Engineering Math II – Exam 2 (Quick) Review

Chapter 3 through Chapter 4, Section 2

3 Chapter 3 – Explicit Functions

3.1 Explicit Linear Functions

- **Definition.** Given a linear function (an equation of a plane) $f[x,y] = mx + ny + b$, the *gradient vector* of the plane is the vector $G = \begin{pmatrix} m \\ n \end{pmatrix}$.

  The gradient is a vector in the xy-plane. Both its *direction* and *magnitude* give us important pieces of information:

  – Direction: Given a fixed point on the plane (x_0, y_0, z_0), the gradient vector tells us how to change the x and y values of this fixed point in order to increase our z-value the fastest. So for example: Let's say we have the gradient $G = (3, -4)$ for a plane $z = f[x,y] = 3x - 4y + 6$ and we have the fixed point $(1,2,1)$ on the plane. The gradient tells us to change the x-value 1 by $+3$ and the y-value 2 by -4 in order to *increase* our z-value *the fastest*. In this case, if we moved from the point $(1,2,1)$ on the plane to the point $(1+3, 2-4, 26) = (4, -2, 26)$, we have increased our z value from 1 to 26 *the fastest*.
  If we look at $-G$, i.e. the negative of the gradient vector, this will give us the direction to *decrease* our z value the fastest.

  – Magnitude: So, the direction of the gradient tells us which way to go in order to increase z the fastest, and so a most natural question to ask is: If we do this, at *what rate* are we increasing our z? The magnitude (or length) of the gradient, $|G|$ tells us the *rate* at which we will change our z value if we go in the direction of the gradient.
  The number $-|G|$ tells us the rate at which we can decrease z the fastest, i.e., the rate at which we decrease z if we go in the direction $-G$. **Caution!!!** This is NOT $|-G|$, but instead $-|G|$. These are two very different things. Indeed, we know $|-G| = |G|$ (Prove this to yourself!), but on the other hand, $-|G| \neq |G|$. So $|-G| \neq -|G|$.

- If we move in a direction $X = (x, y)$ which is perpendicular to G, i.e., $X \cdot G = 0$, then we make no change in our z-value. If $G = (m, n)$, then the two perpendicular directions to G are

$$\begin{pmatrix} -n \\ m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n \\ -m \end{pmatrix}$$

- Local coordinate equation (i.e., (dx, dy, dz)-coordinates) of a plane with gradient $G = (m, n)$ that passes through a point (x_0, y_0, z_0):

$$dz = mdx + ndy \quad \text{where} \quad dx = x - x_0, dy = y - y_0 \quad \text{and} \quad dz = z - z_0.$$

  If we want this equation in (x,y,z)-coordinates, make the substitutions $dx = x - x_0, dy = y - y_0$, and $dz = z - z_0$:

$$z - z_0 = m(x - x_0) + n(y - y_0).$$

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Theorem 3.1.1. *Let $f[x, y] = mx + ny + b$ be a linear function with gradient $G = (m, n)$. Then the rate of change of z for a change of input ΔX is*

$$\frac{\textit{change in } z}{\textit{distance moved in } (x, y)\textit{-direction}} = \frac{G \cdot \Delta X}{|\Delta X|}.$$

The book gives some examples (and a proof!) as to why the change of z in the direction ΔX is in fact $G \cdot \Delta X$. This theorem is saying that the rate of change in z in the direction ΔX is the change in z divided by the length of ΔX.

This is analogous to when we learned about slope in College Algebra: The slope of a line between two points (x_1, y_1) and (x_2, y_2) is the change in y-value divided by the change in x-value:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

This can also be stated: The rate of change in y is

$$\frac{\text{change in y}}{\text{distance moved in } x\text{-direction}},$$

which looks similar to what our theorem says.

I'll refrain from an example here, because this becomes how we understand directional derivatives in §3.3.

3.3 The Derivative Approximation

- The big takeaway from this section is how to find the equation of a tangent plane, both in (dx, dy, dz)-coordinates and (x, y, z)-coordinates: Given a surface $f[x, y]$, the equation of any tangent plane is the "symbolic total differential":

$$dz = \frac{\partial f}{\partial x}[x, y]dx + \frac{\partial f}{\partial y}[x, y]dy.$$

The reason we say "of any tangent plane" is because at this step, we can find the equation of a tangent plane for any point we want. Since the partial derivatives of f are themselves functions of x and y, we can plug in a point (x_0, y_0) into the partials to find the equation of the tangent plane at this particular point. For example, let $f[x, y] = 3x^2 + xy^3$. Then the symbolic total differential for f is

$$\begin{aligned} dz &= \frac{\partial f}{\partial x}[x, y]\ dx + \frac{\partial f}{\partial y}[x, y]\ dy \\ &= (6x + y^3)dx + (3xy^2)dy. \end{aligned}$$

If we wanted to find the equation of the tangent plane at the point $(x_0, y_0) = (-1, 2)$, then we plug this point into the partial derivatives to get:

$$\begin{aligned} dz &= \frac{\partial f}{\partial x}[-1, 2]\ dx + \frac{\partial f}{\partial y}[-1, 2]\ dy \\ dz &= (6(-1) + (2)^3)dx + (3(-1)(2)^2)dy \\ dz &= 2dx - 12dy. \end{aligned}$$

Now, lets convert this to (x, y, z)- coordinates by substituting $dx = x - x_0, dy = y - y_0$, and $dz = z - z_0$. But first, we need to find z_0:

$$z_0 = f[x_0, y_0] = f[-1, 2] = -5.$$

So,

$$\begin{aligned} z - z_0 &= 2(x - x_0) - 12(y - y_0) \\ z + 5 &= 2(x + 1) - 12(y - 2) \\ z &= 2x - 12y + 21. \end{aligned}$$

- *Remark*: Notice that the tangent plane (in (dx, dy, dz)- coordinates or in (x, y, z)-coordinates) is a *linear* function, (i.e., the highest power on x and y is 1, and nothing like $\text{Cos}[xy]$, $\text{Log}[2x^2]$, or e^x is in the function). So if your answer for the equation of a tangent plane has higher powers than 1 or has a trigonometric, logarithmic, exponential, etc. expressions in it, (i.e., not of the form $dz = m dx + n dy$ or $z = mx + ny + b$), then something is wrong.

- Definition of *smoothness*: A function $f[x, y]$ is smooth at a point (x_0, y_0) if its partial derivatives are defined at (x_0, y_0). So for example, the function $f[x, y] = \sqrt{x^2 + y^2}$ is NOT smooth at $(0, 0)$. We have

$$\frac{\partial f}{\partial x}[x, y] = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial f}{\partial y}[x, y] = \frac{y}{\sqrt{x^2 + y^2}}.$$

But the point $(0, 0)$ is not defined for these partial derivatives. Hence, $f[x, y]$ is not smooth at $(0, 0)$.

3.4 Differentiation Skills

- The title of this section says it all: You need to know how to take derivatives and partial derivatives. There are plenty of worksheets online you can do to quiz yourself.

3.5 Directional Derivatives

- In §3.1, we defined the gradient vector of a plane. Given *any* surface described by a function $f[x, y]$ (so now maybe the function has higher powers than 1 and maybe something like Sin[x] in it), we learned in §3.3 how to find an equation for the tangent plane at a point for this surface. Let's put these ideas together:

  Definition. The gradient of a smooth nonlinear function $f[x, y]$ at a point $X = (x_0, y_0)$ is the 2-vector

  $$\nabla f[X] = \begin{pmatrix} \dfrac{\partial f}{\partial x}[x_0, y_0] \\ \\ \dfrac{\partial f}{\partial y}[x_0, y_0] \end{pmatrix}$$

  Basically, the gradient of a function at a point is the gradient of the tangent plane at that point. In our §3.3 example, we found the equation of the tangent plane at the point $(1, -2)$ by first taking partial derivatives and then we evaluated the partial derivatives at $(1, -2)$. When we did this, we obtained numbers that became the coefficients on dx and dy. And even when we converted to (x, y, z)-coordinates, these numbers remained the coefficients on x and y. So when we take partials and evaluate it at a particular point, we are just looking at the coefficients of x and y in our equation of a plane. But that's exactly how we first learned the gradient in §3.1! So this definition makes sense.

- Now that we have a gradient of a general function $f[x, y]$, we would like to use our knowledge about the gradient and "rates of change" to talk about *directional derivatives*:

  Definition. Let $f[x, y]$ be a smooth function. The directional derivative of f at a point (x_0, y_0) in the direction $W = (a, b)$ is

  $$D_W f[x_0, y_0] = \frac{\nabla f[x_0, y_0] \cdot W}{|W|}.$$

  The directional derivative at the point (x_0, y_0) in the direction of W tells us the rate at which we change our z value if we start at the point (x_0, y_0, z_0) and then move in the direction W. So

  $$\text{rate of change in } z = \frac{\text{change in } z}{\text{distance moved}} = \frac{\nabla f[x_0, y_0] \cdot W}{|W|}$$

  Remark: On our notation: $\nabla f[x_0, y_0]$ means that we find the gradient of f FIRST, and THEN plug in the point (x_0, y_0). If you plug in the point (x_0, y_0) into f first, you will get a number. So then if you take the partial derivatives next, you will always get 0. This is wrong! Instead, be sure to take partial derivatives first, then plug in the point.

- Example: Let $f(x,y) = 2x^4 + \cos(xy^2)$. Find the directional derivative of f at the point $(1, \frac{\pi}{2})$ in the direction of the vector $Y = (1,2)$.

  Solution: First, let's find the gradient of f:

$$\nabla f[x,y] = \begin{pmatrix} \dfrac{\partial f}{\partial x}[x,y] \\ \\ \dfrac{\partial f}{\partial y}[x,y] \end{pmatrix} = \begin{pmatrix} 8x^3 - y^2\sin(xy^2) \\ \\ -2xy\sin(xy^2) \end{pmatrix}.$$

  Now, lets find the gradient of f at the given point $(1, \frac{\pi}{2})$:

$$\nabla f\left[1, \frac{\pi}{2}\right] = \begin{pmatrix} 8(1)^3 - (\frac{\pi}{2})^2\sin((1)(\frac{\pi}{2})^2) \\ \\ -2(1)(\frac{\pi}{2})\sin((1)(\frac{\pi}{2})^2) \end{pmatrix} = \begin{pmatrix} 8 - \frac{\pi^2}{4}\sin(\frac{\pi}{4}) \\ \\ -\pi\sin(\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} 8 - \frac{\pi^2}{4}\left(\frac{\sqrt{2}}{2}\right) \\ \\ -\pi\left(\frac{\sqrt{2}}{2}\right) \end{pmatrix}.$$

  Now we can compute the directional derivative of f at the point $(1, \frac{\pi}{2})$ in the direction of Y:

$$D_Y f[1, \frac{\pi}{2}] = \frac{\nabla f[1, \frac{\pi}{2}] \cdot Y}{|Y|} = \frac{\begin{pmatrix} 8 - \frac{\pi^2\sqrt{2}}{8} \\ \\ -\frac{\pi\sqrt{2}}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right|} = \frac{8 - \frac{\pi^2\sqrt{2}}{8} - \pi\sqrt{2}}{\sqrt{5}}.$$

  That was a nasty example.

- Just like in §3.1, we have statements about "rates of change" in z using the gradient: Let $f[x,y]$ be a smooth function with nonzero gradient.

  - If we are at the point (x_0, y_0, z_0) on the surface, the fastest way to increase the z value is to go in the direction of the gradient at this point: $\nabla f[x_0, y_0]$. The largest instantaneous rate of increase in $z = f[x,y]$ at the point (x_0, y_0) is the norm of the gradient at this point: $|\nabla f[x_0, y_0]|$.
  - If we are at the point (x_0, y_0, z_0) on the surface, the fastest way to decrease the z value is to go in the opposite direction of the gradient at this point: $-\nabla f[x_0, y_0]$. The largest instantaneous rate of decrease in $z = f[x,y]$ at the point (x_0, y_0) is the negative of the norm of the gradient at this point: $-|\nabla f[x_0, y_0]|$.
  - The instantaneous rate of change in $z = f[x,y]$ at the point (x_0, y_0) is zero if we go in a direction perpendicular to the gradient at (x_0, y_0), $\nabla f[x_0, y_0]$. The two perpendicular directions are

$$\begin{pmatrix} -\dfrac{\partial f}{\partial y}[x_0, y_0] \\ \\ \dfrac{\partial f}{\partial x}[x_0, y_0] \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dfrac{\partial f}{\partial y}[x_0, y_0] \\ \\ -\dfrac{\partial f}{\partial x}[x_0, y_0] \end{pmatrix}.$$

3.7 Approximation by Differentials

- This is the section about errors and relative errors: Here are the main ideas:
  - The error is "actual − approximate". For a variable t (lets say t is time) the error in t is $dt = t - t_0$, where t is the actual value and t_0 is the approximate value. This can also be interpreted as t =new value, and t_0 =old value in a particular problem.
  - The *relative* error is $\frac{\text{actual}-\text{approximate}}{\text{actual}}$. So our formula becomes $\frac{dt}{t} = \frac{t-t_0}{t}$. An example: $t = 58.6, t_0 = 60$.

$$\text{error in time} = dt = t - t_0 = 58.6 - 60 = -1.4$$
$$\text{relative error in time} = \frac{dt}{t} = \frac{t-t_0}{t} = \frac{58.6-60}{58.6} \approx -.02 = -2\%.$$

    Notice how relative error gives a *percentage* because we are comparing our error with the actual, giving a ratio, which can be thought of as a percentage. The relative error tells us what our error is *relative to* the actual amount.
  - Given the formula $D = R\frac{t}{3600}$, where D is distance, t is time and R is rate, we can use the differential to look at errors and relative errors:

$$\text{error in rate} = dR = \frac{3600}{t}dD - \frac{3600D}{t^2}dt$$
$$\text{relative error in rate} = \frac{dR}{R} = \frac{dD}{D} - \frac{dt}{t}$$

    I skipped computational steps here (do the computations on your own!), but we know that for the error: the coefficient on dD is the partial derivative of R with respect to D, and the coefficient on dt is the partial derivative of R with respect to t. These formulas each tell us something:
    - The error in rate depends on the errors in D and t. So if we were given t, t_0, D, D_0, we could plug in these numbers into the first formula to find the error in rate dR.
    - Similarly, if we were given the *relative error* in time and distance, dt/t and dD/D, we could see how that effects the relative error in rate, dR/R. If I have a 2% relative error in time — maybe your stopwatch timing is off by 2% — then that would correspond to a −2% relative error in speed (i.e. rate)
    - If our formula was something like $dL/L = -2dQ/Q + 3dW/W$, then a 5% relative error in W would correspond to a +15% relative error in L. Similarly, if we had a 4.3% relative error in Q, that would correspond to a −8.6 relative error in L.
  - Be aware: By abuse of language, we sometimes say "error" when we really mean "relative error". Just be careful if you come across this. Let the context of the problem tell you what is meant.

4 Implicit Curves, Surfaces, and Contour Plots

4.1 Implicit Linear Functions

- The contour graph of a linear function $z = mx + ny + b$ looks like a bunch of parallel lines. These parallel lines are all perpendicular to the gradient vector. Each line corresponds to a particular choice of z-value, and the distance between two levels $z = h$ and $z = k$ is given by the formula

$$\frac{|k - h|}{|G|}.$$

- If we write the equation of our line in the form $G \cdot (x, y) = k$, i.e. $(m, n) \cdot (x, y) = k$ or $mx + ny = k$, then this is a line which is perpendicular to the gradient and is a directed distance $k/|G|$ from the origin along G.

4.2 Contour Graphs, Gradients, and Tangents

- The main part of this section is how to find the equation of a tangent *line* given a function $f[x, y]$. This is almost the same as the method for finding the equation of the *tangent plane*, except now we just have $dz = 0$ since we are interested in a line for a particular z-value.

- The general total differential of an equation $f[x, y] = c$ for a fixed number c is

$$0 = \frac{\partial f}{\partial x}[x, y]\ dx + \frac{\partial f}{\partial y}[x, y]\ dy.$$

As before, this is the *general* equation of a tangent line since at this step, we can plug in any point (x_0, y_0) to find the equation of a tangent line at that particular point. Let's use the same example from earlier: $f[x, y] = 3x^2 + xy^3$. The general equation of the tangent line is

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x}[x, y]\ dx + \frac{\partial f}{\partial y}[x, y]\ dy \\ 0 &= (6x + y^3)dx + (3xy^2)dy. \end{aligned}$$

So at the point $(-1, 2)$, the equation of the tangent line is

$$0 = 2dx - 12dy.$$

Converting now to (x, y, z)-coordinates,

$$\begin{aligned} 0 &= 2(x - x_0) - 12(y - y_0) \\ 0 &= 2(x + 1) - 12(y - 2) \\ 0 &= 2x - 12y + 26. \end{aligned}$$