Exam 1 Review
Ordinary Differential Equations

1 Intro to ordinary differential equations

1.4 Solutions to $\dot{x} = A(t)x$ and the transition matrix

- $x(t, t_0, x_0) = R(t, t_0)x_0$ where

$$R(t, t_0) = \left[\begin{array}{c|c|c} x(t, t_0, e_1) & \cdots & x(t, t_0, e_n) \end{array}\right]$$

  and the e_i are the standard basis vectors for $\mathbb{R}^n$.

  - So, $D_{x_0}x(t, t_0, x_0) = R(t, t_0)$.
  - If $A(t)$ is actually autonomous, then $R(t, t_0) = R(t - t_0, 0)$.

1.4.1 The Abel-Jacobi - blah blah blah formula

- **Theorem 1.27** (Abel-Jacobi-blah blah blah formula) *If $A(t) \in \mathbb{R}^{n\times n}$ has continuous trace on an interval I and if $W(t) \in \mathbb{R}^{n\times n}$ defined on I satisfies $\dot{W}(t) = A(t)W(t)$, then*

$$\det(W(t)) = \det(W(t_0))e^{\int_{t_0}^t tr(A(s))ds} \text{ for all } t_0, t \in I.$$

  If, moreover, $A(t)$ is is continuous on I, then the transition matrix $R(t, t_0)$ of the system $\dot{x} = A(t)x$ satisfies the formula

$$\det(R(t, t_0)) = e^{\int_{t_0}^t tr(A(s))ds}.$$

1.4.2 Solutions to $\dot{x} = Ax$ and the exponential of a matrix

- Some exponential matrix formulas for common matrices:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e^{\alpha t J} = \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e^{\alpha t J} = \frac{1}{2}\begin{bmatrix} e^{-\alpha t} + e^{\alpha t} & e^{\alpha t} - e^{-\alpha t} \\ e^{\alpha t} - e^{-\alpha t} & e^{-\alpha t} + e^{\alpha t} \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{\alpha t J} = \begin{bmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e^{\alpha t J} = \begin{bmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{bmatrix}$$

$$J = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}, \quad e^{t J} = \begin{bmatrix} e^{\alpha t} & te^{\alpha t} \\ 0 & e^{\alpha t} \end{bmatrix}.$$

- One way to compute this is to get the matrix A into Jordan canonical form $A = PJP^{-1}$, where P has as its columns the eigenvectors for the eigenvalues of A, written in the order that their corresponding eigenvalues are written in J. If A has distinct eigenvalues λ_1, λ_2, then J is diagonal, so

$$e^{tA} = e^{t(PJP^{-1})} = Pe^{tJ}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}P^{-1}.$$

  If A has a repeated eigenvalue λ then J is upper triangular, so

$$e^{tA} = e^{t(P^{-1}JP)} = Pe^{tJ}P^{-1} = P\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}P^{-1}.$$

- The theorem below says that for IVPs, we calculate the matrix exponential $e^{(t-t_0)A}$ as opposed to e^{tA}; so in all the computations above, just replace t with $(t-t_0)$.

- **Theorem 1.32** *The solution to the IVP $\dot{x} = Ax$, $x(t_0) = x_0$ is*

$$x(t, t_0, x_0) = e^{(t-t_0)A} x_0.$$

So the transition matrix for the system is $R(t, t_0) = e^{(t-t_0)A}$.

1.5 Solutions to $\dot{x} = A(t)x + b(t)$

- **Theorem 1.36** *Consider a system of linear, nonhomogeneous, and nonautonomous ODEs in $\mathbb{R}^n$, $\dot{x} = A(t)x + b(t)$, where $A(t) \in \mathbb{R}^{n\times n}$ and $b(t) \in \mathbb{R}^n$ are both continuous on an interval I. Let $R(t, s)$ be the transition matrix of the system $\dot{y} = A(t)y$. Then*

  1. *The general solution to the system is*

$$\underbrace{x(t, c, v_c)}_{\substack{\text{genral solution to} \\ \dot{x}=A(t)x+b(t)}} = \underbrace{R(t, c)v_c}_{\substack{\text{general solution to} \\ \dot{y}=A(t)y}} + \underbrace{\int_c^t R(t, s)b(s)ds}_{\substack{\text{a particular solution to} \\ \dot{x}=A(t)x+b(t) \text{ satisfying } x(c)=0}} .$$

  where $c \in I$ and $v_c \in \mathbb{R}^n$ are free.

  2. *For $t_0 \in I$ the solution satisfying the initial condition $x(t_0) = x_0$ is*

$$x(t, t_0, x_0) = R(t, t_0)x_0 + \int_{t_0}^t R(t, s)b(s)ds.$$

1.6 Differentiability of solutions with respect to initial conditions

- **Theorem 1.39** *Consider $\dot{x} = f(t, x)$. If both f and $D_x f$ are continuous, then $x(t, t_0, x_0)$ is differentiable with respect to t_0 and x_0. Moreover, the matrix*

$$R(t, t_0, x_0) := D_{x_0}(t, t_0, x_0)$$

is the solution to the matrix variational equation $\dot{R} = A(t, t_0, x_0)R$ with initial conditions $R(t_0, t_0, x_0) = I_n$.

- **Theorem 1.41** *Consider $\dot{x} = f(t, x)$. Suppose both f and $D_x f$ are continuous on an open set $U \subseteq \mathbb{R}^n$, and let $X_{t_0} \subset U$ be a set of initial conditions at t_0 and*

$$X_t := x(t, t_0, X_{t_0}) = \{x(t, t_0, x_0) \mid x_0 \in X_{t_0}\} \subset U,$$

then

  1. $\mathrm{Vol}(X_t) = \int_{X_{t_0}} e^{\int_{t_0}^t tr(D_x f(s, x(s,t_0,x_0)))ds} dx_0$
  2. *If $tr(D_x f(s, x(s, t_0, x_0))) \equiv 0$, then $Vol(X_t) = Vol(X_{t_0})$.*

1.7 Lyapunov stability

1.7.3 Lyapunov stability for $\dot{x} = Ax$

- **Theorem 1.53** *Consider the IVP $\dot{x} = Ax$, $x(t_0) = x_0$. Let $\{\lambda_i\}$ be the eigenvalues of A. Then the solution $x(t, t_0, x_0) = e^{(t-t_0)A}x_0$ to the IVP is:*

  1. *Lyapunov stable at (t_0, x_0) if and only if*
     (a) $\text{Re}(\lambda_i) \leq 0$ *for all i.*
     (b) *If $\text{Re}(\lambda_i) = 0$ for some i, then the geometric and algebraic multiplicities of λ_i are equal: $g(\lambda_i) = a(\lambda_i)$ (equivalently, if λ_i is a multiple eigenvalue with $g(\lambda_i) < a(\lambda_i)$, then* $\text{Re}(\lambda_i) < 0$*).*
  2. *asymptotically Lyapunov stable at (t_0, x_0) if and only if* $\text{Re}(\lambda_i) < 0$ *for all i. In this case the matrix A is said to be Hurwitz.*

- For $n = 2$, let $\delta = \det(A)$ and $\tau = \text{tr}(A)$. Then the solution $x(t, t_0, x_0) = e^{(t-t_0)A}x_0$ to the IVP is:
  - Lyapunov stable at (t_0, x_0) if and only if $\delta \geq 0$ and $\tau \leq 0$ with $\delta = \tau = 0$ only for the zero matrix.
  - asymptotically Lyapunov stable at (t_0, x_0) if and only if $\delta > 0$ and $\tau < 0$.
  - Lyapunov unstable at (t_0, x_0) otherwise.

1.7.4 Lyapunov stability for $\dot{x} = A(t)x$

- **Lemma 1.61**(Logarithmic norms) *Let $A \in \mathbb{R}^{n \times n}$. Then*

$$\mu_1(A) = \max_{1 \leq j \leq n} \left(a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \right)$$

$$\mu_\infty(A) = \max_{1 \leq i \leq n} \left(a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right)$$

$$\mu_2(A) = \max_{1 \leq i \leq n} \lambda_i(S(A)) \quad \text{where} \quad S(A) = \frac{1}{2}(A + A^T).$$

- **Theorem 1.65** *Consider a logarithmic norm $\mu(\cdot)$ and a system $\dot{x} = A(t)x$ with $A(t) \in \mathbb{R}^{n \times n}$ continuous on $[t_0, \infty)$. Suppose $\mu(A(t)) \leq \alpha$ for all $t \in [t_0, \infty)$.*
  - *If $\alpha \leq 0$, then $x(t, t_0, x_0)$ is Lyapunov stable at (t_0, x_0).*
  - *If $\alpha < 0$, then $x(t, t_0, x_0)$ is asymptotically Lyapunov stable at (t_0, x_0).*

1.7.5 Lyapunov stability for $\dot{x} = A(t)x + b(t)$

- **Theorem 1.68** *Consider a system $\dot{x} = A(t)x + b(t)$, with $A(t) \in \mathbb{R}^{n \times n}$ and $b(t) \in \mathbb{R}^n$, both continuous on $[t_0, \infty)$. Then the Lyapunov stability of the solution $x(t, t_0, x_0)$ at (t_0, x_0) is the same as the stability of the solution $y(t, t_0, 0) \equiv 0$ to the system $\dot{y} = A(t)y$ at $(t_0, 0)$ (whether Lyapunov stable, asymptotically stable, or unstable).*
  - So basically, the stability of the system $\dot{x} = A(t)x + b(t)$ is the same as the stability of the system $\dot{x} = A(t)x$.

1.7.6 Lyapunov stability for $\dot{x} = f(t, x)$

- **Lemma 1.70** *Suppose $\mu_2(D_x f(t, x)) \leq \alpha$ for all $t \in [t_0, \infty)$.*
  - *If $\alpha \leq 0$, then $x(t, t_0, x_0)$ is Lyapunov stable at (t_0, x_0).*
  - *If $\alpha < 0$, then $x(t, t_0, x_0)$ is asymptotically Lyapunov stable at (t_0, x_0).*

2 Introduction to continuous dynamical systems

2.2 The flow

- **Definition 2.2** The **flow** of $\dot{x} = f(x)$ is a family of maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ given by $x_0 \mapsto \varphi_t(x_0) := x(t, 0, x_0)$ parametrized by t. For a fixed t, φ_t is called the **flow map** at t.

2.4 Fixed points

- **Definition 2.10** x^* is a **fixed point** of $\dot{x} = f(x)$ if $f(x^*) = 0$.
- **Definition 2.12** Let x^* be a fixed point of $\dot{x} = f(x)$ and suppose f is differentiable at x^*. The **linearized system** associated to $\dot{x} = f(x)$ is $\dot{y} = Df(x^*)(y - x^*)$.
- **Definition 2.14** Let x^* be a fixed point of $\dot{x} = f(x)$.
  - x^* is a **sink** if $x(t) = x^*$ is an asymptotically Lyapunov stable solution to $\dot{x} = f(x)$.
  - x^* is a **source** if $x(t) = x^*$ is a sink for $\dot{y} = -f(y)$.
  - If f is differentiable at x^*, then x^* is a **saddle** if there exists distinct eigenvalues λ_j and λ_k of the matrix $Df(x^*)$ such that $\mathrm{Re}(\lambda_j) < 0 < \mathrm{Re}(\lambda_k)$.
  - x^* is a **center** if there exists a neighborhood U of x^* such that the solutions to the system $\dot{x} = f(x)$ in U remain in U and are periodic, except the fixed point solution $x(t) = x^*$.
- **Theorem 2.15** *Let x^* be a source fixed point of $\dot{x} = f(x)$. Then $x(t) = x^*$ is Lyapunov unstable.*
- **Definition 2.16** Let x^* be a fixed point of $\dot{x} = f(x)$ and suppose f is differentiable at x^*.
  - x^* is a **hyperbolic** fixed point if $\mathrm{Re}(\lambda_i(D(f(x^*))) \neq 0$ for all i.
  - x^* is a **nonhyperbolic** fixed point if there exists j such that $\mathrm{Re}(\lambda_j(D(f(x^*))) = 0$.

2.4.1 The Hartman-Grobman Theorem

- **Theorem 2.17** (Hartman-Grobman) *Let x^* be a hyperbolic fixed point of $\dot{x} = f(x)$ where f is $\mathcal{C}^1$ in a neighborhood of x^*. Let φ_t be the local flow of $\dot{x} = f(x)$ and let ψ_t be the local flow of the associated linearized system. Then there exists a homeomorphism h of a ball $B = B(x^*, \rho)$ such that $h(x^*) = x^*$ and for all x_0 in B, we have $\varphi_t(x_0) = (h^{-1} \circ \psi_t \circ h)(x_0)$.*
  - So basically, if x^* is a hyperbolic fixed point of a system $\dot{x} = f(x)$ and f is differentiable at x^*, then the linearized system $\dot{y} = Df(x^*)(y - x^*)$ is a good approximation of the original system.

2.4.3 The direct method of Lyapunov

- **Definition 2.24** Let x^* be a fixed point of $\dot{x} = f(x)$ with f defined on a ball $B = B(x^*, \rho)$, $\rho > 0$. Let $V : B \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that
  1. $V(x^*) < V(x)$ for all $x \in B$, (so x^* is a strict local minimizer for V).
  2. $(L_f V)(x) := DV(x)f(x) = \nabla V(x)^T f(x) \leq 0$ for all $x \in B$. ($L_f V$ is the **Lie derivative** of V in the direction of f.)

  Then V is called a **weak Lyapunov function** (if $(L_f V)(x) < 0$, it is a **strict Lyapunov function**) at x^* of $\dot{x} = f(x)$.
- **Theorem 2.26** (The direct method of Lyapunov) *Let x^* be a fixed point of $\dot{x} = f(x)$ with f continuous.*
  1. *If there exists a weak Lyapunov function V at x^* of $\dot{x} = f(x)$, then $x(t) = x^*$ is Lyapunov stable.*
  2. *If there exists a strict Lyapunov function V at x^* of $\dot{x} = f(x)$, then $x(t) = x^*$ is asymptotically Lyapunov stable, i.e., x^* is a sink.*

- **Theorem 2.28** *Let x^* be a fixed point of $\dot{x} = f(x)$ where f is defined on a neighborhood of x^* and differentiable at x^*.*

  1. *If there exists j such that $\mathrm{Re}(\lambda_j(Df(x^*))) > 0$, then $x(t) = x^*$ is Lyapunov unstable.*
  2. *Equivalently, if $x(t) = x^*$ is Lyapunov stable, then $\mathrm{Re}(\lambda_i(Df(x^*))) \leq 0$ for all i.*

- **Theorem 2.29** *Let x^* be a hyperbolic fixed point of $\dot{x} = f(x)$.*

  1. *$x(t) = x^*$ is Lyapunov unstable if and only if there exists j such that $\mathrm{Re}(\lambda_j(Df(x^*))) > 0$.*
  2. *x^* is a sink if and only if $\mathrm{Re}(\lambda_i(Df(x^*))) < 0$ for all i.*
  3. *x^* is a source if and only if $\mathrm{Re}(\lambda_i(Df(x^*))) > 0$ for all i.*
  4. *x^* is either a sink, source, or saddle.*

2.4.4 Gradient systems

- **Definition 2.32** A **gradient system** is a system of ODEs of the form $\dot{x} = -\nabla U(x)$.
- **Theorem 2.36** *Let x^* be a fixed point of $\dot{x} = -\nabla U(x)$ (so $U(x^*) = 0$) with $U \in \mathcal{C}^2$ and suppose the Hessian matrix $\nabla^2 U(x^*)$ is (strictly) positive definite ($p^T \nabla^2 U(x^*) p > 0$ for all $p \in \mathbb{R}^n \setminus \{0\}$). Then x^* is a sink.*
- **Lemma 2.37** (Integrability Lemma) *Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$. Then $f(x) = -\nabla U(x)$ if and only if $Df(x)^T = Df(x)$, i.e. $\partial_{x_i} f_j = \partial_{x_j} f_i$ for all $i \neq j$.*
- If we have a gradient system, we can find the specific $U(x)$ such that $\dot{x} = -\nabla U(x)$ by the formula

$$U(x) = -\int_0^1 \sum_i^n x_i f_i(sx) ds.$$

2.6 Periodic orbits

- **Definition 2.45** A point x_0 is **periodic** of **period** $0 < T < \infty$ if $\varphi_T(x_0) = x_0$ and $\varphi_t(x_0) \neq x_0$ for all $0 < t < T$. The solution $x(t) = \varphi_t(x_0)$ is called a **periodic solution** and the set $\Gamma_{x_0} = \{\varphi_t(x_0) \mid 0 \leq t \leq T\}$ is called a **periodic orbit**.

2.6.1 Existence of periodic orbits in $\mathbb{R}^2$

- **Theorem 2.47** (Poincaré - Bendixon) Let $\dot{x} = f(x)$ such that f is $\mathcal{C}^1$ on an open set U. If there exists $K \subset U$ compact containing x_0 such that $\Gamma^+_{x_0} = \{\varphi_t(x_0) \mid t \geq 0\} \subseteq K$, and K contains no other fixed point of $\dot{x} = f(x)$, then the ω-limit set $\omega(x_0) \subseteq K$ is a periodic orbit. (same is true if we replace $\Gamma^+_{x_0}$ with $\Gamma^-_{x_0}$ and ω with α).
- **Theorem 2.51** (Liénard systems). Consider $\ddot{y} + g(y)\dot{y} + h(y) = 0$ or equivalently

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = f(x) = \begin{bmatrix} x_2 \\ -g(x_1)x_2 - h(x_1) \end{bmatrix}$$

  for $x_1 := y$ and $x_2 := \dot{y}$. Suppose the following are satisfied:

  - $g, h \in \mathcal{C}^1$
  - h is odd ($h(-y) = -h(y)$)
  - $h(y) > 0$ for $y > 0$
  - g is even ($g(-y) = g(y)$)
  - $G(y) := \int_0^y g(u) du$ (G is odd) satisfies
    - $G(a) = 0$ for some $a > 0$;

* $G(y) < 0$ for $0 < y < a$;
* $G(y) > 0$ for $a < y$;
* $G'(y) = g(y) \geq 0$ for $a < y$ (i.e., G is nondecreasing for $a < y$);
* $\lim_{y \to +\infty} G(y) = +\infty$

Then, there exists a unique (thus isolated) periodic orbit Γ and it surrounds the origin in $\mathbb{R}^2$. Moreover, Γ is asymptotically orbitally stable and thus is an ω-limit cycle.

2.6.2 Nonexistence of periodic orbits

- **Theorem 2.54** *Gradient systems $\dot{x} = -\nabla U(x)$ with $U \in \mathcal{C}^1$ have no periodic orbit.*

- **Theorem 2.55** (Dulac's criterion) *Let*

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

with f defined on an open set $U \subseteq \mathbb{R}^2$ and $f \in \mathcal{C}^1$. Let $D \subseteq U$ be simply connected and suppose $B : D \to \mathbb{R}$ is $\mathcal{C}^1$ such that

$$\mathrm{div}(B(x)f(x)) = D_{x_1}(B(x_1, x_2)f_1(x_1, x_2)) + D_{x_2}(B(x_1, x_2)f_2(x_1, x_2)) \neq 0 \text{ for all } x \in D$$

Then D does not contain any periodic orbit.