_m) \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

and so $\{\rho(x_n, y_n)\}$ is a Cauchy sequence of real numbers and therefore converges. ☕

(ii) Define X' to be the set of Cauchy sequences in X. For two Cauchy sequences in X, $\{x_n\}$ and $\{y_n\}$, define $\rho'(\{x_n\}, \{y_n\}) = \lim \rho(x_n, y_n)$. Show that this defines a pseudometric ρ' on X'.

Proof. Since ρ is a metric, it is nonnegative. Therefore, $\lim \rho(x_n, y_n)$ will be nonnegative, i.e. ρ' is nonnegative. Symmetry of ρ' follows from symmetry of ρ. We have the triangle inequality for ρ' since if $\rho(x_n, y_n) \leq \rho(x_n, z_n) + \rho(z_n, y_n)$ then

$$\begin{aligned}\rho'(\{x_n\}, \{y_n\}) = \lim \rho(x_n, y_n) \leq \lim[\rho(x_n, z_n) + \rho(z_n, y_n)] &= \lim \rho(x_n, z_n) + \lim \rho(z_n, y_n) \\ &= \rho'(\{x_n\}, \{z_n\}) + \rho'(\{z_n\}, \{y_n\}).\end{aligned}$$

Although $\{x_n\} = \{y_n\} \implies \{\rho(x_n, y_n)\} \to 0$, it may not be the case that $\lim \rho(x_n, y_n) = 0$ gives $\{x_n\} = \{y_n\}$. Indeed, if $\{x_n\} = (x_1, x_2, x_3, \dots)$ and $\{y_n\} = (y_1, x_2, x_3, \dots)$ for some $y_1 \neq x_1$, then $\lim \rho(x_n, y_n) = 0$ but $\{x_n\} \neq \{y_n\}$. Therefore, ρ' is a pseudometric on X'. ☕

(iii) Define two members of X', that is, two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X, to be equivalent provided $\rho'(\{x_n\}, \{y_n\}) = 0$. Show that this is an equivalence relation in X' and denote by $\widehat{X}$ the set of equivalence classes. Define the distance $\widehat{\rho}$ between two equivalence classes to be the ρ' distance between representatives of the classes. Show that $\widehat{\rho}$ is properly defined and is a metric on $\widehat{X}$.

Proof. Reflexivity of this relation follows from the fact that $\lim \rho(x_n, x_n) = 0$. Symmetry follows from symmetry of ρ. Finally, this relation is transitive since if $0 = \lim \rho(x_n, y_n) = \lim(y_n, z_n)$, then by the triangle inequality of ρ'

$$\lim \rho(x_n, z_n) \leq \lim \rho(x_n, y_n) + \lim \rho(y_n, z_n) = 0,$$

and since ρ' is nonnegative, we have $\{x_n\} \sim \{z_n\}$.

To show that $\hat{\rho}$ is well-defined, we must show that for any two representatives $\{x_n\}$ and $\{x'_n\}$ of the equivalence class $[\{x_n\}]$ and a fixed representative $\{y_n\}$ of $[\{y_n\}]$,

$$\rho'(\{x_n\}, \{y_n\}) = \rho'(\{x'_n\}, \{y_n\}).$$

This follows from the following two inequalities:

$$\lim \rho(x_n, y_n) \leq \lim \rho(x_n, x'_n) + \lim \rho(x'_n, y_n) = \lim \rho(x'_n, y_n),$$

$$\lim \rho(x'_n, y_n) \leq \lim \rho(x'_n, x_n) + \lim \rho(x_n, y_n) = \lim \rho(x_n, y_n).$$

Similarly, we get that if $\{x_n\}$ is a fixed representative of $[\{x_n\}]$ and if $\{y_n\}$ and $\{y'_n\}$ are two representatives of $[\{y_n\}]$, then

$$\rho'(\{x_n\}, \{y_n\}) = \rho'(\{x_n\}, \{y'_n\}).$$

Since $\hat{\rho}$ is properly defined in terms of ρ' and ρ' is a pseudometric, then nonnegativity, symmetry, and the triangle inequality all hold for $\hat{\rho}$. Thus it remains to show that

$$\hat{\rho}([\{x_n\}], [\{y_n\}]) = 0 \iff [\{x_n\}] = [\{y_n\}]. \tag{6}$$

Since $\hat{\rho}$ is well defined, we can work with any representatives. So

$$0 = \hat{\rho}([\{x_n\}], [\{y_n\}]) \iff \lim \rho(x_n, y_n) = 0 \iff \{x_n\} \sim \{y_n\} \iff [\{x_n\}] = [\{y_n\}].$$

(iv) Show that the metric space $(\widehat{X}, \widehat{p})$ is complete. (Hint: If $\{x_n\}$ is a Cauchy sequence from X, we may assume [by taking subsequences] that $\rho(x_n, x_{n+1}) < 2^{-n}$ for all n. If $\{\{x_{n,m}\}_{n=1}^{\infty}\}_{m=1}^{\infty}$ is a sequence of such Cauchy sequences that represents a Cauchy sequence in $\widehat{X}$, then the sequence $\{x_{n,n}\}_{n=1}^{\infty}$ is a Cauchy sequence from X that represents the limit of the Cauchy sequences from $\widehat{X}$.)

Proof. We have $[\{x_n\}] \in \hat{X}$ and $\{[\{x_i\}_j]\}$, a sequence in $\hat{X}$. Choose representative for each class, and we get $\{x_{i,j}\}$, a sequence in $\hat{X}$, which is Cauchy.

$$\begin{aligned}
x_{i,1} &= \{x_{11}, x_{21}, x_{31} \ldots, \} \\
x_{i,2} &= \{x_{12}, x_{22}, x_{32} \ldots, \} \\
x_{i,3} &= \{x_{13}, x_{23}, x_{33} \ldots, \} \\
&\vdots \\
x_{i,n} &= \{x_{1n}, x_{2n}, x_{3n} \ldots, \} \\
&\vdots
\end{aligned}$$

Let $z = \{x_{11}, x_{22}, x_{33}, \cdots\}$. Want to show z is a Cauchy sequence. let $\epsilon > 0$. We claim

$$\rho(x_{nn}, x_{mm}) \leq \rho(x_{nn}, x_{jn}) + \rho(x_{jn}, x_{jm}) + \rho(x_{jm}, x_{mm}) < \epsilon. \tag{7}$$

Notice that there exists $N_1, N_2 \in \mathbb{N}$ so that $\rho(x_{nn}, x_{jn}) < \epsilon/3$ if $n, j \geq N_1$ and $\rho(x_{jm}, x_{mm}) < \epsilon/3$ if $m, j \geq N_2$.

Moreover, there exists N_3 such that for all $k, \ell \geq N_3$,

$$\hat{\rho}(\{x_{ik}\}, \{x_{i\ell}\}) < \epsilon/3$$

i.e.,

$$\lim \rho(x_{ik}, x_{i\ell}\}) = 0.$$

So there exists N_4 such that if $i \geq N_4$,

$$\rho(x_{ik}, x_{i\ell}) < \epsilon/3.$$

So, we choose $N = \max\{n_1, N_2, N_3, N_4\}$ and choose $n, j, m \geq N$ so that (7) holds.

We now must show that z is the limit of $\{[\{x_i\}_j]\}$. To that end,

$$\begin{aligned}\lim_{n\to\infty} \hat{\rho}(\{x_i\}_n, z) &= \lim_{n\to\infty}\left(\lim_{i\to\infty}\rho(x_{in}, x_{ii})\right) \text{ (we know this converges!, so we go to subsequence...)}\\ &= \lim_{n\to\infty}\lim_{i\to\infty}(x_{ii}, x_{ii}) = 0\end{aligned}$$

(v) Define the mapping h from X to $\widehat{X}$ by defining, for $x \in X$, $h(x)$ to be the equivalence class of the constant sequence all of whose terms are x. Show that $h(X)$ is dense in $\widehat{X}$ and that $\widehat{\rho}(h(u), h(v)) = \rho(u, v)$ for all $u, v \in X$.

Proof. Want to show that for any open ball around $\hat{y} = \{y_1, \dots, y_n, \dots\} \in \hat{X}$ there exists $\{x, x, \dots, x, \dots\}$ that is also in the ball. The ball of radius r around $\hat{y}$ is

$$B_r(\hat{y}) = \{\{x_n\} \mid \lim_{n\to\infty}\rho(x_n, y_n) < r\}.$$

Pick $x = y_k$ for sufficiently large k. Then, it follows that

$$\lim_{n\to\infty}\rho(x_k, y_n) < r$$

To show $\widehat{\rho}(h(u), h(v)) = \rho(u, v)$, note that $h(u) = [\{u\}] = [\{u, u, u, \dots\}]$ and $h(v) = [\{v\}] = [\{v, v, v, \dots\}]$. Then

$$\widehat{\rho}(h(u), h(v)) = \rho'(\{u\}, \{v\}) = \lim \rho(u, v) = \rho(u, v),$$

where the last equality holds because $\lim \rho(u, v)$ is a limit of the constant $\rho(u, v)$.

(vi) Define the set $\tilde{X}$ to be the disjoint union of X and $\widehat{X} \sim h(X)$. For $u, v \in \tilde{X}$, define $\tilde{\rho}(u, v)$ as follows: $\tilde{\rho}(u, v) = \rho(u, v)$ if $u, v \in X$; $\tilde{\rho}(u, v) = \widehat{u, v}$ for $u, v \in \hat{X} \sim h(X)$; and $\tilde{\rho}(u, v) = \hat{\rho}(h(u), v)$ for $u \in X, v \in \hat{X} \sim h(X)$. From the preceding two parts conclude that the metric space $(\tilde{X}, \tilde{\rho})$ is a complete metrics space containing (X, ρ) as a dense subspace.

Proof.

Problem 1 Show that any two norms on a finite dimensional vector space are equivalent.

Proof. **Proof obtained from the following source:**

`https://math.berkeley.edu/~sarason/Class_Webpages/solutions\_202B_assign10.pdf`

Since every finite dimensional vector space is equivalent to $\mathbb{R}^n$, we show that any norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm $||\cdot||_2$ on $\mathbb{R}^n$.

To show this, let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $||\cdot||$ be any norm on $\mathbb{R}^n$. Then for $x = x_1e_2 + \dots x_ne_n$, we have by the Cauchy-Schwarz inequality

$$||x|| \leq |x_1|\ ||e_1|| + \cdots + |x_n|\ ||e_n|| \leq ||x||_2\sqrt{||e_1||^2 + \dots ||e_n||^2}.$$

Therefore, for $C = \sqrt{||e_1||^2 + \dots ||e_n||^2}$, we have $||x|| \leq C||x||_2$.

To get the other inequality, we observe that since $|\ ||x|| - ||y||\ | \leq ||x - y|| < C||x - y||_2$, then $||\cdot||$ is a continuous function with respect to the usual topology. Therefore, since $S^{n-1} = \{x \mid ||x||_2 = 1\}$ is compact, $||x||$ achieves a minimum values $c > 0$ on S^{n-1}. Thus if $x \neq 0$, we have $\frac{x}{||x||_2} \in S^{n-1}$, and

$$||x|| = ||x||_2 \left|\left|\frac{x}{||x||_2}\right|\right| \geq c||x||_2.$$

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