Homework for
Introduction to Algebraic Topology

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Exercises are from
Basic Concepts of Algebraic Topology by Croom.
Beware: Some solutions may be incorrect!

Nicholas Camacho
Topology - Discussion
Homework 1
September 1, 2016

1-3 Prove that a set $A = \{a_0, a_1, \dots, a_k\}$ of points in $\mathbb{R}^n$ is geometrically independent if and only if the set of vectors $\{a_1 - a_0, \dots, a_k - a_0\}$ is linearly independent.

Proof. We prove the contrapositive statement. Assume the points in the set A are geometrically dependent. This occurs if and only if there exists a $(k-1)$ hyperplane, say P, that contains all of the points in A. So, for some $k-1$ linearly independent vectors $v_1, v_2, \dots, v_{k-1}$,

$$P = \left\{ a_0 + \sum_{i=1}^{k-1} \mu_i v_i \middle| \mu_i \in \mathbb{R}, 1 \leq i \leq m \right\}.$$

So, given any point $a_j \in A$, we can write

$$\begin{aligned} a_j &= a_0 + \mu_{j_1} v_1 + \mu_{j_2} v_2 + \cdots + \mu_{j_{(k-1)}} v_{k-1} \\ a_j - a_0 &= \mu_{j_1} v_1 + \mu_{j_2} v_2 + \cdots + \mu_{j_{(k-1)}} v_{k-1} \end{aligned}$$

if and only if $\text{Span}\{a_1 - a_0, a_2 - a_0, \dots, a_k - a_0\} \subseteq \text{Span}\{v_1, v_2, \dots, v_{k-1}\}$. Now,

$$\dim\{\text{Span}\{a_1 - a_0, a_2 - a_0, \dots, a_k - a_0\}\} = k > k-1 = \dim\{\text{Span}\{v_1, v_2, \dots, v_{k-1}\}\},$$

if and only if $\{a_1 - a_0, a_2 - a_0, \dots, a_k - a_0\}$ are linearly dependent.

■

1-5 A subset B of $\mathbb{R}^n$ is *convex* provided that B contains every line segment having two of its members as end points.

(a) If a and b are points in $\mathbb{R}^n$, show that the line segment L joining a and b consists of all points of the form

$$x = ta + (1-t)b$$

where t is a real number with $0 \leq t \leq 1$.

Proof. Let$\{a, b\}$ be a set of geometrically independent points. (Otherwise, we would have $a = b$, and we could not begin to consider a line segment L joining a and b). So, we can define a 1-simplex (a closed line) spanned by $\{a, b\}$ as the line L where

$$L = \{x \in \mathbb{R}^n | x = ta + sb, t + s = 1, \text{ and } t, s \in \mathbb{R} \text{ are non-negative}\}$$

Since $s = 1 - t$, then all the points on the 1-simplex can be written as

$$x = ta + (1-t)b$$

Since $s = 1 - t$ is non-negative then we must have $0 \leq t \leq 1$. ■

(b) Prove that every simplex is a convex set.

Proof. Let a and b be any two points in a k-simplex, σ^k, spanned by the set $\{a_0, a_1, \ldots, a_k\}$. Then, a and b can be written as

$$a = \lambda_0 a_0 + \lambda_2 a_2 + \cdots + \lambda_k a_k \text{ where } \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \ \forall i$$

and

$$b = \mu_0 a_0 + \mu_2 a_2 + \cdots + \mu_k a_k \text{ where } \sum_{i=0}^{k} \mu_i = 1 \text{ and } \mu_i \geq 0 \ \forall i$$

From part (a), the line segment joining a and b consists of all points of the form

$$c = ta + (1-t)b \text{ where } 0 \leq t \leq 1$$

Thus,

$$\begin{aligned} c &= t(\lambda_0 a_0 + \lambda_2 a_2 + \cdots + \lambda_k a_k) + (1-t)(\mu_0 a_0 + \mu_2 a_2 + \cdots + \mu_k a_k) \\ &= [t\lambda_0 + (1-t)\mu_0]a_0 + [t\lambda_1 + (1-t)\mu_1]a_1 + \cdots + [t\lambda_k + (1-t)\mu_k]a_k \end{aligned}$$

Now, as t, λ_i, and μ_i are all non-negative for all i, then $t\lambda_i + (1-t)\mu_i$ is non-negative for all i. Also, notice that

$$\sum_{i=0}^{k}[t\lambda_i + (1-t)\mu_i] = t\sum_{i=0}^{k} \lambda_i + (1-t)\mu_i = t(1) + (1-t)(1) = 1$$

and so $c \in \sigma^k$, and thus σ^k is a convex set. ■

(c) Prove that a simplex σ is the smallest convex set which contains all vertices of σ.

Proof. Let σ^n be an n-simplex. We proceed by induction on the dimension of the faces of σ^n. Suppose $C \subseteq \mathbb{R}^n$ is a convex set that contains all the vertices of σ^n, $< a_0 a_1 \ldots a_n >$. For the base case, it is by definition of C that the 0-faces of $< a_0 a_1 \ldots a_n >$, which are $< a_0 >, < a_1 > \cdots < a_n >$, are all in C. Now, suppose that C contains all of the $(k-1)$-faces, $k > 0$. Without loss of generality, consider the k-face $< a_0 a_1 \ldots a_k >$. If x is a point in $< a_0 a_1 \ldots a_k >$, then we can write

$$x = \sum_{i=0}^{k} \lambda_i a_i$$

where the numbers $\lambda_0, \ldots \lambda_k$ are the barycentric coordinates of x. Then,

$$
\begin{aligned}
x &= \sum_{i=0}^{k} \lambda_i a_i \\
&= \left[\lambda_0 a_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i\right] + \left[\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i + \lambda_k a_k\right] \\
&= \left(\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i\right)\left[\left(\frac{1}{\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i}\right)\left(\lambda_0 a_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i\right)\right] \\
&\qquad + \left(\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k\right)\left[\left(\frac{1}{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k}\right)\left(\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i + \lambda_k a_k\right)\right] \\
&= \left(\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i\right)\left[\left(\frac{\lambda_0 a_0}{\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i}\right) + \left(\frac{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i}{\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i}\right)\right] \\
&\qquad + \left(\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k\right)\left[\left(\frac{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i}{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k}\right) + \frac{\lambda_k a_k}{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k}\right]
\end{aligned}
$$

Let

$$
t = \left(\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i\right) \quad \text{and} \quad s = \left(\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k\right).
$$

Then, $s + t = 1$ and

$$
\begin{aligned}
x = t &\left[\left(\frac{\lambda_0 a_0}{\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i}\right) + \left(\frac{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i}{\lambda_0 + \frac{1}{2}\sum_{i=1}^{k-1} \lambda_i}\right)\right] \\
&+ s\left[\left(\frac{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i a_i}{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k}\right) + \frac{\lambda_k a_k}{\frac{1}{2}\sum_{i=1}^{k-1} \lambda_i + \lambda_k}\right],
\end{aligned}
$$

which shows that x can be written as a point on a line between two $(k-1)$-faces. Since we assumed the $(k-1)$-faces are in C, and C is convex, then $x \in C$, and so C contains the k-faces of σ. Therefore, by mathematical induction, C contains all of the k-faces of σ^n for each $k = 1, 2, \ldots, n$. Thus, σ^n is always a subset of any convex set C which contains its vertices, and thus σ^n is the smallest convex set containing all of its vertices. ■

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Homework 2
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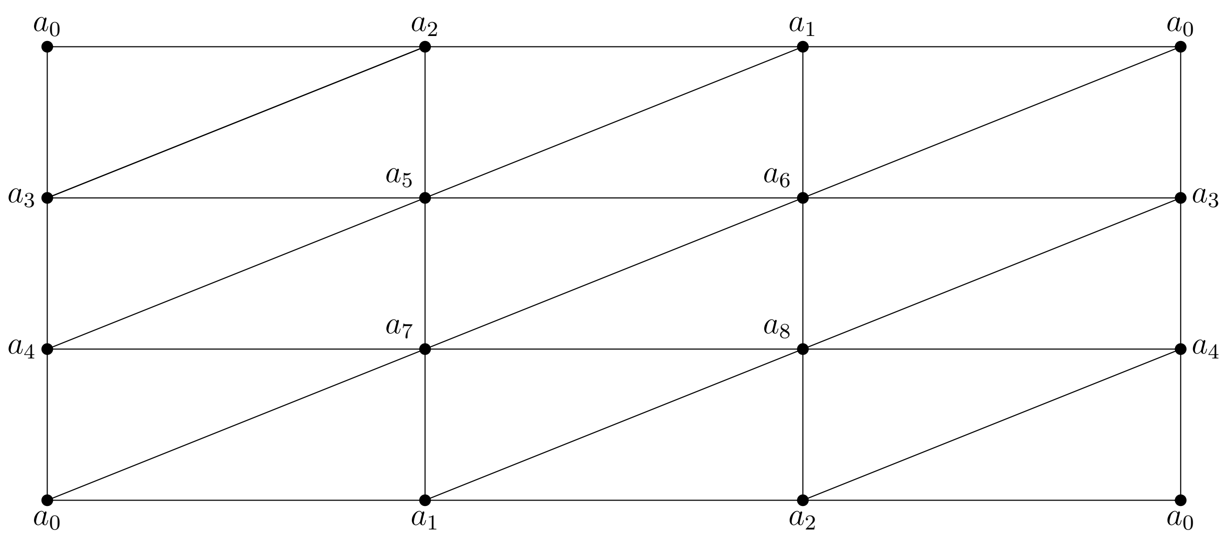
1-6 How many faces does an n-simplex have?
Solution: Let σ^n be an n-simplex. Since σ^n has $n+1$ vertices, there are $\binom{n+1}{1}$ 0-faces of σ^n. Likewise, σ^n has $\binom{n+1}{2}$ 1-faces. In general, we can say that the number of $(k-1)$-faces is $\binom{n+1}{k}$. So, we have

$$\sum_{k=1}^{n+1} \binom{n+1}{k}$$

total faces of σ^n.

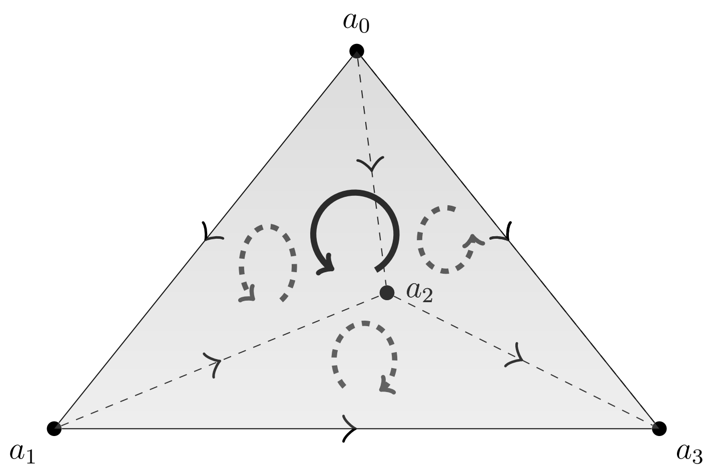
1-8 Triangulation of the Klein Bottle.



9. Let K denote the closure of a 3-simplex $\sigma^3 = \langle a_0 a_1 a_2 a_3 \rangle$ with vertices ordered by

$$a_0 < a_1 < a_2 < a_3.$$

Use this given order to induce an orientation on each simplex of K, and determine all incidence numbers associated with K.
Solution:



Per the given order, we list the positive orientation for each simplex in K:

0-simplices:

$$+\sigma^0_\alpha = \langle a_0 \rangle,\ +\sigma^0_\beta = \langle a_1 \rangle,\ +\sigma^0_\gamma = \langle a_2 \rangle,\ +\sigma^0_\delta = \langle a_3 \rangle,$$

1-simplices:

$$+\sigma^1_\alpha = \langle a_0 a_1 \rangle,\ +\sigma^1_\beta = \langle a_0 a_2 \rangle,\ +\sigma^1_\gamma = \langle a_0 a_3 \rangle,$$
$$+\sigma^1_\delta = \langle a_1 a_2 \rangle,\ +\sigma^1_\epsilon = \langle a_1 a_3 \rangle,\ +\sigma^1_\zeta = \langle a_2 a_3 \rangle$$

2-simplices:

$$+\sigma^2_\alpha = \langle a_0 a_1 a_2 \rangle,\ +\sigma^2_\beta = \langle a_0 a_2 a_3 \rangle,\ +\sigma^2_\gamma = \langle a_0 a_1 a_3 \rangle,\ +\sigma^2_\delta = \langle a_1 a_2 a_3 \rangle,$$

3-simplex:

$$+\sigma^3 = +\langle a_0 a_1 a_2 a_3 \rangle,$$

We compute the incidence number of σ^3 and all 2-simplices algebraically:

$$[\sigma^3, \sigma^2_\alpha] = -1 \text{ since } \langle a_3 a_0 a_1 a_2 \rangle = -\langle a_0 a_1 a_2 a_3 \rangle$$
$$[\sigma^3, \sigma^2_\beta] = -1 \text{ since } \langle a_1 a_0 a_2 a_3 \rangle = -\langle a_0 a_1 a_2 a_3 \rangle$$
$$[\sigma^3, \sigma^2_\gamma] = +1 \text{ since } \langle a_2 a_0 a_1 a_3 \rangle = +\langle a_0 a_1 a_2 a_3 \rangle$$
$$[\sigma^3, \sigma^2_\delta] = +1 \text{ since } \langle a_0 a_1 a_2 a_3 \rangle = +\langle a_0 a_1 a_2 a_3 \rangle$$

Now, using the orientations shown by arrows in the figure above, we compute the incidence numbers associated with K:

$[\sigma^2_\alpha, \sigma^1_\alpha] = +1$	$[\sigma^2_\beta, \sigma^1_\alpha] = 0$	$[\sigma^2_\gamma, \sigma^1_\alpha] = +1$	$[\sigma^2_\delta \sigma^1_\alpha] = 0$
$[\sigma^2_\alpha, \sigma^1_\beta] = -1$	$[\sigma^2_\beta, \sigma^1_\beta] = +1$	$[\sigma^2_\gamma, \sigma^1_\beta] = 0$	$[\sigma^2_\delta \sigma^1_\beta] = 0$
$[\sigma^2_\alpha, \sigma^1_\gamma] = 0$	$[\sigma^2_\beta, \sigma^1_\gamma] = -1$	$[\sigma^2_\gamma, \sigma^1_\gamma] = -1$	$[\sigma^2_\delta \sigma^1_\gamma] = 0$
$[\sigma^2_\alpha, \sigma^1_\delta] = +1$	$[\sigma^2_\beta, \sigma^1_\delta] = 0$	$[\sigma^2_\gamma, \sigma^1_\delta] = 0$	$[\sigma^2_\delta \sigma^1_\delta] = +1$
$[\sigma^2_\alpha, \sigma^1_\epsilon] = 0$	$[\sigma^2_\beta, \sigma^1_\epsilon] = 0$	$[\sigma^2_\gamma, \sigma^1_\epsilon] = +1$	$[\sigma^2_\delta \sigma^1_\epsilon] = -1$
$[\sigma^2_\alpha, \sigma^1_\zeta] = 0$	$[\sigma^2_\beta, \sigma^1_\zeta] = +1$	$[\sigma^2_\gamma, \sigma^1_\zeta] = 0$	$[\sigma^2_\delta \sigma^1_\zeta] = +1$

$[\sigma^1_\alpha, \sigma^0_\alpha] = -1$	$[\sigma^1_\beta, \sigma^0_\alpha] = -1$	$[\sigma^1_\gamma, \sigma^0_\alpha] = -1$	$[\sigma^1_\delta \sigma^0_\alpha] = 0$	$[\sigma^1_\epsilon \sigma^0_\alpha] = 0$	$[\sigma^1_\zeta \sigma^0_\alpha] = 0$
$[\sigma^1_\alpha, \sigma^0_\beta] = +1$	$[\sigma^1_\beta, \sigma^0_\beta] = 0$	$[\sigma^1_\gamma, \sigma^0_\beta] = 0$	$[\sigma^1_\delta \sigma^0_\beta] = -1$	$[\sigma^1_\epsilon \sigma^0_\beta] = -1$	$[\sigma^1_\zeta \sigma^0_\beta] = 0$
$[\sigma^1_\alpha, \sigma^0_\gamma] = 0$	$[\sigma^1_\beta, \sigma^0_\gamma] = +1$	$[\sigma^1_\gamma, \sigma^0_\gamma] = 0$	$[\sigma^1_\delta \sigma^0_\gamma] = +1$	$[\sigma^1_\epsilon \sigma^0_\gamma] = 0$	$[\sigma^1_\zeta \sigma^0_\gamma] = -1$
$[\sigma^1_\alpha, \sigma^0_\delta] = 0$	$[\sigma^1_\beta, \sigma^0_\delta] = 0$	$[\sigma^1_\gamma, \sigma^0_\delta] = +1$	$[\sigma^1_\delta \sigma^0_\delta] = 0$	$[\sigma^1_\epsilon \sigma^0_\delta] = +1$	$[\sigma^1_\zeta \sigma^0_\delta] = +1$

11. In the triangulation M of the Möbius strip, let us call a 1-simplex *interior* if it is a face of two 2-simplexes. For each interior simplex σ_i, let $\bar{\sigma}_i$ and $\bar{\bar{\sigma}}_i$ denote the two 2-simplexes of which σ_i is a face. Show that it is not possible to orient M so that

$$[\overline{\sigma_i}, \sigma_i] = -[\overline{\overline{\sigma_i}}, \sigma_i] \tag{1}$$

for each interior simplex σ_i.

Proof. We start by proving the following Lemma:

Lemma. *Let $\sigma^2_1 = \langle a_i a_j a_k \rangle$ and $\sigma^2_2 = \langle a_i a_j a_\ell \rangle$ be any two 2-simplices meeting along the common 1-simplex $\sigma^1 = \langle a_i a_j \rangle$ in a coherently oriented 2-complex. Suppose*

$$+\sigma^2_1 = \langle a_i a_j a_k \rangle$$

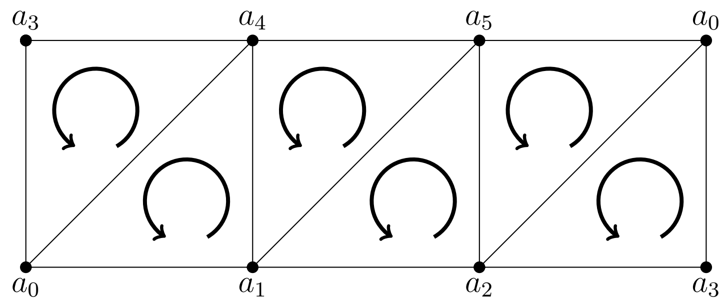
Then, no matter the positive orientation of σ^1, we must have

$$+\sigma^2_2 = \langle a_j a_i a_\ell \rangle$$

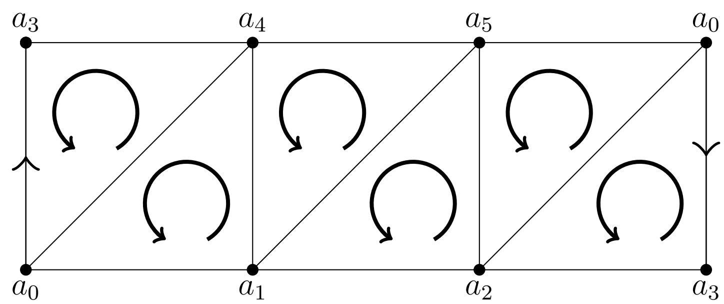
In other words, in the triangulation of a coherently oriented 2-complex containing two 2-simplices, we must have that the two swirls — which are drawn in the triangulation to show positive orientation — of each 2-simplex must both be clockwise or counter-clockwise.

Proof of Lemma. Let σ^2_1, σ^2_2, & σ^1 be as above. Suppose $+\sigma^1 = \langle a_i a_j \rangle$. Then $[\sigma^2_1, \sigma^1] = +1$. Because we are in a coherently oriented complex, we must have $[\sigma^2_2, \sigma^1] = -1$. This implies $+\sigma^2_2 = \langle a_j a_i a_\ell \rangle$.
Now suppose $+\sigma^1 = \langle a_j a_i \rangle$. Then $[\sigma^2_1, \sigma^1] = -1$. Now, because we're in a coherently oriented complex, we must have $[\sigma^2_2, \sigma^1] = +1$. This implies $+\sigma^2_2 = \langle a_j a_i a_\ell \rangle$. ■

To get a contradiction, assume that the Möbius strip is orientable. By the Lemma, we can draw a counterclockwise swirl in each 2-simplex.



This means $+\langle a_0a_3a_4\rangle = \langle a_0a_4a_3\rangle$ and $+\langle a_0a_2a_3\rangle = \langle a_0a_2a_3\rangle$. Without loss of generality, suppose $+\langle a_0a_3\rangle = \langle a_0a_3\rangle$.



Then, we have

$$[\langle a_0a_3a_4\rangle, \langle a_0a_3\rangle] = -1 \quad \text{and} \quad [\langle a_0a_2a_3\rangle, \langle a_0a_3\rangle] = -1,$$

which is a contradiction. Thus, the Möbius strip is nonorientable. ■

2-1 Suppose that K_1 and K_2 are two triangulations of the same polyhedron. Are the chain groups $C_p(K_1)$ and $C_p(K_2)$ isomorphic? Explain.

Solution:

Although K_1 and K_2 are two triangulations of the same polyhedron, they may not have the same number of p-simplexes. So, suppose $C_p(K_1)$ has α_p p-simplexes and $C_p(K_2)$ has β_p p-simplexes. Then,

$$C_p(K_1) \cong \mathbb{Z}^{\alpha_p} \quad \text{and} \quad C_p(K_2) \cong \mathbb{Z}^{\beta_p}$$

Then, $C_p(K_1) \cong C_p(K_2)$ if and only if $\alpha_p = \beta_p$.

2-2 Suppose that complexes K_1 and K_2 have the same simplexes but different orientations. How are the chain groups $C_p(K_1)$ and $C_p(K_2)$ related?

Solution:

$C_p(K_1)$ and $C_p(K_2)$ are isomorphic because K_1 and K_2 have the same simplexes. However, since some (if not all) of the simplexes of K_1 have different orientations from that of K_2, the p-chain of a given p-simplex in K_1 will be the negative of the p-chain of that same p-simplex in K_2. More precisely, suppose σ^p is a p-simplex in K_1 and K_2 such that the orientation of σ^p in K_1 is not the same orientation of σ^p in K_2. Let c_p be the p-chain for σ^p in K_1 and d_p be the p-chain for σ^p in K_2. Then, $c_p = -d_p$.

2-3 Prove Theorem 2.2.

Theorem 2.2. *If K is an oriented complex, then $B_p(K) \subset Z_p(K)$ for each integer p such that $0 \leq p \leq n$ where n is the dimension of K.*

Proof. Let $b_p \in B_p(K)$. Then, there exists $c_{p+1} \in C_{p+1}(K)$ such that $\partial(c_{p+1}) = b_p$. So,

$$\partial(b_p) = \partial(\partial(c_{p+1})) = \partial^2(c_{p+1}) = 0$$

by Theorem 2.1. Thus, $b_p \in Z_p(K)$. ☕

Ch2-16 Let K be a complex and K^r its r-skeleton. Show that $H_p(K)$ and $H_p(K^r)$ are isomorphic for $0 \leq p < r$. How are $H_r(K)$ and $H_r(K^r)$ related?

Proof. Since K^r contains all m-simplexes of K for all $0 \leq m \leq r$, then $C_m(K) = C_m(K^r)$ for all $0 \leq m \leq r$. Since $p < r$, then $p+1 \leq r$ and so $C_{p+1}(K) = C_{p+1}(K^r)$. Then,

$$B_p(K) = \partial_{p+1}(C_{p+1}(K)) = \partial_{p+1}(C_{p+1}(K^r)) = B_p(K^r)$$

and since $\partial_p : C_p(K) \to C_{p-1}(K)$, then

$$Z_p(K) = \ker(\partial_p) = Z_p(K^r).$$

So,

$$H_p(K) = Z_p(K)\big/B_p(K) = Z_p(K^r)\big/B_p(K^r) = H_p(K^r).$$

Let $n = \dim(K)$ and notice that for any $r \leq n$, $B_r(K^r) = \{0\}$ since K^r contains no $r+1$-chains and so $H_r(K^r) = Z_r(K^r)$. Since K and K^r contain the same r-simplexes, then we always have $Z_r(K) = Z_r(K^r)$. Notice

$$H_r(K) = Z_r(K)\big/B_r(K)$$

and

$$H_r(K^r) = Z_r(K^r) = Z_r(K).$$

So $H_r(K)$ is in fact a quotient group of $H_r(K^r)$. ☕

1. Homology groups of the Klein Bottle. Let K be the triangulation given on the next page.

   (a) $H_2(K)$
   Since K contains no 3-chains, $B_2(K) = \{0\}$. Label edges in K as follows:
   Type I: $\langle 06\rangle, \langle 36\rangle, \langle 03\rangle$.
   Type II: All other 1-simplexes.
   Notice that for any Type II edge σ', we have

   $$[\overline{\sigma'}, \sigma'] = [\overline{\overline{\sigma'}}, \sigma'],$$

   while for Type I edges, we get

   $$\begin{aligned}
   [\langle 083\rangle, \langle 03\rangle] &= [\langle 043\rangle, \langle 03\rangle] = -1\\
   [\langle 356\rangle, \langle 36\rangle] &= [\langle 376\rangle, \langle 36\rangle] = -1\\
   [\langle 062\rangle, \langle 06\rangle] &= [\langle 061\rangle, \langle 06\rangle] = 1.
   \end{aligned}$$

   Now let $z^2 = \sum g_{ijk}\langle ijk\rangle \in Z_2(K)$. Since the coefficients of $\partial(z^2)$ are all 0, then $g_{ijk} = g$ for all i, j, k. So

   $$0 = \partial(z^2) = -2g\langle 03\rangle - 2g\langle 36\rangle + 2\langle 06\rangle \tag{1}$$

   which implies $g = 0$ and so $z^2 = 0$. Thus, $Z_2(K) = \{0\}$, therefore $H_2(K) = \{0\}$.

   (b) $H_0(K)$
   Since all 0-cycles have boundary 0, then $Z_0(K) = C_0(K)$. For all $i \in \{1, 2, 3, 4, 6, 8\}$, let $c_i = \langle 0i\rangle$. Then let $c_5 = \langle 06\rangle - \langle 56\rangle$ and $c_7 = \langle 01\rangle + \langle 17\rangle$. Notice that for all $i \in \{1, 2, 3, 4, 6, 8\}$, we have $\langle i\rangle \sim \langle 0\rangle$:

   $$\langle i\rangle = \langle 0\rangle + \partial(\langle c_i\rangle) = \langle 0\rangle + \langle i\rangle - \langle 0\rangle,$$

   and also:

   $$\langle 5\rangle = \langle 0\rangle + \partial(\langle c_5\rangle) = \langle 0\rangle + \langle 6\rangle - \langle 0\rangle - (\langle 6\rangle - \langle 5\rangle).$$

   and

   $$\langle 7\rangle = \langle 0\rangle + \partial(\langle c_7\rangle) = \langle 0\rangle + \langle 1\rangle - \langle 0\rangle + \langle 7\rangle - \langle 1\rangle.$$

   Thus, given $z^0 = \sum_{i=0}^{8} g_i\langle i\rangle \in Z_0(K)$ we have

   $$z^0 = \sum_{i=0}^{8} g_i\langle i\rangle = g_0\langle 0\rangle + \sum_{i=1}^{8} g_i\left(\langle 0\rangle + \partial(c_i)\right) = \left(\sum_{i=0}^{8} g_i\right)\langle 0\rangle + \partial\left(\sum_{i=1}^{8} g_i c_i\right).$$

   Thus, $z^0 \sim \langle 0\rangle$, which means all cycles in $Z_0(K)$ fall into the same homology class. Thus,

   $$H_0(K) = \{g\langle 0\rangle + B_0(K) \mid g \in \mathbb{Z}\} \cong \mathbb{Z}.$$

(c) $H_1(K)$
Let $z_0^1 = \langle 01\rangle + \langle 12\rangle - \langle 02\rangle$ and $z_1^1 = \langle 03\rangle + \langle 36\rangle - \langle 06\rangle$. Notice that z_1^1 has order 2 in homology, since by (1), for any $c^2 \in C_2(K)$, we have

$$\partial(c^2) = 2g\langle 03\rangle - 2g\langle 36\rangle + 2\langle 06\rangle$$

Now, notice that z_1^1, z_0^1, and $z_1^1 - z_0^1$ are not boundaries. If, for example, z_1^1 were a boundary, then since all edges not in z_1^1 have coefficient 0, the boundary formula says that if

$$z_1^1 = \sum g_{ijk}\langle ijk\rangle$$

we get $g_{ijk} = g$ for all i, j, k.. Thus $[z_1^1] \neq [z_0^1]$ and both classes are nontrivial. Now, let

$$z^1 = \sum_{\langle ij\rangle \in K} g_{ij}\langle ij\rangle \in Z_1(K).$$

We perform "the trick" as indicated in the triangulation below to build a $c^2 \in C_2(K)$ such that

$$\begin{aligned} z^1 + \partial(c^2) = p_{01}\langle 01\rangle + p_{12}\langle 12\rangle + p_{02}\langle 02\rangle \\ + q_{03}\langle 03\rangle + q_{36}\langle 36\rangle + q_{06}\langle 06\rangle \\ + h_{17}\langle 17\rangle + h_{47}\langle 47\rangle + h_{45}\langle 45\rangle + h_{68}\langle 68\rangle. \end{aligned}$$

Observe that $\langle 5\rangle$ is isolated. So, $h_{17} = h_{47} = h_{45} = 0$. Moreover, $\langle 8\rangle$ is isolated, which gives $h_{68} = 0$. Computing $\partial(z^1 + \partial(c^2))$ yields

$$\begin{aligned} p &:= p_{01} = p_{12} = -p_{02} \\ q &:= q_{03} = p_{36} = -q_{06}. \end{aligned}$$

Therefore, $z^1 + \partial(c^2) = p(z_1^1) + q(z_0^1)$, and so

$$[z^1] = p[z_0^1] + q[z_1^1]$$

which implies

$$H_1(K) = \langle [z_0^1], [z_1^1]\rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Ch 2, # 13 Prove that the geometric carriers of the combinatorial components of the complex K and the components of the polyhedron $|K|$ are identical.

Proof. Let $K_1, K_2, \ldots, K_r$ denote the combinatorial components of K. Let $C_1, C_2, \ldots, C_m$ denote the path components of $|K|$. It suffices to show that $r = m$, and up to reordering of indices, $|K_i| = C_i$, for all $i = 1, 2, \ldots, r$.
Let K_i be a combinatorial component of K. We need to find a path component C_j for which $C_j \supset |K_i|$. Let $x, y \in |K_i|$. Then, x and y belong to some simplices σ_x and σ_y, respectively. First suppose $\sigma_x \cap \sigma_y = \emptyset$. Since σ_x and σ_y belong to the same combinatorial component, there is a path of 1-simplices $\sigma_1^1, \sigma_2^1, \ldots, \sigma_n^1$ which creates a path between σ_x and σ_y. Let

$$\sigma_x^0 = \sigma_x \cap (\sigma_1^1 \cup \sigma_n^1),$$

or in other words, σ_x^0 is the vertex of σ_x which is also a vertex of one of the "endpoints" of the path created by the 1-simplices. Likewise, define σ_y^0. Recall the simplices are convex. In particular, there exists a path in σ_x from x to σ_x^0. Similarly, there exists a path from y to σ_y^0. So, we have a path from x to y, so that $|K_i|$ is contained in some path component C_j.
Now suppose $\sigma_x \cap \sigma_y \neq \emptyset$. Since we are in a properly joined complex, there is at least one vertex σ_{xy}^0 for which $\sigma_{xy}^0 \in \sigma_x \cap \sigma_y$. Again since simplices are convex, there exists a path between x and σ_{xy}^0, and also between σ_{xy}^0 and y. Therefore, there is a path between x and y. So again, $|K_i|$ is contained in some path component C_j.
Could not figure out opposite inclusion. (Guess I should have went to office hours). Alex helped me here:
Let C_j be a path component of $|K|$. We claim that C_j is a union of simplices of K. That is, if σ is a simplex which intersects C_j, then $\sigma \subset C_j$. To see this note that the intersection of σ and C_j is nonempty, then they share at least one point in common, say x. Since simplices are convex, there is a path from every point in σ to x. Also, there exists a path from every point in C_j to x. By transitivity of path connectedness, every point in σ is path connected to C_j, and hence, $\sigma \subset C_j$. Now we claim that any two simplices in C_j are combinatorially connected. So suppose $\sigma_1, \sigma_2 \subset C_j$. Let v be a vertex os σ_1 and w a vertex of σ_2. Suppose first that $\sigma_1 \cap \sigma_2 = \emptyset$. Since C_j is path connected, there exists a path in C_j from v to w. Since $C_j \subset |K|$, there exist simplices $\tau_1, \tau_2, \ldots, \tau_n$ of K so that the path is connected in $\bigcup_{i=1}^n \tau_i$ and $\tau_i \cap \tau_j \neq \emptyset$ for all i. So there exists a simplex τ_1 (up to reordering) so that $x \in \sigma_1 \cap \tau_1$. Let r_1 be the largest integer for which $\tau_1 \cap \tau_2 \cap \cdots \cap \tau_{r_1} \neq \emptyset$. Let v_1 be the shared vertex. Then there exists (by the definition of a simplex) a 1-simplex σ_1^1 so that σ_1^1 contains both v and v_1. If $r_1 = n$, we are almost done. If not, continue in this manner: while $2 \leq \ell \leq n$, let r_ℓ be the largest integer for which $\tau_{r_{\ell-1}} \cap \tau_{r_{\ell-1}+1} \cap \cdots \cap \tau_{r_\ell} \neq \emptyset$. Let v_ℓ be the shared vertex. Then there exists a 1-simplex σ_ℓ^1 so that σ_ℓ^1 contains both $v_{\ell-1}$ and v_ℓ. Once $r_\ell = n$ for some ℓ, stop the process. Then since v_ℓ and w are vertices of σ_ℓ^1 there exists a 1-simplex $\sigma_{\ell+1}$ containing both v_ℓ and w. Thus there exists a sequence of 1-simplices $\sigma_1^1 \sigma_2^1, \ldots \sigma_{\ell+1}^1$ so that $v \in \sigma_1 \cap \sigma_1^1$, $w \in \sigma_{\ell+1}^1 \cap \sigma_2$ and $\sigma_i^1 \cap \sigma_{i+1}^1 \neq \emptyset$ for all $1 \leq i \leq \ell$. This tells us precisely that σ_1 and σ_2 are combinatorially connected. If however we had $\sigma_1 \cap \sigma_2 \neq \emptyset$, then they'd still be combinatorially connected. Therefore, there exists K_i so that $C_j \subset |K_i|$. ☕

11. Derive the possibilities for (m, n, F) referred to in the proof of Theorem 2.7. How do you rule out the cases $m = 1$ and $m = 2$?

Proof. We use the following relations from the theorem:

$$F(2n - mn + 2m) = 4m, \quad n \geq 3, m < 6$$

If $m = 1$, then $F(n + 2) = 4$. Since $n \geq 3$, then $4 = F(n + 2) \geq 5F$, which cannot happen. If $m = 2$, then $4F = 8$ which means $F = 2$. Then By Euler's Theorem, $V - E = 0 \implies V = E$, which cannot happen.

$$\begin{aligned}
m = 3 &\implies F(6 - n) = 12 \\
&\implies n = 3, F = 4 \\
&\quad \text{or } n = 4, F = 6 \\
&\quad \text{or } n = 5, F = 12 \\
&\implies (m, n, F) = (3, 3, 4) \\
&\quad \text{or } (m, n, F) = (3, 4, 6) \\
&\quad \text{or } (m, n, F) = (3, 5, 12) \\
m = 4 &\implies F(8 - 2n) = 16 \\
&\implies n = 3, F = 8 \\
&\implies (m, n, F) = (4, 3, 8) \\
m = 5 &\implies F(10 - 3n) = 20 \\
&\implies n = 3, F = 20 \\
&\implies (m, n, F) = (5, 3, 20)
\end{aligned}$$

☕

14. Prove that the pth Betti Number of a complex K is the rank of the free part of the pth homology group $H_p(K)$.

Proof. Suppose $R_p(K) = r$ and $H_p(K) \cong \mathbb{Z}^s \oplus T_1 \oplus T_2 \oplus \cdots \oplus T_m$ where each T_i is a finite cyclic group. Thus, r is the largest integer for which there exists cycles $\{z_1^p, \ldots, z_r^p\} \subseteq Z_p(K)$ which are linearly independent with respect to homology. That is,

$$\sum_{i=1}^{r} g_i z_i^p = \partial(c^{p+1})$$

for some $c^{p+1} \in C_{p+1}(K)$ if and only if $g_i = 0$ for all $1 \leq i \leq r$. This is equivalent to

$$\sum_{i=1}^{r} g_i [z_i]^p = [0] \iff g_i = 0 \quad \forall\ 1 \leq i \leq r. \tag{$*$}$$

We claim $r = s$.

Notice that $(x) \in T_1 \oplus \cdots \oplus T_m$ if and only if the first s coordinates of (x) are zero and there exists a non-negative integer g so that $g \cdot (x) = 0$ (In particular, $g = \max_i\{|T_i|\}$). So, for a fixed homology class $[z_i^p]$, if $g \cdot [z_i^p] = 0$ then by $(*)$, $g = 0$. Thus, $[z_i^p] \notin T_1 \oplus \cdots \oplus T_m$ so that $[z_i^p] \in \mathbb{Z}^s$ for all $1 \leq i \leq r$. We also have from $(*)$ that the collection $\{[z_i]^p\}$ is linearly independent in $\mathbb{Z}^s$, so that

$$r = rank(span\{z_1^p, \ldots, z_r^p\}) \leq rank(\mathbb{Z}^s) = s.$$

Let $e_1, \ldots e_s$ be the standard basis elements of $\mathbb{Z}^s$. That is, for each $1 \leq i \leq s$, we have $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 appears in the ith coordinate. Pick representative x_i^p from the equivalence classes corresponding to e_i for all $1 \leq i \leq s$. We claim $\{x_1^p, \ldots, x_s^p\}$ are linearly independent with respect to homology. Once this is verified, then $s \leq r$ since r is the *maximal* integer so that there exists cycles which are linearly independent with respect to homology. Suppose there exists integers $g_1, \ldots g_s$ and $c^{p+1} \in C_{p+1}(K)$ such that $\sum_{i=1}^r g_i x_i^p = \partial(c^{p+1})$. Then

$$\begin{aligned}
\sum_{i=1}^{r} g_i x_i^p = \partial(c^{p+1}) &\iff \sum_{i=1}^{r} g_i [x_i^p] = [0] \\
&\iff \sum_{i=1}^{r} g_i e_i = 0 \\
&\iff g_i = 0 \quad \forall i \text{ since the } e_i\text{'s are linearly independent.} \\
&\implies \{x_1^p, \ldots, x_s^p\} \text{ are linearly independent w.r.t. homology.}
\end{aligned}$$

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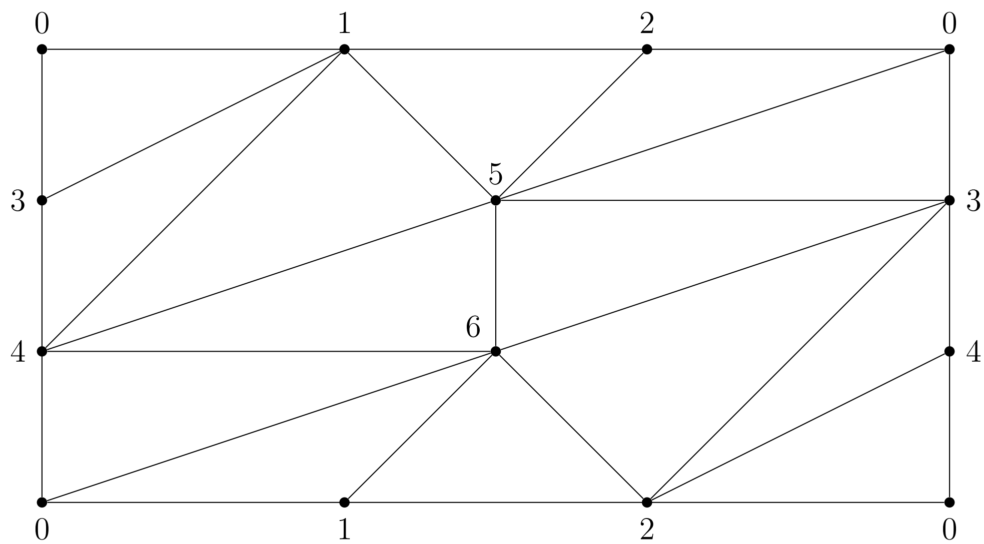
Ch2, # 15 Find a minimal triangulation for the torus T

Proof. First note that T is a 2-pseudomanifold. Also, $H_0(T) \cong \mathbb{Z}, H_1(T) \cong \mathbb{Z}^2$, and $H_2(T) \cong \mathbb{Z}$. So $\chi(T) = 1 - 2 + 1 = 0$. Therefore by Theorem 2.8, the number of $0, 1$, and 2 simplexes are

$$a_0 \geq \frac{1}{2}\left(7 + \sqrt{49 - 24(0)}\right) = 7$$
$$a_1 \geq 3((7) - (0)) = 21$$
$$a_2 \geq \frac{2}{3}(21) = 14.$$

Hence a minimal triangulation of T will have 7 vertices, 21 edges, and 14 faces.



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Ch2, # 22 Show that an orientable n-pseudomanifold has exactly two coherent orientations for its n-simplexes.

Proof. Let K be an orientable n-pseudomanifold and σ_1^n, σ_k^n, be any two n-simplexes in K. Since K is an n-psuedomanifold, there exists a sequence of n-simplexes

$$\sigma_1^n, \sigma_2^n, \dots \sigma_k^n$$

so that $\sigma_i^n \cap \sigma_{i+1}^n = \sigma_i^{n-1}$ for some $n-1$ simplex σ_i^{n-1} for all $1 \leq i \leq k-1$. Since K is orientable,

$$[\sigma_i^n, \sigma_i^{n-1}] = -[\sigma_{i+1}^n, \sigma_i^{n-1}] \quad \forall \;\; 1 \leq i \leq k-1.$$

Therefore, once an orientation on σ_1^n is determined, the orientation on σ_2^n is determined, and thus the orientation on σ_3^n is determined, and so on. Since σ_1^n has exactly two possible orientations, and σ_k^n was arbitrary, all of the n-simplexes of K have one of two possible coherent orientations.

Ch2, # 23 If K is an orientable n-pseudomanifold, prove that $H_n(K) \cong \mathbb{Z}$.

Proof. Since K is orientable, give it a coherent orientation. Since $B_n(K) = \{0\}$, then $H_n(K) = Z_n(K)$. Let

$$z^n = \sum_i g_i \sigma_i^n \in Z_n(K).$$

Since $\partial(z^n) = 0$, then

$$0 = \partial\left(\sum_i g_i\sigma_i^n\right) = \sum_i \partial(g_i\sigma_i^n) = \sum_i \left(\sum_{\sigma_j^{n-1}\in K} [\sigma_i^n, \sigma_j^{n-1}] g_i \cdot \sigma_i^n\right) = \sum h_\ell \sigma_\ell^{n-1}.$$

Since K is a n-pseudomanifold and z^n is a cycle, then each $n-1$ simplex lies on exactly two n simplexes so that the coefficient on each σ_ℓ^{p-1} is a linear combination of two integers, i.e., $h_\ell = (g_r \pm g_s)$ where σ_r^n and σ_s^n are adjacent n simplexes. Moreover, since K is coherently oriented, then $h_\ell = \pm(g_r - g_s)$ for all ℓ. Since $0 = \sum h_\ell \sigma_\ell^{n-1}$, then $h_\ell = 0$ for all ℓ. This show that each pair of adjacent simplexes have the same coefficient in z^n. We now show that in fact all n-simplexes have the same coefficient in z^n so that $z^n = \sum g\sigma_i^n$ for some integer g, and hence $H_n(K) \subseteq \mathbb{Z}$.

Fix two n simplexes σ_1^n and σ_i^n, $i > 1$. Then there exists a sequence of n simplexes

$$\sigma_i^n = \sigma_{i_0}^n, \sigma_{i_1}^n, \dots, \sigma_{i_m}^n = \sigma_i^n$$

so that $\sigma_{i_p}^n \cap \sigma_{i_{p+1}}^n = \sigma_{\ell_p}^{n-1}$ for all $0 \leq p \leq m-1$. Since $0 = h_{j_p} = \pm(g_{j_p} - g_{j_{p+1}})$ for all $0 \leq p \leq m-1$, then

$$g_1 = g_{i_0} = g_{i_1} = g_{i_2} = \cdots = g_{i_{m-1}} = g_{i_m} = g_i.$$

Since σ_i^n was arbitrary then all n simplexes in z^n have the same coefficient.

Conversely, suppose

$$c^n = \sum g\sigma_i^n \in C_p(K)$$

for some integer g. If the $n-1$ simplex σ^{n-1} is a face of σ_1^n and σ_2^n, then since K is coherently oriented

$$[\sigma_1^n, \sigma^{n-1}] = -[\sigma_2^n, \sigma^{n-1}]$$

which implies

$$\partial(c^n) = \sum (g-g)\sigma_\ell^{p-1} = 0$$

and thus c^n is a cycle. Therefore, $\mathbb{Z} \subseteq H_n(K)$.