Engineering Math II – Exam 3 (Quick) Review

Chapter 8, Sections 1,2,3;

Chapter 13, Section 2;

Chapter 9, Sections 1,2,4,6,7,8

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8 Chapter 8 – Integration in Cartesian Coordinates

8.1 Review of Integration in 1D

- The big thing is this section is the Fundamental Theorem of Calculus, which describes the familiar technique we use when computing integrals:

  Theorem 1 (The Fundamental Theorem of Calculus: Part I)**.** *Suppose we want to find* $\int_a^b f(x)dx$*. If we can find another function* $F(x)$ *so that* $F'(x) = f(x)$ *for every* $a \leq x \leq b$ *(i.e., for every* x *between* a *and* b*, including* a *and* b*), then*

  $$\int_a^b f(x)dx = F(b) - F(a).$$

  – So when we want to compute an integral, we find the antiderivative $F(x)$ of the integrand $f(x)$ we are integrating and compute $F(b) - F(a)$. However, remember that in order to apply the theorem, both $f(x)$ and $F(x)$ need to be defined and continuous for ALL x between (and including) a and b, not just at a and b. Compare the following two examples:

  $$\int_1^2 \frac{1}{x^2}dx = \int_1^2 x^{-2}dx = (-x^{-1})\Big|_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = \frac{1}{2}$$

  and

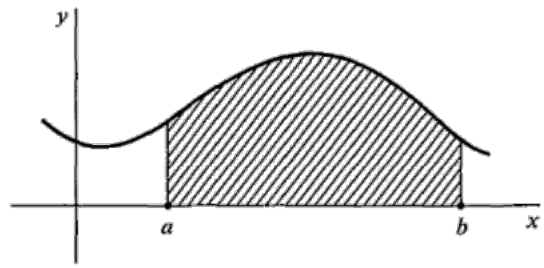
  $$\int_{-2}^1 \frac{1}{x^2}dx = \int_{-2}^1 x^{-2}dx = (-x^{-1})\Big|_{-2}^1 = -\frac{1}{1} - \left(-\frac{1}{-2}\right) = -\frac{3}{2}.$$

  In both cases, we computed the antiderivative just fine, and applied the theorem as we think we should. However, the function $f(x) = x^{-2}$ and the function $F(x) = -x^{-1}$ are NOT defined for EVERY x value between -2 and 1. Sure, they are defined at -2 and 1, but not for ALL values between them; namely, at 0. So the first example above is fine, but the second example is an incorrect application of the theorem.

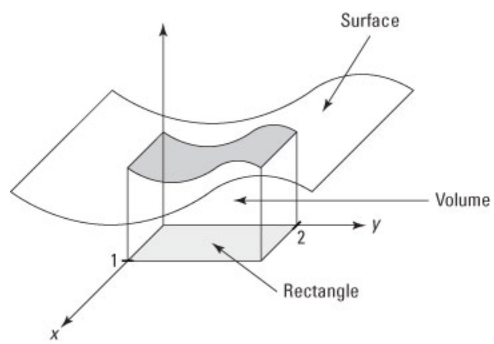
- Also, you should really review your integration rules. There are some reviews at the end of this section in the textbook. Alternatively, there are charts and lists online that you can use to study antiderivatives. The easiest part of the exam should be actually computing integrals. Usually setting up the integrals the correct way is the hardest part.

8.2 Integration in 2D

- Back in Calculus I, we integrated functions of one variable, (so functions that have only one variable x appearing in them) and learned that this gave the "area under the curve", or in other words, the area between the graph of a function and the x-axis. In this case, we just had to specify a range of x-values that we want to integrate the function over, usually we said "integrate $f(x)$ between a and b".



- Bumping it up one dimension, we learned about how integrating a function of two variables, (so functions that have two variables x and y appearing in them), is really about finding the "volume under the surface". Now, we have to specify an entire *region* over which we want to integrate a function*in the xy-plane*, not just an interval of x-values. So now we say "integrate $f(x, y)$ over a region $\mathbb{D}$".



(Now, the region $\mathbb{D}$ may not be a rectangle as in the above picture, and this is where it can get tricky. See the review sheet "integrating over non-rectangular regions" for examples showing how to set up such integrals.)

- In the case that we are integrating over a rectangle $\mathbb{D} = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, we can easily compute

$$\int_a^b \left(\int_c^d f(x, y) dy \right) dx \quad \text{or} \quad \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$$

- *Suggestion:* Practice taking antiderivatives of functions of two variables. This can be more difficult because we have to do a "reverse operation" to taking partial derivatives, (so maybe we should call it "partial antiderivation"? ☺)

8.3 Iterated Integrals in 2D

- This section is all about giving intuition behind how we find the volume under a surface using (double) integration, and about how to set up and compute double integrals over non-rectangular regions. See the review sheet on this topic for "plenny" of examples.

- Center of Mass: We are introduced in the exercises of this section to the center of mass (or center of gravity) formulas. If we want to find the center of mass of a region $\mathbb{D}$, the x and y values of the center of mass are called $\bar{x}$ and $\bar{y}$, and are given by the formulas

$$\begin{aligned} \text{Area}[\mathbb{D}] &= \int_{\mathbb{D}} 1 \; dxdy, \\ \bar{x} &= \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} x \; dxdy, \\ \bar{y} &= \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} y \; dxdy. \end{aligned}$$

  Important! The center of mass $(\bar{x}, \bar{y})$ of a region does NOT necessarily need to be a point on the region itself. To see an example of this, see Example 13.2.7 in §13.2.
  Important! A common mistake is to forget to divide by the area of the domain (or equivalently, to multiply by 1 over the area) in $\bar{x}$ and $\bar{y}$. Remember to compute the above integrals, then divide by the area of the domain!

- Moment of Inertia: We also see moment of inertia formula for a region $\mathbb{D}$ in this section. It is a quantity that measures how much energy it takes to spin the domain $\mathbb{D}$ in the xy-plane around a vertical axis through the origin. It is given by the formula

$$\int_{\mathbb{D}} x^2 + y^2 \; dxdy.$$

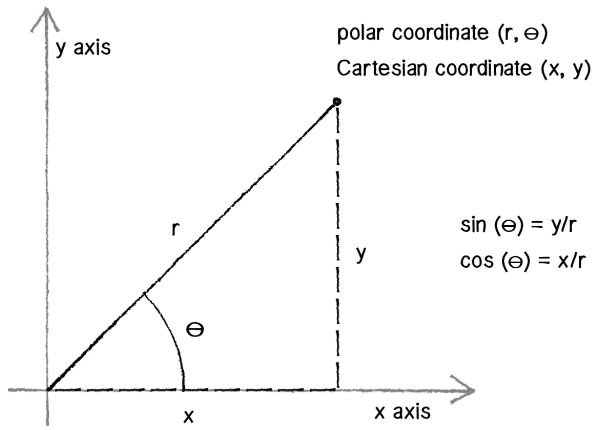
- Radius of Gyration: There was only one homework problem on this, but just to be safe: The radius of gyration for a region $\mathbb{D}$ is given by the formula

$$\rho = \sqrt{\frac{\int_{\mathbb{D}} x^2 + y^2 dxdy}{\text{Area}[\mathbb{D}]}}.$$

13 Chapter 13 – Coordinate Systems in 2D

13.2 Area Integrals in Polar Coordinates

- For all of your (adult) math life up to this point, you've been describing points in the plane by their x and y values, (hence the name "the xy-plane"). We simply say how far left or right the point is (the x value), and how far up or down the point is (the y-value) to describe a point. These are called "Cartesian coordinates". In this section, we learn a new way to describe points in the plane.



- Polar Coordinates! Using the formulas

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

  we now have a new way to describe a point in the plane: First, by the point's distance from the origin: $r = \sqrt{x^2 + y^2}$ (Check that this formula is true using the formulas given above!). This should look familiar. Indeed, it is the length (or "norm", or "magnitude") of the vector/point (x, y)! Second, we look at the angle between the positive x-axis, and the line between the origin and the point. With these two pieces of information, we can describe a point in the plane.

- Why do this? Basically, because *sometimes* it's easier to compute integrals if we use the different letters (r, θ). Of course, there are about ten bagillion[1] other reasons for polar coordinates, but for right now, this is the only reason we really care about.

- You've got to know how to describe regions in the plane using polar coordinates. If you know how to describe the regions in Exercise 1 of §13.2, you will be fine.

- Next thing: Integrating using polar coordinates. If we have an integral $\int_{\mathbb{D}} f(x, y)da$ in Cartesian coordinates, we can convert it to polar coordinates by the following steps:
  - Describe the domain $\mathbb{D}$ in polar coordinates r and θ.
  - Change the function $f(x, y)$ in terms of Cartesian coordinates into a function in polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$.
  - Replace $dxdy$ with $r\ drd\theta$. Notice that we DO NOT just replace $dxdy$ with $drd\theta$, but instead by $r\ drd\theta$.

  Here's two examples:

[1] this is a precise number

Example 1. Compute $\int_{\mathbb{D}} \frac{y}{\sqrt{x^2+y^2}} da$ where $\mathbb{D}$ is the right half disk of radius 3.

Solution:

First, we describe our region in polar coordinates:

$$\mathbb{D} = \{(r, \theta) : 0 \leq r \leq 3,\ -\pi/2 \leq \theta \leq \pi/2\}.$$

Next, we write our function in terms of r and θ:

$$f(x, y) = \frac{y}{\sqrt{x^2 + y^2}} \quad \xrightarrow{\text{convert to polar}} \quad f(r\cos\theta, r\sin\theta) = \frac{r\sin\theta}{r}.$$

Notice that we used the equation $r = \sqrt{x^2 + y^2}$ here. Be sure the know formulas like this to save time on the exam.

Next, we replace $dxdy$ with $r\ drd\theta$.

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \frac{r\sin\theta}{r} r\ drd\theta = \int_{-\pi/2}^{\pi/2} \int_0^3 r\sin\theta drd\theta.$$

(The computation of the above integral is left as an exercise for the reader)

Example 2. Suppose we shift the region $\mathbb{D}$ from the previous example by 2 to the right, and -1 down. (Another way to obtain this is to start with a circle of radius 3 centered at $(2, -1)$ and let the region $\mathbb{D}$ be the right half of this shifted circle.)

Solution:

There's a small trick to this one. We have to make a change to something: either our bounds of integration (which would be difficult, to say the least) or make a change to our function. Let's do the second thing.

So, our limits of integration stay the same, but we have to shift our function. Here's what to do: Instead of the substitutions $x = r\cos\theta$ and $y = \sin\theta$, we make the substitutions $x = r\cos\theta + 2$ and $y = r\sin\theta - 1$. So our function is now

$$\begin{aligned} f(r\cos\theta + 2, r\sin\theta - 1) &= \frac{r\sin\theta - 1}{\sqrt{(r\cos\theta + 2)^2 + (r\sin\theta - 1)^2}} \\ &= \frac{r\sin\theta - 1}{\sqrt{(r^2\cos^2\theta + 4r\cos\theta + 4) + (r^2\sin^2\theta - 2r\sin\theta + 1)}} \\ &= \frac{r\sin\theta - 1}{\sqrt{r^2 + 4r\cos\theta - 2r\sin\theta + 5}}. \end{aligned}$$

We no longer have the formula $r = \sqrt{x^2 + y^2}$, since we are substituting different things for x and y. So our integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \frac{r\sin\theta - 1}{\sqrt{r^2 + 4r\cos\theta - 2r\sin\theta + 5}} r\ drd\theta.$$

Don't worry, you won't have to integrate something this gross on the exam (cartesian coordinates may be easier for this one actually!). I'm just using this terrible function as an example to show how to change the integral appropriately in a situation like this.

- I would highly recommend that you read the examples given in this section of the textbook. They're good ones. Try to do them on your own first, and then check how the textbook does it.

- Center of Mass: We already have the formulas given above for finding the center of mass of a region. Substituting appropriately, we get center of mass formulas in polar coordinates:

$$\begin{aligned}
\text{Area}[\mathbb{D}] &= \int_{\mathbb{D}} 1 dxdy = \int_{\mathbb{D}} r\ drd\theta, \\
\bar{x} &= \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} x\ dxdy = \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} (r\cos\theta)\ r\ drd\theta = \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} r^2 \cos\theta\ drd\theta, \\
\bar{y} &= \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} y\ dxdy = \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} (r\sin\theta)\ r\ drd\theta = \frac{1}{\text{Area}[\mathbb{D}]} \int_{\mathbb{D}} r^2 \sin\theta\ drd\theta.
\end{aligned}$$

  (And although we can't tell from the way it is written above, we need to make sure to describe the region $\mathbb{D}$ in polar coordinates when converting our functions into polar coordinates.)

- Moment of Inertia: Again, we already have a formula here, but lets substitute to find our moment of inertia formula in polar coordinates:

$$\int_{\mathbb{D}} x^2 + y^2 dxdy = \int_{\mathbb{D}} r^2 r\ drd\theta = \int_{\mathbb{D}} r^3\ drd\theta.$$

9 Parametric Curves

9.1 Parametric Lines

- One main equation in this section, an equation of a line: $X = P + tV$. Let's understand what it's saying a bit more: To describe each point on a given line, we need to give its x-value, y-value, and z-value. If we want to describe a line in three-space, we do so parametrically, i.e., we give the x, y, z-values of each point on the line in terms of an independent variable t.

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+t \\ 3-5t \\ -1+4t \end{pmatrix}.$$

  The equation $X = P + tV$ can be thought of intuitively in this way: The vector P has its tail at the origin, and its tip on the line we are trying to describe. The vector V is a vector with its tail at the tip of P, and sits in the line we are trying to describe. So, $P + tV$ means that we start at the origin, go along P, then do some scalar multiple t of V. Doing this for all possible t values will describe each point on the line. This isn't rigorous, but it helps understand intuitively where the equation comes from.

- Example 9.1.5 in §9.1 is a helpful example.

9.2 Parametric Curves in 2D

- The main thing we picked up from this section was the equation to parametrically describe a circular motion in 2 dimensions:

$$X = rU\cos\theta + rV\sin\theta + C,$$

  where U, V, and C are each 2-vectors (for now). The number r is the radius of the circle. The vector C is the center of the circle. The vector U is a **unit vector *which has its tail at*** C and its tip pointing in the direction of the point on the circle corresponding to when $t = 0$ (the starting point). The vector V is a **unit vector *which has its tail at*** C and its tip pointing in the direction of the point on the circle corresponding to when $t = \pi/2$ (a quarter turn from our start, in either a clockwise or counterclockwise rotation, based on what the problem asks). Therefore, U **and** V **are perpendicular to each other**.

Example 3. Find parametric equations for the following motion: A counterclockwise motion of radius 3, centered at $(2, -1)$, starting at the point $(5, -1)$.

Solution:

Immediately, we know that $r = 3$ and $C = (2, -1)$. Let's find U and V. The vector U needs to be a unit vector *which has its tail at* C and points in the direction of our starting vector $(5, -1)$. The vector $(5, -1) - (2, -1) = (3, 0)$ is a vector that points from C to our start. Let's normalize it (divide by its norm) to find U:

$$U = \frac{(3,0)}{|(3,0)|} = \frac{(3,0)}{\sqrt{3^2 + 0^2}} = \frac{(3,0)}{3} = (1,0).$$

Now for V, we need a vector that points in the direction corresponding to a quarter turn counterclockwise. On the circle, if we make a counterclockwise turn starting at $(5, -1)$, then we are at the point $(2, 2)$ (draw a picture to see this!). So

$$V = \frac{(2,2) - (2,-1)}{|(2,2) - (2,-1)|} = \frac{(0,3)}{|(0,3)|} = (0,1).$$

Putting it all together,

$$\begin{aligned} X &= rU\cos\theta + rV\sin\theta + C \\ \begin{pmatrix} x \\ y \end{pmatrix} &= 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos\theta + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\theta + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3\cos\theta + 2 \\ 3\sin\theta - 1 \end{pmatrix} \end{aligned}$$

9.4 Parametric Curves in 3D

- I think the most helpful thing to do for this section is just some examples. I would recommend that you read the textbook's examples, too.

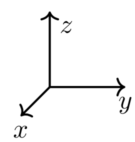
  Example 4. Find a parametric equation for a helix of radius 2 that winds around the x-axis with a right-hand orientation with your thumb on the positive x-axis and has a "pitch" of 5 units forward in x for each full rotation in yz.

  Solution:

  We are essentially drawing a circle in the yz-plane, but moving along in the x direction as we draw it, making a helix. A circle in the yz-plane of radius 2 with a right hand orientation with your thumb on the positive x axis is given parametrically by the equations

$$\begin{aligned} x &= 0 \\ y &= 2\cos t \\ z &= 2\sin t \end{aligned}$$

  Why is $y = 2\cos t$ and $z = 2\sin t$? Why not the other way around? You already saw how to draw a circle in the previous section. You need two unit vectors U and V. I decided here to start at the vector $U = (0, 1, 0)$ and rotate towards $V = (0, 0, 1)$. So, using our formula in the previous section, we get $y = 2\cos t$ and $z = 2\sin t$. Drawing a picture helps:



  This difficulty with these problems is that we could have also started at the positive z axis, $U = (0, 0, 1)$ and then, rotating with the right hand orientation, we get the negative y unit vector: $V = (0, -1, 0)$. So in this case we get $y = -2\sin t$ and $z = 2\cos t$. And these give the same parametrizations, but different starting locations. On the exam, you'll just have to be comfortable with the different choices you could make.

  We are not done; we have only given the equations of a circle of radius 2 in the yz-plane. This is not what we want; we want a helix. So now, we need to change our equation for x. If our pitch was 1, then we know $x = \frac{1}{2\pi}t$. In this problem, our pitch is 5, so our parametric equation for x is: $x = \frac{5}{2\pi}t$. Our final answer is:

$$\begin{aligned} x &= \frac{5}{2\pi}t \\ y &= 2\cos t \\ z &= 2\sin t \end{aligned}$$

  Example 5. Find a parametric equation for a helix of radius 3 that winds around the y-axis with a left-hand orientation with your thumb on the positive y-axis and has a "pitch" of $\frac{1}{3}$ units forward in y for each full rotation in xz.

  Solution:

  Now we want to draw a circle in the xz-plane with a left-hand orientation, and then vary our y direction. The left-hand orientation says to rotate from the positive x axis towards the positive z axis, making a clockwise rotation. So $U = (1, 0, 0)$ and $V = (0, 0, 1)$, giving $x = 3\cos\theta$ and $z = 3\sin t$. Since our pitch is given to be $\frac{1}{3}$, our final answer is

$$\begin{aligned} x &= 3\cos t \\ y &= \frac{1}{3} \cdot \frac{1}{2\pi}t = \frac{1}{6\pi}t \\ z &= 3\sin t \end{aligned}$$

Example 6. Find equations for a circular motion of radius 3, centered at the origin, that rotates counterclockwise in the plane which is perpendicular to the vector $(1,2,1)$ when viewed from $(-3,-6,-3)$. (Hint: the vectors $(1,2,1),(1,-1,1),(3,0,-3)$ are mutually perpendicular.

Solution:

There's a lot going on here. Basically, we need to describe a circle that sits in a particular plane. So, let's use our equation that describes a circle:

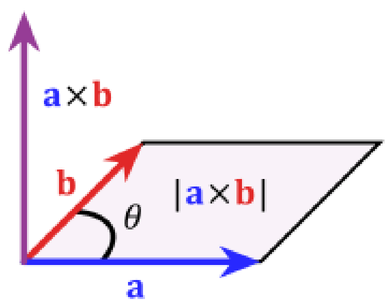
$$X = rU\cos\theta + rV\sin\theta + C,$$

We see immediately that $r = 3$ and $C = (0,0,0)$. Now we just need to find U and V.

Again, we are looking for a circle that lives in a given plane. So our U and V must also live in that plane. The plane (just like all planes) is described by what all the vectors in the plane are perpendicular to: $(1,2,1)$. So U and V must be vectors that are perpendicular to $(1,2,1)$. We are given two vectors in the hint that are perpendicular to $(1,2,1)$, AND are perpendicular to each other. So (after we make them unit vectors), these are our U and V.

So which one is U and which one is V? The other piece of information in the problem is that, when we view the circle from the vector $(-3,-6,-3)$, the rotation is counterclockwise.

Now, think back to the cross product of two vectors.



If we take the cross product of two vectors, we get a new vector $a \times b$ such that $a \times b$ is perpendicular to a and b, AND if we look down on the vectors a and b from the perspective of $a \times b$, we ALWAYS have a right hand orientation, i.e., we always have a rotating towards b in a counterclockwise fashion.

So if we compute

$$(1,-1,1) \times (3,0,-3) = (3,6,3),$$

then this tells us that if we view the vectors $(1,-1,1)$ and $(3,0,-3)$ from the perspective of their cross product, $(3,6,3)$, then the vector $(1,-1,1)$ is rotating towards $(3,0,-3)$ in a counterclockwise manner. (Another way to interpret this is to say: If we view the vectors from the negative of their cross product $(-3,-6,-3)$, then $(1,-1,1)$ is rotating toward $(3,0,-3)$ in a clockwise manner.) BUT, the problem says that, when we view from $(-3,-6,-3)$, then we are rotating in a counterclockwise manner. So this doesn't help us. But if we compute the other way [2]

$$(3,0,-3) \times (1,-1,1) = (-3,-6,-3),$$

then this tells us that if we view the vectors $(3,0,-3)$ and $(1,-1,1)$ from the perspective of their cross product, $(-3,-6,-3)$, then the vector $(3,0,-3)$ is rotating towards $(1,-1,1)$ in a counterclockwise

[2] Remember that if we compute the cross product one way, then we can compute the cross product the other way by just negating the entries! In other words $-(a \times b) = b \times a$.

manner. The question says we want a counterclockwise rotation when we view from $(-3,-6,-3)$, so $U = \frac{(3,0,-3)}{|(3,0,-3)|} = \frac{(3,0,-3)}{3\sqrt{2}}$ and $V = \frac{(1,-1,1)}{|(1,-1,1)|} = \frac{(1,-1,1)}{\sqrt{3}}$. Giving the final result:

$$\begin{aligned}\begin{pmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{pmatrix} &= \frac{3}{3\sqrt{2}} \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \cos\theta + \frac{3}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \sin\theta + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{\sqrt{2}}\cos\theta + \frac{3}{\sqrt{3}}\sin\theta \\ -\frac{3}{\sqrt{3}}\sin\theta \\ -\frac{3}{\sqrt{2}}\cos\theta + \frac{3}{\sqrt{3}}\sin\theta \end{pmatrix}\end{aligned}$$

Example 7. Find equations for a circular motion of radius 2, centered at the origin, that rotates counterclockwise in the plane perpendicular to the vector $(2,1,2)$ when viewed from $(2,1,2)$. (Hint: $(2,1,2), (1,2,-2)$ and $(-2,2,1)$ are mutually perpendicular and right-hand oriented.

Solution:

Let's use similar techniques from the previous problem:

$$(1,2,-2) \times (-2,2,1) = (6,3,6).$$

Although this is not the vector given when the problem says "... when viewed from $(2,1,2)$", it is however a positive scalar multiple of that vector: $(6,3,6) = 3(2,1,2)$. So the perspective is the same. In other words, when we view the vectors $(1,2,-2)$ and $(-2,2,1)$ from the perspective of $(6,3,6)$ (or from $(2,1,2)$; it's the same perspective), then we get that $(1,2,-2)$ rotates toward $(-2,2,1)$ in a counterclockwise manner. Since the problem asks for a counterclockwise rotation from this perspective, we get $U = \frac{(1,2,-2)}{|(1,2,-2)|} = \frac{(1,2,-2)}{3}$ and $V = \frac{(-2,2,1)}{|(-2,2,1)|} = \frac{(-2,2,1)}{3}$. So,

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= \frac{2}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cos\theta + \frac{2}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \sin\theta + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3}\cos t - \frac{4}{3}\sin t \\ \frac{4}{3}\cos t + \frac{4}{3}\sin t \\ -\frac{4}{3}\cos t + \frac{2}{3}\sin t \end{pmatrix}.\end{aligned}$$

Important! If we change the problem a little, and require that the rotation is clockwise from the same given perspective, then we literally just switch the U and V that we found.

Review for sections 9.6, 9.7, and 9.8 are not included. Please review the main theorems and examples in these sections on your own.