Engineering Math II – Exam 1 (Quick) Review

Chapters 1 & 2

1 Chapter 1 – Basic Graphs

1.1 Explicit

- The equation is given *explicitly* in terms of y. In other words, y is by itself on one side, and some expression involving x is on the other side. We are familiar with this type of equation from previous math courses.

Examples

- $y = x^2$, $y = 3x+17$, $y = -4x^2+13x+95$, $y = x^{2017}+14x^{2011}+130x^{96}-100x^2-15$.

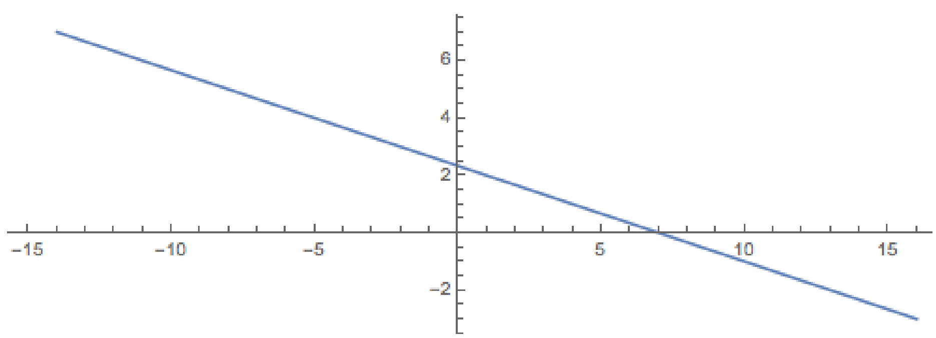
1.2 Implicit

- The equation is given with x and y on the same side, and a number on the other side. Sure, maybe we could solve for y and make it an explicit equation, but sometimes we can't solve for y so we leave it in this form.

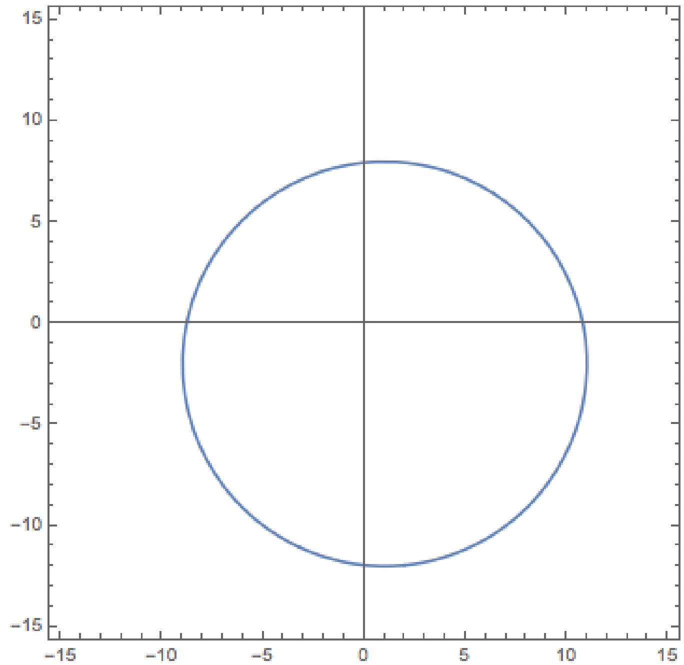
- When graphing, we can just pick values of x and solve for y to determine what y is (the same way we are use to with explicit equations).

Examples

- $2x - 3y = 6 \rightarrow$ Line that is perpendicular to the line from $(0, 0)$ to $(2, -3)$ and that is "pushed away" from the origin in the direction of $(2, -3)$ by a distance of $\frac{6}{\sqrt{(2)^2+(-3)^2}} = \frac{6}{\sqrt{13}}$.



- $(x-1)^2 + (y+2)^2 = 100 \to$ Circle centered at $(1, -2)$ of radius 10.



1.3 Parametric Equations

- We are use to explicit graphs which have one "input" variable x, and then the "output" variable y is determined by whatever x is. (That's another way to say that x is the "independent" variable because we can "input anything" so it's not dependent on anything; and y is the dependent variable because y is *dependent* on x). In this way, the x-axis as being the"input axis" and the y-axis as being the "output axis".

- But now, for *parametric* equations, we have the x and y values *both dependent* on another variable, t. We don't graph any t values; we still only graph x and y values. But this time, the x and y are both dependent on t and so both x and y are now dependent variables. Now, we view both the x-axis and y-axis are "output axes".

- When graphing, we pick (easy) values for t to find points (x, y) in the graph.

Examples

- $\begin{cases} x &= 3t+1 \\ y &= -t+2 \end{cases}$. $\to$ Line passing through $(1, 2)$ in the direction of the vector $(3, -1)$.

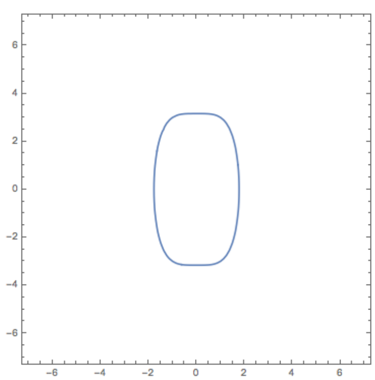
- $\begin{cases} x &= \cos[t] \\ y &= \sin[t] \end{cases}$. $\to$ Circle of radius 1.

1.4 Sliding & Squashing

- **Sliding**: If we want to slide (i.e. translate) a graph/equation by a in the x direction and by b in the y direction, we *replace* x with $(x - a)$ and *replace* y with $(y - b)$.
- **Stretching/Squashing:** Same idea here, but now we just "replace" by something different. If we want to stretch (i.e. expand) by a in the x direction, we *replace* x by $\left(\frac{x}{a}\right)$. If we want to squash (i.e. shrink or compress) by b in the x direction, we *replace* x by $\left(\frac{x}{\frac{1}{b}}\right) = (bx)$. Similarly, if we want to stretch by a in the y direction, we *replace* y by $\left(\frac{y}{a}\right)$. If we want to squash by b in the y direction, we *replace* y by $\left(\frac{y}{\frac{1}{b}}\right) = (by)$.

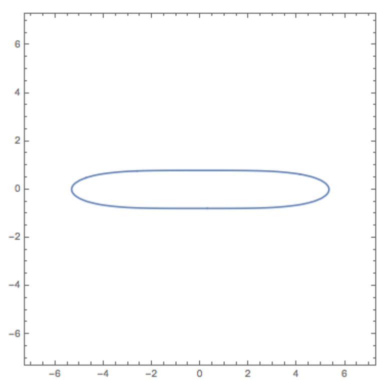
Examples

- If the equation is $x^2 + y^4 = 17$, and we want to slide it by -2 in the x direction and slide by 5 in the y direction, we get $(x+2)^2 + (y-5)^4 = 17$.
- If the equation is $x^4 + y^2 = 10$, and we want to stretch in the x direction by 3 and squash in the y direction by -4 then we get $\left(\frac{x}{3}\right)^4 + (-4y)^2 = 10$.



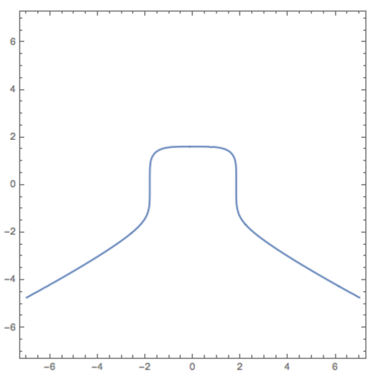
$x^4 + y^2 = 10$

$\longrightarrow$

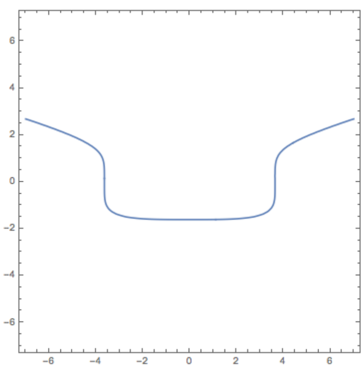


$\left(\frac{x}{3}\right)^4 + (-4y)^2 = 10$

Remark: In the last example, squashing by -4 in the y direction flips the graph *across the x-axis* since -4 is negative. We can't see the flip in the picture above, since the graph is symmetric about the x axis, so here's another example that's more clear: We stretch by 2 in the x direction, and "squash" by -1 in the y direction (Stretching/squashing by 1 or -1 really means that we didn't actually squash or stretch the graph at all!).



$\longrightarrow$



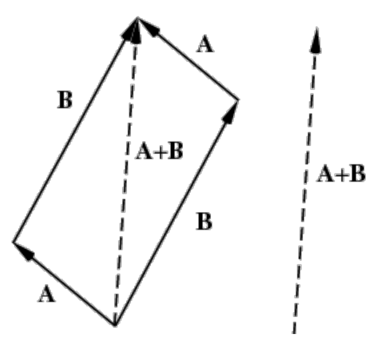
$x^4 + y^5 = 10$ $\qquad$ $\left(\frac{x}{2}\right)^4 + (-y)^5 = 10$

If we were to stretch/squash in the x direction by a negative number, we would reflect the graph *across the y-axis.*

2 Chapter 2 – Vectors

2.2 Sums

- We add (and subtract) two vectors exactly the way we think we should. Just add the like components of the vectors. Geometrically, when we add two vectors A and B with the same tail, we obtain the vector $A + B$ that is the diagonal of the parallelogram created by A and B.



Example

$$\begin{pmatrix} 1 \\ -10 \\ 14 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 13 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -10 \\ 14 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -15 \\ 15 \end{pmatrix}.$$

2.3 Displacement

- The displacement vector from X to Y is given by $Y - X$. It is the vector that has its tail at X, and its tip at Y.

Examples

- Let $X(1, 5, -1)$ and $Y(4, -3, 1)$ be vectors.

$$\text{Displacement vector from } X \text{ to } Y\text{:} \quad Y - X = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 2 \end{pmatrix}.$$

$$\text{Displacement vector from } Y \text{ to } X\text{:} \quad X - Y = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -2 \end{pmatrix}.$$

Remark: Notice that $X - Y$ is just the negative of $Y - X$, and vice versa. Algebraically,

$$-(X - Y) = -X + Y = Y - X \quad \text{and} \quad -(Y - X) = -Y + X = X - Y.$$

2.4 Scalar Products

- If c is a number and X is a vector, to compute cX, we just multiply each component of X by c. Notice that we do NOT write $c \cdot X$, to make sure not to confuse the scalar product (of a number and a vector) with the dot product (of two vectors).
- When we multiply a vector by a number bigger than 1, we stretch the vector to make it longer. If we multiply a vector by a number between 0 and 1, we shrink it.
- If we multiply a vector by a negative number that is less than -1, then we make the vector longer AND flip its direction. If we multiply a vector by a negative number that is between -1 and 0, then we shrink the vector AND flip its direction.

Example

- Let $X(1, 2, -1)$ and $c = -3$. Then $cX = (-3)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix}$.

2.5 Angles & Projections

- **Dot Product:** The dot product is one way to "multiply" vectors and get a number. To calculate the dot product, we add the products of like coordinates. In other words, if $X = (a, b, c)$ and $Y = (d, e, f)$ then their dot product is

$$X \cdot Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf.$$

  The dot product has the following properties:

  1. We can do the dot product in any order: $X \cdot Y = Y \cdot X$.

2. The dot product distributes in the same way that we're use to with numbers: $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$.
3. If we multiply the dot product of two vectors by a number c, that's the same as scaling the first vector by c, and its also that same as scaling the second vector by c: $c(X \cdot Y) = (cX) \cdot Y = X \cdot (cY)$.

- **Example** Let $X = (1, 2, 3)$ and $Y = (4, 5, 6)$. Then

$$X \cdot Y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 4 + 10 + 18 = 32.$$

- **Length:** The length (or "magnitude", or "norm") of a vector $X = (a, b, c)$ is given by the formula

$$|X| = \sqrt{a^2 + b^2 + c^2}.$$

- Notice that formula for the length of a vector $X = (a, b, c)$ is closely related to the dot product. Since $X \cdot X = a^2 + b^2 + c^2$, then

$$|X| = \sqrt{a^2 + b^2 + c^2} = \sqrt{X \cdot X}.$$

  Squaring both sides, we get $|X|^2 = X \cdot X$.

- **Example** Let $X = (-1, 3, 5)$. Then

$$|X| = \sqrt{(-1)^2 + (3)^2 + (5)^2} = \sqrt{1 + 9 + 25} = \sqrt{30}.$$

- **Angle Between Vectors:** We have a formula for the angle between two vectors X and Y. The angle θ between X and Y is the number between 0 and π which satisfies the equation

$$\cos[\theta] = \frac{X \cdot Y}{|X||Y|}.$$

  Remark: It's important to note here that the vectors X and Y *must have their tails in the same place* for this formula to work.

- If the dot product of X and Y is equal to 0, then X and Y are perpendicular. It's true the other way, too: If X and Y are perpendicular, then the dot product of X and Y is equal to 0. We can prove this fact by using our formula:

$$\cos[\theta] = \frac{X \cdot Y}{|X||Y|} = \frac{0}{|X||Y|} = 0.$$

  When is $\cos[\theta] = 0$? Well, when $\theta = \pi/2$, which is $90°$.

- **Direction Cosines & Direction Angles:** The *direction cosines* of a vector X are just the components of the vector $\frac{X}{|X|}$. Once we have this vector, the *direction angles* of X are found by setting $\cos[\theta]$ equal to each of the direction cosines.
  Remark: For any vector X, the vector $\frac{X}{|X|}$ is the vector of length 1 (i.e. unit vector) that points in the same direction as X.

- **Examples** Let $X = (1, 2, 3)$. Then

$$\frac{X}{|X|} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \frac{(1,2,3)}{\sqrt{14}} = \begin{pmatrix} \dfrac{1}{\sqrt{14}} \\ \dfrac{2}{\sqrt{14}} \\ \dfrac{3}{\sqrt{14}} \end{pmatrix}.$$

So, the direction cosines of X are the numbers $1/\sqrt{14}, 2/\sqrt{14}$, and $3/\sqrt{14}$. The direction angles for X are the numbers

$$\alpha_1 = \arccos\left(\frac{1}{\sqrt{14}}\right),$$
$$\alpha_2 = \arccos\left(\frac{2}{\sqrt{14}}\right),$$
$$\alpha_3 = \arccos\left(\frac{3}{\sqrt{14}}\right).$$

- Let $X = (\sqrt{3}, 1)$. Then

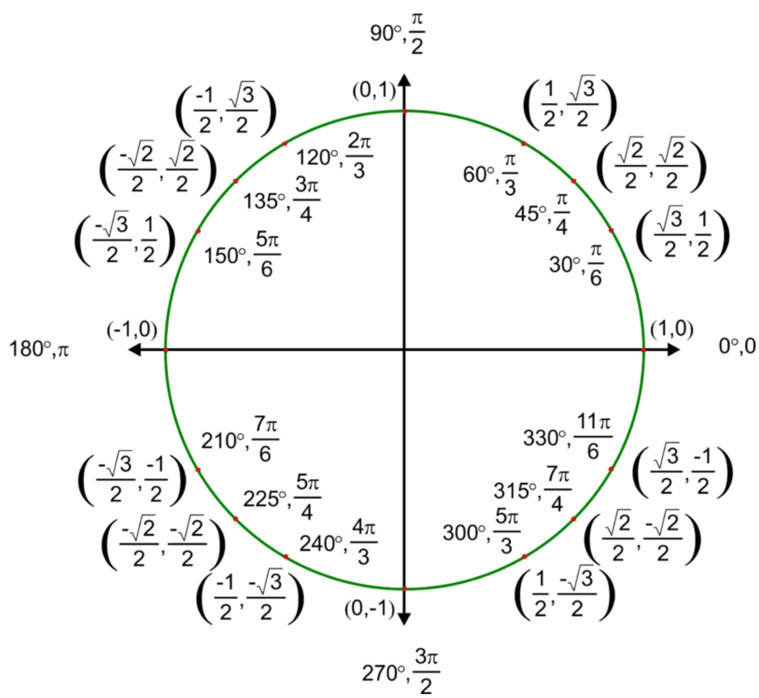
$$\frac{X}{|X|} = \frac{(\sqrt{3},1)}{\sqrt{(\sqrt{3})^2+1^2}} = \frac{(\sqrt{3},1)}{\sqrt{4}} = \begin{pmatrix} \dfrac{\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix}$$

So, the direction cosines of X are the numbers $\sqrt{3}/2$ and $1/2$. The direction angles for X are the numbers θ_1 and θ_2 satisfying the equations

$$\cos[\theta_1] = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos[\theta_2] = \frac{1}{2}.$$

So $\theta_1 = \pi/6$ and $\theta_2 = \pi/3$.

As we can see, it's important that we remember the unit circle. Here it is for reference:



- **Perpendicular Projection of Vectors:** The perpendicular projection of the vector X onto another vector Y is the vector

$$Z = \left(\frac{X \cdot Y}{Y \cdot Y}\right) Y.$$

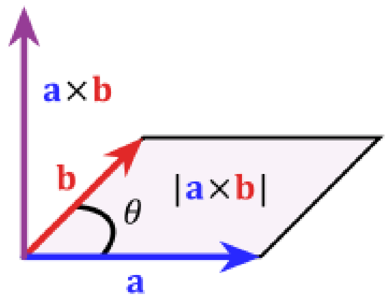
The vector Z just a scalar multiple of the vector Y. The displacement from Z to X, $X - Z$, is perpendicular to Y. The vector X is the sum of Z and the displacement vector from Z to X: $X = Z + (X - Z)$.

- **Example** The perpendicular projection of $A = (4, -2, 3)$ onto $B = (3, 1, 2)$ is the vector

$$\left(\frac{A \cdot B}{B \cdot B}\right) B = \frac{12 - 2 + 6}{9 + 1 + 4} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \frac{8}{7} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{24}{7} \\ \frac{8}{7} \\ \frac{16}{7} \end{pmatrix}.$$

2.6 Cross Product:

- The cross product of two vectors a and b is another way to "multiply" vectors to get a new vector, $a \times b$. The vector $a \times b$ is perpendicular to a and is also perpendicular to b. A neat property of the cross product is that the length of the cross product of a and b, $|a \times b|$, is the area of the parallelogram with sides a and b.



If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, then the cross product of a and b is

$$\begin{aligned} a \times b &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= i(a_2b_3 - a_3b_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1) \\ &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \end{pmatrix}, \end{aligned}$$

where $i = (1, 0, 0)$, $j = (0, 1, 0)$, and $k = (0, 0, 1)$.

Remark: It matters in which order we take the cross product! In other words, $X \times Y \neq Y \times X$. However it IS true that $X \times Y = -(Y \times X)$. So if we happen to take the cross product the wrong way, we can just change the signs on all the components of the vector to get the right answer.

- We have the "$BAC - CAB$" rule which relates the cross product and the dot product:

$$A \times (B \times C) = (C \cdot A)B - (B \cdot A)C$$

- **Example** If $X = (2, -3, 5)$ and $Y = (1, -1, 1)$, then

$$\begin{aligned} X \times Y = \begin{vmatrix} i & j & k \\ 2 & -3 & 5 \\ 1 & -1 & 1 \end{vmatrix} &= i(-3 - (-5)) - j(2 - 5) + k(-2 - (-3)) \\ &= 2i + 3j + k = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}. \end{aligned}$$

So, we know that $Y \times X = -(X \times Y) = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$. We also know that the area of the parallelogram with sides X and Y is $|X \times Y| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$. We can also find the area of this same parallelogram by $|Y \times X| = \sqrt{(-2)^2 + (-3)^2 + (-1)^2} = \sqrt{14}$. Additionally, if we wanted to find the area of the *triangle* with sides X, Y, and $Y - X$, then we can just divide the area of the parallelogram by 2.