2017 Qualifying Exams Preparation

Complex Analysis

Students of Intro to Analysis I and II

Summer 2017

Contents

Instructions

We will be focusing on previous qualifying exams from various institutions (Texas A& M, UIUC, Kansas, and others). The questions on these exams will be at the level of what you are expected to perform We will be making a master file which will contain all problems and solutions we write this summer. You are expected to write all final solutions in Tex. In the past, students have been plagued with one problem: how do I come up with this solution on my own? To address this concern, your solutions will be comprised of two parts: your previous attempts/rationale this solution; and the finalized solution.

1 How to Use This File

Collaboration is can be hard, especially so when working with a large number of people. To keep each student's work simple, I've created a master document which will read from smaller files in which each of you will write your solutions. Since this document will quickly grow in size, I ask you write your keep your explanations terse. I also want ease readability of the source documents, which leads to the following guidelines.

1.1 Ground Rules

I ask that you all adhere to the following rules:

1. Close the documents when you have completed your task
2. Test individual files before compiling the master document
3. Save Often
4. Keep a copy of your own work in another directory

1.2 Folder Structure

The `Analysis Solutions` folder is divided into smaller portions, which I hope are useful for our purposes. At the top level we have the following files

- `Analysis_2017_Solutions.tex`, the maser file
- `Analysis_2017_Solutions.pdf`, your PDF
- `CommonFiles`, see §1.3
- `RealSolutions`, files for solutions pertaining to real analysis
- `ComplexSolutions` see above

- `Images` Repository for all images included in this document
- Less important files

The folders `RealSolutions` and `ComplexSolutions` each contain several subfolders:

- `TAMU`
- `Kansas`
- `UIUC`

each which finally contain a tex file, e.g. `RTAMU.tex`. Each file is a standalone document which can be compiled independently of the master file.

1.3 User Defined Commands

You will come to find every TeX user has a unique style, in particular with creating shortcuts. Who really wants to write `\mathbb{R}` when one can use `\R`? If you have your own shortcuts, copy and paste these into the file `CommonFiles\UserCommands.tex` in the following manner:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Rolando's Commands
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\newcommand{\C}{\mathbb{C}}
\newcommand{\N}{\mathbb{N}}
\newcommand{\Q}{\mathbb{Q}}
\newcommand{\R}{\mathbb{R}}
\newcommand{\Z}{\mathbb{Z}}
\newcommand{\norm}[1]{\| #1 \|}
\newcommand{\set}[1]{ \left\lbrace #1 \right\rbrace }
\newcommand{\generator}[1]{\langle #1 \rangle}
```

This labeling will make it easier for me to help debug your code if necessary. Make sure you save and close this file immediately.

1.4 Your Solutions

You are to write your solutions in the proper file. Are you writing a solution to a real analysis question from a Texas A & M qualifying exam? Then you best be writing your *finalized* solution in `RSolutions/TAMU/RTAMU.tex`.

To keep a consistent look, I ask you write your solutions in the following manner:

```
\begin{prob}[Other University, Fall 2016 Q4]
Here is some question
\end{prob}
```

```
\begin{soln}[My Name]   $ $\newline
\textbf{Attempt:}\\
I tried nothing... How sad

\noindent\textbf{Solution:}\\
Math!
\end{soln}
```

This fomula will produce the following output for each problem:

Problem 1.1 (Other University Fall, 2016 Q4)**.** *Here is some question*

Solution. [My Name]
Attempt:
I tried nothing... How sad
Solution:
Math! □

If you apply a theorem from our textbooks to solve your problem, I ask that you include the theorem name/number and page where it may be found. It is imperative that you provide your solutions in a timely manner so that they may be made available to all.

1.5 Images

Let's suppose you'd like to include a picture you've made. Scan and save that image (with a meaningful name) into the `Images` folder. Once you've added your image to the Dropbox folder, you can embed it into the master document by using the following code:

```
\begin{figure}[h!]
\includegraphics[scale=.75]{MeanfulPictureTitle.jpeg}
\end{figure}
```

and adjust the number `.75` to be as large or as small as you'd like. Take care to type the file name and extension exactly as it appears in the folder. If you'd like a professional-looking document, you may add labels and reference images, but that is not required.

The images in the following section were uploaded using this scheme.

2 Compiling the Document

I've designed this file with multiple users in mind. Want to test the look of your work without affecting the main document? Good news!

2.1 Compiling Individual Files

The document is formatted in such a way that allows each source file to behave independently. In essence, you can compile each individual file to check the look of your work. Since my method isn't perfect, you need to do the following

If you are testing your work, make sure the top of the file looks like so:

```
\documentclass[../../Analysis_2017_Solutions.tex]{subfiles}
%Remember to comment this line when compiling master
document
\input{../../CommonFiles/UserCommands}
```

This will ensure you can compile this document without changing the main pdf. Once you are finished, make sure you comment out the top line so the file looks as follows:

```
\documentclass[../../Analysis_2017_Solutions.tex]{subfiles}
%Remember to comment this line when compiling master
document
%\input{../../CommonFiles/UserCommands}
```

2.2 The Master File

The file joins all our work is `Analysis_2017_Solutions.tex` which creates the document which you are currently reading. When you are happy with the look of your work, save the file you are working on and compile `Analysis_2017_Solutions.tex` to produce `Analysis_2017_Solutions.pd`

Part I
Real Analysis

3 Kansas Qual

Problem 3.1 (Kansas, Spring 2004 Q7)**.** *Let $p \geq 1$ be a real number and let $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R}, \lambda)$ be a sequence with $\lim_{n\to\infty} ||f_n||_p = 0$. Prove there exists integers $1 \leq n_1 \leq n_2 \leq \cdots$ such that $\lim_{k\to\infty} f_{n_k} = 0\lambda$-a.e.*

Solution. [Sara Reed] Note the following will be used in the proof:

- Chebyshev's Inequality (p.80): Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,
$$m\{x \in E|\ f(x) \geq \lambda\} \leq \frac{1}{\lambda}\int_E f.$$
- Convergence of p-series: If $p > 1$, $\sum_{k=1}^{\infty} \frac{1}{n^p}$ converges.
- The Borel-Cantelli Lemma (p. 46): Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.
  - Note that another way to say this statement is:
$$m\Big(\bigcap_{n=1}^{\infty}\Big[\bigcup_{k=n}^{\infty} E_k\Big]\Big) = 0.$$

Also note that an alternate proof can be found as a part of the Riesz-Fischer Theorem on page 148.

Proof. Let $\epsilon > 0$. Since $\lim_{n\to\infty} ||f_n||_p = 0$, choose $\{n_k\}_k$ such that $1 \leq n_1 \leq n_2 \leq \cdots$ and $||f_{n_k}||_p < \frac{1}{k^2}$. Consider the set
$$E_k^{(\epsilon)} = \{x|\ |f_{n_k}(x)|^p \geq \epsilon\} = f_{n_k}^{-1}(\epsilon^{\frac{1}{p}}, \infty).$$

By Chebyshev's Inequality, we know
$$\begin{aligned}\lambda(E_k^{(\epsilon)}) &\leq \frac{1}{\epsilon}\int_{\mathbb{R}} |f_{n_k}|^p d\lambda \\ &= \frac{1}{\epsilon}||f_{n_k}||_p^p \\ &\leq \frac{1}{\epsilon}\frac{1}{k^{2p}}.\end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty} \lambda(E_k^{(\epsilon)}) \leq \sum_{k=1}^{\infty} \frac{1}{\epsilon}\frac{1}{k^{2p}} = \frac{1}{\epsilon}\sum_{k=1}^{\infty} \frac{1}{k^{2p}} < \infty$$

since $2p \geq 2 > 1$. Since $\sum_{k=1}^{\infty} \lambda(E_k^{(\epsilon)}) < \infty$, the Borel Cantelli Theorem tells us that

$$\lambda\Big(\bigcap_{n=1}^{\infty}\Big[\bigcup_{k=n}^{\infty} E_k^{(\epsilon)}\Big]\Big) = 0.$$

This statement holds for all $\epsilon > 0$. Define $\epsilon_j = \frac{1}{j}$. Note that $\lim_{j\to\infty} \epsilon_j = 0$. Since Lebesgue measure is countably subadditive (p.34), we have

$$\lambda\Big(\bigcup_{j=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k^{(\epsilon_j)}\Big) \leq \sum_{j=1}^{\infty}\lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k^{(\epsilon_j)}\Big) = 0$$

and therefore

$$\lambda\Big(\bigcup_{j=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k^{(\epsilon_j)}\Big) = 0.$$

Note that the set $E = \bigcup_{j=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k^{(\epsilon_j)}$ can be described in the following way: all $x \in \mathbb{R}$ such that there exists $\epsilon_j > 0$ such that for all $n \geq 0$ there exists $k \geq n$ satisfying $|f_{n_k}(x)| > \epsilon_j^{\frac{1}{p}}$. We can excise this set of measure zero. Then, for $x \in \mathbb{R} \sim E$, we know for all $\epsilon_j > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, we have $|f_{n_k}(x)| < \epsilon_j^{\frac{1}{p}}$. Therefore, we can conclude that $\lim_{k\to\infty} f_{n_k}(x) = 0$ almost everywhere. □

□

Problem 3.2 (Nicholas Camacho, Spring 2009 Q6)**.** *Let E be a Lebesgue measurable set in $\mathbb{R}^n$. Prove that*

$$E = A_1 \cup N_1 = A_2 \sim N_2$$

where A_1 is an F_σ set, A_2 is a G_δ set, and $m(N_1) = m(N_2) = 0$.

Proof. We claim that in fact there exists a G_δ set G containing E such that $m(G \sim E) = 0$, and hence we can write $E = G \sim (G \sim E)$. If we can show this, then since E^c is also measurable, we can apply this result to E^c to find a G_δ set M containing E^c such that $E^c = M \sim N$ where $m(N) = 0$, and then

$$E = (E^c)^c = (M \cap N^c)^c = M^c \cup N,$$

and since the complement of a G_δ is an F_σ, we have obtained both desired forms for E. Hence it remains only to show that such a set G exists.

First suppose that E has finite measure and let $\epsilon > 0$. Since m is outer regular, there exists an open set $\mathscr{O}$ containing E such that

$$m(\mathscr{O} \sim E) = m(\mathscr{O}) - m(E) < \epsilon. \qquad (*)$$

Now, if $m(E) = \infty$, we can write E as a countable disjoint union of finite measure sets $\{E_k\}_{k\in\mathbb{N}}$. Then by $(*)$, for each $n \in \mathbb{N}$, we can choose an open set $\mathscr{O}_n$ containing E_n such that $m(\mathscr{O}_n \sim E_n) < \epsilon/2^n$. Then $\mathscr{O} = \bigcup_{n\in\mathbb{N}}$ is open, contains E, and

$$m(\mathscr{O} \sim E) = m\left(\bigcup_{n\in\mathbb{N}} \mathscr{O}_n \sim E\right) \leq m\left(\bigcup_{n\in\mathbb{N}} \mathscr{O}_n \sim E_n\right) \leq \sum_{n\in\mathbb{N}} m(\mathscr{O}_n \sim E_n) = \epsilon.$$

Hence, for each $n \in \mathbb{N}$, we can pick an open set $\mathscr{O}_n$ containing E such that $m(\mathscr{O}_n \sim E) < 1/n$. Then $G = \bigcap_{n\in\mathbb{N}} \mathscr{O}_n$ is a G_δ set containing E, and since $G \sim E \subseteq \mathscr{O}_n \sim E$ for each n, we get

$$m(G \sim E) \leq \lim_{n\to\infty} m(\mathscr{O}_n \sim E) = 0.$$

□

Problem 3.3 (Elaina Aceves, August 2008 Q8)**.** *Decide which space is bigger,* $L^1([0,1])$ *or* $L^2([0,1])$*? Explain why.*

Solution.
Attempt:
I looked at Corollary 3 on page 142 to approach this problem, which tells us that $L^{p_2}(E) \subset L^{p_1}(E)$ when E is a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$. Thus, this problem is a special case when $E = [0,1]$, $p_1 = 1$, and $p_2 = 2$.
Solution:
Let $f \in L^2([0,1])$ and let λ denote the Lebesgue measure. Then $\int_0^1 |f(x)|^2 dx < \infty$. Let $g = \chi_{[0,1]}$. Then $g \in L^2([0,1])$ because $\lambda([0,1]) = 1 < \infty$. We have the following inequalities.

$$\begin{aligned}
\int_0^1 |f(x)| dx &= \int_0^1 |f(x)| \cdot g(x)\, dx \\
&\leq ||f(x)||_2 \cdot ||g(x)||_2 \quad \text{by Holder Inequality} \\
&= ||f(x)||_2 \cdot \sqrt{\int_0^1 |g(x)|^2\, dx} \text{ by definition} \\
&= ||f(x)||_2 \cdot (\lambda([0,1])^{1/2} \\
&= ||f(x)||_2 < \infty \quad \text{since } f \in L^2([0,1]).
\end{aligned}$$

Hence $f \in L^1([0,1])$.

Example to show strict subset: $f \in L^1([0,1]) \sim L^2([0,1])$ is $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^\alpha & \text{if } 0 < x \leq 1 \end{cases}$ for $-1 < \alpha < -1/2$ which is given on page 143.

□

Problem 3.4 (Nicholas Camacho, Fall 2014 Q9)**.** *Is it true that every closed and bounded set in* $L^2([0,1])$ *is compact?*

Proof. **Attempt:** First thought: "Well, what does it mean to be compact in $L^2([0,1])$? Is it the standard "Every open cover has a finite subcover"?. And yes, it is. But this may not be the most helpful definition. And then I remembered: $L^2([0,1])$ is a metric space! So we have equivalent notions of compactness. Let's try one of those: Complete and totally bounded, or, sequentially compact. So after asking Rolando for help, here's what we have:

Solution: Consider the closed unit ball in $L^2([0,1])$,

$$B := \{f \in L^2([0,1]) : \ ||f||_2 \leq 1\}.$$

Then certainly B is closed and bounded by it's definition. Now define a sequence of functions $f_n(x) = \sin(n\pi x)$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} ||f_n||_2^2 = \int_0^1 |\sin(n\pi x)|^2 dx &= \int_0^1 \sin^2(n\pi x) dx \\ &= \frac{1}{n\pi}\int_0^{n\pi} \frac{1-\cos 2u}{2} du \qquad (u = n\pi x, du = n\pi dx) \\ &= \frac{1}{2}. \end{aligned}$$

and so $\{f_n\} \subset B$. However, for distinct n, m we have

$$\begin{aligned} ||f_n - f_m||_2^2 &= \int_0^1 |\sin(n\pi x) - \sin(m\pi x)|^2 \\ &= \int_0^1 [\sin(n\pi x) - \sin(m\pi x)]^2 \\ &= \int_0^1 \sin^2(n\pi x) - 2\sin(n\pi x)\sin(m\pi x) + \sin^2(m\pi x) \\ &= \frac{1}{4} - 2\int_0^1 \sin(n\pi x)\sin(m\pi x), \\ &= \frac{1}{4} - 2\left(\frac{1}{2}\int_0^1 \cos((n-m)\pi x) - \frac{1}{2}\int_0^1 \cos((n+m)\pi x)\right) \\ &= \frac{1}{4} - \frac{1}{n-m}\left[\sin((n-m)\pi x)\right]_0^1 - \frac{1}{n+m}\left[\sin((n+m)\pi x)\right]_0^1 \\ &= \frac{1}{4}. \end{aligned}$$

In other words, the the f_n's do not get close to one another, and so no subsequence of $\{f_n\}$ converges.

I think it might be good to give another example: Credit this one to Tyler Reynolds. His idea is to create functions that all have the same "area under the curve", but are each nonzero on their own interval.

$$f_n(x) = \begin{cases} \sqrt{2^{n+1}} & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$||f_n||_2^2 = \int_0^1 |f_n|^2 = \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \left|\sqrt{2^{n+1}}\right|^2 = 2^{n+1}\left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) = 1$$

and so $\{f_n\} \subset B$, but for distinct m, n,

$$\begin{aligned} ||f_n - f_m||_2^2 = \int_0^1 |f_n - f_m|^2 &= \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} |f_n - f_m|^2 + \int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} |f_n - f_m|^2 \\ &= \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \left|\sqrt{2^{n+1}}\right|^2 + \int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} \left|\sqrt{2^{m+1}}\right|^2 \\ &= 2, \end{aligned}$$

and so again the f_n's do not get close to one another, and so no subsequence of $\{f_n\}$ converges.

Remark: Similar functions will in fact work in $L^p([a,b])$ for any $1 \leq p < \infty$. Just define

$$f_n(x) = \begin{cases} \sqrt[p]{\dfrac{2^{n+1}}{b-a}} & \text{if } x \in \left(\dfrac{b-a}{2^{n+1}}, \dfrac{b-a}{2^n}\right) \\ 0 & \text{otherwise.} \end{cases}$$

□

Problem 3.5 (Kansas, Spring 2015 Q1)**.** *Is there a Borel set* $A \subset \mathbb{R}$ *such that* $0 < \lambda(A \cap I) < \lambda(I)$ *for every interval* $I \subset \mathbb{R}$

Solution. [Jared Grove]
Attempt:
I initially thought the answer was no, so I tried to disprove it and it obviousy didn't go so well. I tried going through each type of Borel set and showing it wouldn't work, but there are so many wierd types of sets its almost impossible to test every type of case.
Solution:
The quick answer is yes, but here comes the justification. We need to construct something like a collection of nowhere dense sets that cover $\mathbb{R}$. I believe a term that some people use is a fat Cantor set if that means anything to you. We will begin by letting $\{r_n\}$ be an enumeration of the rational numbers. Next let V_1 be a segment of finite length centered at r_1 and V_n be a segment of length $\frac{\lambda(V_{n-1})}{3}$ centered at r_n. Notice that each v_n is a third the length of the previous set and that the collection of V_n will cover $\mathbb{R}$ since $\mathbb{Q}$ is dense in the reals. Next we will define the set

$$W_n = V_n - \cup_{k=1}^{\infty} V_{n+k}$$

and notice that

$$\lambda(W_n) \geq \lambda(V_n) - \sum_{k=1}^{\infty} \lambda(V_{n+k}) = \lambda(V_n) - \lambda(V_n)\sum_{k=1}^{\infty} 3^{-k} = \frac{\lambda(V_n)}{3} > 0$$

For each n choose a Borel set $A_n \subset W_n$ with $0 < \lambda(A_n) < \lambda(W_n)$. You can do this as for every set W_n with $\lambda(W_n) > 0$ there exists F closed (so Borel) with $F \subset W_n$ such that $\lambda(W_n \sim F) > \epsilon$. Now define $A = \cup_{n=1}^{\infty} A_n$. Since $A_n \subset W_n$ and the W_n are disjoint (they still conver $\mathbb{R}$ though) $\lambda(A \cap W_n) = \lambda(A_n)$. Thus

$$0 < \lambda(A \cap W_n) = \lambda(A_n) < \lambda(W_n)$$

for every n. Since every interval will contain at least a part of a W_n we can extend this fact above to any interval and A is the set we are looking for.

□

Problem 3.6 (Kansas, Spring 2015 Q3). *Compute the following limit and justify your calculation:*

$$\lim_{n\to\infty} \int_0^1 \frac{1 + \frac{n}{2}x}{(1 + x)^{n/2}} dx.$$

Solution. [Michael Kratochvil]
Attempt:
I first thought that I would have to pass the limit under the integral sign somehow, but even if I were to do that, I would still have to compute the limit of the integrand. Then I realized that for all n the integral could be computed in its closed form using grade school calculus techniques, and the limit could be performed after the computation, so that is what I did.
Solution:

Letting $p = \frac{n}{2}$ (for ease of writing $n/2$ over and over again) and $p > 2$, we obtain

$$\begin{aligned}
\int_0^1 \frac{1 + px}{(1 + x)^p} dx &= \int_1^2 \frac{1 + p(u - 1)}{u^p} \\
&= \int_1^2 (1 - p)u^{-p} + pu^{1-p} du \\
&= u^{1-p} + \frac{p}{2 - p} u^{2-p}|_1^2 \\
&= 2^{1-p} - 1 + \frac{p}{2 - p}(2^{2-p} - 1) \\
&= 2^{1-n/2} - 1 + \frac{n/2}{2 - n/2}(2^{2-n/2} - 1).
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int_0^1 \frac{1 + \frac{n}{2}x}{(1 + x)^{n/2}} dx = \lim_{n\to\infty} 2^{1-n/2} - 1 + \frac{n/2}{2 - n/2}(2^{2-n/2} - 1) = 0.$$

□

Problem 3.7 (Kansas, Fall 2011, Q11)**.** *Denote by λ_1 the Lebesgue measure on $\mathbb{R}$. Let E be a λ_1 set of positive measure. Show that for every $\alpha < 1$ there is an open interval $I = I(\alpha)$ such that $\lambda_1(E \cap I) > \alpha\lambda_1(I)$.*

Solution. [W. Tyler Reynolds]
Attempt:
My first attempt was to straightforwardly use the definition of measure and outer approximation by open sets. This didn't quite work, but Chapter 2 Theorem 12 of Royden seemed to suggest that the use of the symmetric difference might help. It did, if one assumed that E had finite measure. The infinite measure case followed readily from this. To avoid having to write out the two cases separately, I reduced E to a (possibly smaller) finite measure set at the outset of the solution.
Solution:
If $\alpha \leq 0$, let I be any open interval such that $\lambda_1(E \cap I) > 0$ (such an interval exists since $\lambda_1(E) > 0$). Then $\lambda_1(E \cap I) > 0 \geq \alpha\lambda_1(I)$.

Now let $0 < \alpha < 1$. Let $F \subset E$ with $0 < \lambda_1(F) < \infty$. Since $0 < (1-\alpha)\lambda_1(F) < \infty$, we can find a finite disjoint collection $\{I_k\}_{k=1}^n$ of open intervals such that if $O = \bigcup_{k=1}^n I_k$, then $\lambda_1(O \setminus F) + \lambda_1(F \setminus O) < (1-\alpha)\lambda_1(F)$. (This is Chapter 2 Theorem 12 of Royden). Notice that

$$\begin{aligned}\lambda_1(F) = \lambda_1(F \cap O) + \lambda_1(F \setminus O) &\leq \lambda_1(F \cap O) + \lambda_1(F \setminus O) + \lambda_1(O \setminus F)\\ &< \lambda_1(F \cap O) + (1-\alpha)\lambda_1(F),\end{aligned}$$

so that $\lambda_1(F) < \lambda_1(F \cap O)/\alpha$. Thus,

$$\begin{aligned}\lambda_1(O) = \lambda_1(F \cap O) + \lambda_1(O \setminus F) &< \lambda_1(F \cap O) + (1-\alpha)\lambda_1(F)\\ &< \lambda_1(F \cap O) + \frac{1-\alpha}{\alpha}\lambda_1(F \cap O) = \frac{\lambda_1(F \cap O)}{\alpha}.\end{aligned}$$

Rewriting this yields

$$\sum_{k=1}^n \lambda_1(I_k) = \lambda_1(O) < \frac{\lambda_1(F \cap O)}{\alpha} = \sum_{k=1}^n \frac{\lambda_1(F \cap I_k)}{\alpha}.$$

It follows that $\lambda_1(I_k) < \dfrac{\lambda_1(F \cap I_k)}{\alpha}$ for some k. Letting $I = I_k$, we obtain $\alpha\lambda_1(I) < \lambda_1(F \cap I) \leq \lambda_1(E \cap I)$.

□

Problem 3.8 (Kansas, Fall 2013, Q1)**.** *Let $\{f_n\}$ be a sequence of integrable function on $X, \mathscr{M}, \mu$, s.t. $f_n \to f$ a.e. where $f \in L^1(\mu)$. Prove that $\int_X |f - f_n| d\mu \to 0$ iff $\int_X |f_n| d\mu \to \int_X |f| d\mu$*

Solution. [Yanqing Shen]
Attempt:
Check the theorems about the Lebesgue integral, and how to bound functions with triangular inequality.
Solution:
1. "$\Longrightarrow$". We assume $\int_X |f - f_n| d\mu \to 0$.

Since $\{f_n\}$, $f \in L^1(\mu)$, hence $\{|f - f_n|\}$, $\{|f_n| - |f|\} \in L^1(\mu)$.

By reversed triangular inequality $\big||x| - |y|\big| \le |x - y||$, hence

$$\big||f| - |f_n|\big| \le |f - f_n||$$

Therefore,

$$\int_X \big||f| - |f_n|\big| d\mu \le \int_X |f - f_n| d\mu,$$

since $\int_X |f - f_n| d\mu \to 0$, and $\big|\int_X (|f| - |f_n|)\, d\mu\big| \le \int_X \big||f| - |f_n|\big| d\mu$.

Therefore,

$$\big|\int_X (|f| - |f_n|)\, d\mu\big| \to 0 \Longrightarrow \int_X (|f| - |f_n|)\, d\mu \to 0,$$

and

$$\int_X (|f| - |f_n|)\, d\mu = \int_X |f| d\mu - \int_X |f_n| d\mu$$

Thus

$$\int_X |f_n| d\mu \to \int_X |f| d\mu.$$

2. "$\Longleftarrow$". We assume $\int_X |f_n| d\mu \to \int_X |f| d\mu$.

Since $f_n \to f$ a.e. on X. $\exists\, X_0 \subseteq X$ s.t. $f_n(x) = f(x)$, $\forall\, x \in X_0$, $\mu(X \backslash X_0) = 0$.

Define $|g_n| = |f| + |f_n| - |f - f_n|$ on X_0, then $\lim_{n\to\infty} g_n(x) = 2|f(x)|$, $\forall\, x \in X_0$,

and define $g(x) := 2|f(x)|$ on X.

Then $g_n \to g$ a.e. on X, since $\{g_n\}$ are nonnegative and integrable functions, by *Fatou's Lemma,*

$$\int_X g\, d\mu \le \liminf \int_X g_n\, d\mu.$$

This implies

$$
\begin{aligned}
2\int_X |f|\,d\mu &\le \liminf \left\{ \int_X \big[|f| + |f_n| - |f - f_n|\big]\,d\mu \right\} \\
&\le \liminf \int_X |f|\,d\mu + \liminf \left\{ \int_X \big[|f_n| - |f - f_n|\big]\,d\mu \right\} \\
&= \int_X |f| d\mu + \liminf \left\{ \int_X \big[|f_n| - |f - f_n|\big]\,d\mu \right\} \\
&\le \int_X |f| d\mu + \liminf \int_X |f_n| d\mu + \liminf \left\{ \int_X -|f - f_n| d\mu \right\} \\
&= \int_X |f| d\mu + \int_X |f| d\mu - \limsup \left\{ \int_X |f - f_n| d\mu \right\} \\
&= 2\int_X |f| d\mu - \limsup \left\{ \int_X |f - f_n| d\mu \right\}
\end{aligned}
$$

Therefore, $\limsup \left\{ \int_X |f - f_n| d\mu \right\} \le 0.$

Given that $\int_X |f| d\mu \le \infty$,

since

$$0 \le \liminf \left\{ \int_X |f - f_n| d\mu \right\} \le \limsup \left\{ \int_X |f - f_n| d\mu \right\} \le 0,$$

thus

$$\lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

□

Problem 3.9 (Kansas, Spring 2013, Q5)**.** *Let g be a bounded Lebesgue measurable function on $\mathbb{R}$ which has the property that $\lim_{n\to\infty} \int_I g(nx)dx = 0$ for every interval $I \subset [0,1]$. Prove that for every $f \in L^1([0,1])$,*

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = 0.$$

Solution. [Alex Bates]
Attempt:
My first attempt, defining $g_n(x) := g(nx)$, I tried to use the fact that $f \in L^1([0,1])$ to apply Hölder's Inequality to obtain $\int_0^1 |f(x) \cdot g(nx)| dx \le ||f||_1 \cdot ||g_n||_\infty$. But getting an *esssup* on g_n (this is how $||\cdot||_\infty$ is defined) would have been difficult, if not impossible.
Solution:
Let's first prove the claim under the assumption that f is a simple function, i.e., $f =$

$\sum_{k=1}^m c_k \cdot \chi_{I_k}$, where $\{I_k\}_{k=1}^m$ is a collection of disjoint intervals whose union is $[0,1]$. Then:

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx &= \lim_{n\to\infty} \int_0^1 \sum_{k=1}^m (c_k \cdot \chi_{I_k})\, g(nx)dx \\
&= \lim_{n\to\infty} \sum_{k=1}^m c_k \int_{I_k} g(nx)dx \\
&= \sum_{k=1}^m c_k \lim_{n\to\infty} \int_{I_k} g(nx)dx \\
&= \sum_{k=1}^m c_k \cdot 0 \\
&= 0.
\end{aligned}$$

Let $\epsilon > 0$. Since g is bounded, there is $M > 0$ for which $|g| \leq M$ on all of $\mathbb{R}$. By Problem 4.6.44 of Royden (pg. 95), there is a step function ϕ defined on $[0,1]$ for which $\int_0^1 |f - \phi| < \frac{\epsilon}{2M}$. Further, by the work above let $N \in \mathbb{N}$ be such that $n \geq N$ implies $\left|\int_0^1 \phi(x)g(nx)dx\right| < \epsilon/2$. Then for any $n \geq N$,

$$\begin{aligned}
\left|\int_0^1 f(x)g(nx)dx\right| &= \left|\int_0^1 (f(x) - \phi(x) + \phi(x))g(nx)dx\right| \\
&\leq \left|\int_0^1 (f(x) - \phi(x))g(nx)dx\right| + \left|\int_0^1 \phi(x)g(nx)dx\right| \\
&\leq \int_0^1 |f(x) - \phi(x)||g(nx)|dx + \left|\int_0^1 \phi(x)g(nx)dx\right| \\
&\leq M \cdot \int_0^1 |f(x) - \phi(x)|dx + \left|\int_0^1 \phi(x)g(nx)dx\right| \\
&< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

so that $\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = 0$.

□

Problem 3.10. *, Fall 2014 Q8]*

Suppose (X, M, μ) is a measure space, μ a positive measure, $f_n \in L^p(X), \forall n \in \mathbb{N}$ and $f \in L^p(X)$, where $1 \leq p < \infty$. Prove the following;

(i). If $\|f_n - f\|_p \to 0$ as $n \to \infty$ then $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.
(ii). If $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$ then $\|f_n - f\|_p \to 0$

Solution. This problem is very similar to theorem 7 on page 148. For part (i) I used a similar proof as in the book, however, proof of (ii) is not the same as what is given in the book.

Proof. (i). By Minkowski's inequality (pg 141)

$$|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p \quad \forall n \in \mathbb{N}$$

By taking the limit on both sides we get that

$$\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$$

(ii). Suppose $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$. Observe that

$$|f_n - f|^p \leq 2^p(|f_n|^p + |f|^p) \tag{1}$$

$$\implies \int_X |f_n - f|^p \leq \int_X 2^p(|f_n|^p + |f|^p) \tag{2}$$

$$\implies \|f_n - f\|_p^p \leq 2^p(\|f_n\|_p^p + \|f\|_p^p) \tag{3}$$

$$\implies \lim_{n\to\infty} \|f_n - f\|_p^p \leq 2^p\Big(\lim_{n\to\infty} \|f_n\|_p^p + \lim_{n\to\infty} \|f\|_p^p\Big) = 2^p\Big(\|f\|_p^p + \|f\|_p^p\Big) \tag{4}$$

$$\implies \lim_{n\to\infty} \int_X |f_n - f|^p = \lim_{n\to\infty} \|f_n - f\|_p^p \leq 2^{p+1}\|f\|_p^p = 2^{p+1}\int_X |f|^p \tag{5}$$

Now let $h_n = |f_n - f|^p$and $g_n = 2^p(|f_n|^p + |f|^p)$. Then $|h_n| \leq g_n$ and

$$\{h_n\} \to 0, \ \ \{g_n\} \to 2^{p+1}|f|^p \text{ pointwise a.e on X.} \tag{6}$$

Since

$$\lim_{n\to\infty} \int_X g_n = 2^{p+1}\int_X |f|^p < \infty$$

we can apply the generalized Lebesgue dominated convergence theorem (pg 89) to get that

$$\lim_{n\to\infty} \int_X |f_n - f|^p = \lim_{n\to\infty} \int_X h_n = 0$$

$$\implies \|f_n - f\|_p^p \to 0$$

$$\implies \|f_n - f\|_p \to 0$$

□

□

Problem 3.11 (Kansas, Fall 2014 Q4)**.** *Let f be entire and bounded on the strip $0 \leq Re(z) \leq 1$, with the property that*

$$f(z+1) = \frac{f(z)}{2} \qquad \forall z \in \mathbb{C}.$$

Prove that $f(z) = a2^{-z}$ for some $a \in \mathbb{C}$.

Solution. [Sara Reed] Define K to be the strip $0 \leq \mathrm{Re}(z) \leq 1$. Note that $|f(z)| \leq M$ for all $z \in K$ and for some $M \in \mathbb{R}$. Let $z \in \mathbb{C}$. We will now consider two cases: $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(z) < 0$.

1. $\mathrm{Re}(z) \geq 0$: We can write $z = x + iy = \lfloor * \rfloor x + x^* + iy$ where $z^* = x^* + iy \in K$. Note that $z^* + \lfloor x \rfloor = z$. It follows

$$\begin{aligned} f(z) &= f(z^* + \lfloor * \rfloor x) \\ &= \frac{f(z^*)}{2^{\lfloor * \rfloor x}} \\ &\leq \frac{M}{2^{\lfloor * \rfloor x}} \end{aligned}$$

since $z^* \in K$.

2. $\mathrm{Re}(z) < 0$: We can write $z = x + iy = \lfloor * \rfloor x + 1 - x^* + iy$ where $x^* \in [0, 1)$. Then $z^* = 1 - x^* + iy \in K$. Note that $z + \lfloor x \rfloor = z^*$ since $\lfloor x \rfloor < 0$. It follows

$$\begin{aligned} f(z^*) &= f(z + (-\lfloor x \rfloor)) \\ &= \frac{f(z)}{2^{-\lfloor * \rfloor x}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f(z) &= f(z^*)2^{-\lfloor * \rfloor x} \\ &\leq M2^{-\lfloor * \rfloor x}. \end{aligned}$$

since $z^* \in K$.

Consider $g(z) = f(z)2^z$. Since f is entire, we know g is entire. Again, we consider three cases. In each case, we will use the following fact:

$$|2^z| = |2^{x+iy}| = |2^x||2^{iy}| = |2^x||e^{iy\ln(2)}| = |2^x|$$

since $y \ln 2 \in \mathbb{R}$ so $|e^{iy\ln(2)}| = 1$.

1. $\mathrm{Re}(z) \geq 0$:

$$\begin{aligned} |g(z)| &= |f(z)2^z| \\ &\leq \frac{M|2^z|}{|2^{\lfloor * \rfloor x}|} \\ &= M|2^{z^*}| \\ &= M|2^{x^*}| \\ &= 2M. \end{aligned}$$

2. $\mathrm{Re}(z) < 0$:

$$\begin{aligned}
|g(z)| &= |f(z)2^z| \\
&\leq M|2^{-\lfloor x \rfloor}||2^z| \\
&= M|2^{-\lfloor x \rfloor}||2^x| \\
&= M|2^{-\lfloor x \rfloor + x}| \\
&= M|2^{1-x^*}| \\
&\leq 2M.
\end{aligned}$$

So for all $z \in \mathbb{C}$, we have shown $|g(z)| \leq 2M$. Therefore, g is an entire bounded function. By Liouville's Theorem, we know g is constant. Therefore, there exists $a \in \mathbb{C}$ such that $g(z) = f(z)2^z = a$. We conclude $f(z) = a2^{-z}$ as desired. □

Problem 3.12 (Elaina Aceves, Fall 2014 Q6)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space. Suppose $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are measurable functions. Prove that the sets $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.*

Solution.
Attempt:
These are special cases of Theorem 1 and Proposition 6 from p.55-57.
Solution:
Let $x \in X$. If $f(x) < g(x)$, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $f(x) < q < g(x)$. Then

$$\{x : f(x) < g(x)\} = \bigcup_{q \in \mathbb{Q}} \{x : f(x) < q\} \cap \{x : g(x) > q\}$$

Since f is measurable, $\{x : f(x) < q\}$ is measurable and since g is measurable, $\{x : g(x) > q\}$. Recall that the intersection of two measurable sets is measurable. Thus, $\{x : f(x) < g(x)\}$ is measurable as the countable union of measurable sets. Furthermore, $\{x : f(x) \leq g(x)\}$ and $\{x : f(x) \geq g(x)\}$ are measurable since $\{x : f(x) < g(x)\}$ is measurable. Then

$$\{x : f(x) = g(x)\} = \{x : f(x) \geq g(x)\} \cap \{x : f(x) \leq g(x)\}$$

Hence $\{x : f(x) = g(x)\}$ is measurable as the intersection of two measurable sets. □

4 Texas A & M Quals

Problem 4.1 (Nicholas Camacho, Page 1, Q1). *Let* (X, ρ) *be a metric space,* $E \subset X$*, and* $f(x) = \inf_{y \in E} \rho(x, y)$*. Show that* f *is continuous on* X*, and that* $\overline{E} = \{x \in X : f(x) = 0\}$*.*

Proof. **Attempt:**
I recently solved the first part of this problem while studying for the Topology qual. It's about understanding basic properties of inf. Namely: the inf is less than or equal to any member of the set you're inf-ing over, and, if you find something less than or equal to a member of the set, (i.e., a lower bound), then the inf will be bigger than or equal to that thing. The second part is using definitions and showing set inclusion both ways. So, I don't have much in the way of an "attempt", but only the polished solution. This problem wasn't too difficult.
Solution:
Define $\inf_{y \in E} \rho(x, y) = \operatorname{dist}(x, E)$. For $x_1, x_2 \in X$ and $y \in E$,

$$\operatorname{dist}(x_1, E) \leq \rho(x_1, y) \leq \rho(x_1, x_2) + \rho(x_2, y)$$

This gives

$$\operatorname{dist}(x_1, E) - \rho(x_1, x_2) \leq \rho(x_2, y)$$

and so

$$\operatorname{dist}(x_1, E) - \rho(x_1, x_2) \leq \operatorname{dist}(x_2, E),$$

which yeilds

$$\operatorname{dist}(x_1, E) - \operatorname{dist}(x_2, E) \leq \rho(x_1, x_2).$$

Switching roles of x_1 and x_2 gives

$$|\operatorname{dist}(x_1, E) - \operatorname{dist}(x_2, E)| \leq \rho(x_1, x_2).$$

So in fact $f(x)$ is Lipschitz, and hence continuous.

Now, let $x \in \overline{E}$. That is, any neighborhood of x intersects E. If $f(x) = \operatorname{dist}(x, E) = \epsilon > 0$, let $1/n < \epsilon$. Then $B(x, 1/n) \cap E = \emptyset$, a contradiction. Conversely, let $f(x) = 0$ and suppose $x \notin \overline{E}$, i.e., there exists a neighborhood U of x not intersecting E. If n is such that $B(x, 1/n) \subset U$, then we have $\rho(x, y) > 1/n$ for all $y \in E$, which means $0 = f(x) = \operatorname{dist}(x, E) \geq 1/n > 0$, a contradiction. □

Problem 4.2 (Kaitlin Healy, Page 1 Q4). *Let* $\{f_n\}_{n \geq 1}$ *be a sequence of functions in* $L^p(\mathbb{R}, \mu)$ *where* $1 < p < \infty$ *and* μ *is Borel measure on* $\mathbb{R}$*. Suppose that*

$$\sup_{n \geq 1} ||f_n||_p < \infty$$

Show that $\{f_n\}_{n \geq 1}$ *is 'uniformly integrable', that is, for all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that when* $\mu(E) < \delta$*, we have*

$$\sup_{n \geq 1} \int_E |f_n| d\mu < \varepsilon$$

Solution.
Attempt:

My initial thought was to try to divide up $\mathbb{R}$ into intervals of size slightly less than δ but could not see where to go past that point. Then, I talked to Rolando and got a good solution.
Solution:

Let $\varepsilon > 0$. We want to find a $\delta_\varepsilon > 0$ that satisfies the definition of uniformly integrable. To find this δ, notice the following for any n:

$$\begin{aligned}
\int_E |f_n| &= \int_{\mathbb{R}} |f_n| \cdot \chi_E \\
&\leq \left(\int_{\mathbb{R}} |f_n|^p\right)^{1/p} \cdot \left(\int_{\mathbb{R}} \chi_E\right)^{1/q} \qquad \text{by Hölder's inequality} \\
&= \left(\int_{\mathbb{R}} |f_n|^p\right)^{1/p} \cdot \left(\int_E 1\right)^{1/q} \\
&= ||f_n||_p \cdot [\mu(E)]^{1/q}
\end{aligned}$$

Since $\sup_{n\geq 1} ||f_n||_p < \infty$, we know that the supremum is finite. Thus, let $c = \sup_{n\geq 1} ||f_n||_p$. This gives us

$$||f_n||_p \cdot [\mu(E)]^{1/q} \leq c \cdot [\mu(E)]^{1/q}$$

We need that $c \cdot [\mu(E)]^{1/q} < \varepsilon$. This happens when $\mu(E) < (\varepsilon/c)^q$. Therefore, let $\delta_\varepsilon = (\varepsilon/c)^q$ where q is the conjugate of p. Since the above equalities and inequalities were true for any n, they must hold for $\sup_{n\geq 1} \int_E |f_n| d\mu$. Hence, we conclude for our δ_ε that

$$\sup_{n\geq 1} \int_E |f_n| d\mu < \varepsilon$$

and thus we have that $\{f_n\}_{n\geq 1}$ is uniformly integrable.

□

Problem 4.3 (W. Tyler Reynolds, Texas A&M, Page 2 Q5)**.** *Consider the function* $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ *defined by*

$$f(x, y) = \sum_{n=1}^{\infty} \frac{1}{nx^2 + n^3 x^{-2} y}.$$

Find the limit $g(y) := \lim_{x\to\infty} f(x, y)$ *for* $y > 0$*, with a proof.*

Solution.
Attempt:

Solving this problem revolved around the observation that series and integration are closely related. I realized that I would have exactly what I needed if I could "pass the limit under the summation sign". This led to a bit of research in which I discovered that many of our integration theorems such as Fatou's Lemma and the Dominated Convergence Theorem have analogues for series. I believe that this parallel theory could be probably be used for this problem without too much trouble; however, building the proofs of the required theorems from scratch would be tedious and take too much time on an exam. Thus, I ended up looking for a way to change the summations into integrals which would play nicely with the theorems we all know and love (for better or worse, you *do* love them). I initially tried to use the Monotone Convergence Theorem, only to realize that my sequence of functions wasn't increasing in the proper sense - it was only *intervalwise eventually increasing*, meaning that the index past which the sequence became increasing could vary depending on the interval under examination. As a workaround, I formulated a very artificial but useful result: the Measurewise Eventually Monotone Convergence Theorem. While it's about as overzealous as using $\frac{\epsilon}{100}$ instead of ϵ, it works.

Lemma 4.4 (The Measurewise Eventually Monotone Convergence Theorem)**.** *Let $\{f_n\}$ a sequence of measurable functions on E, where $E = \bigsqcup_{k=1}^{\infty} A_k$ for $A_k \subset E$ such that $m(A_k) > 0$ for all k. Suppose that $\{f_n\}$ is measurewise eventually nonnegative and increasing in the following sense: for each k, there is an index N_k such that $f_n(x) \geq 0$ and $f_{n+1}(x) \geq f_n(x)$ when $x \in A$ and $n \geq N_k$. If $\{f_n\} \to f$ pointwise on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$.*

Proof. Suppose first that $\{f_n\}$ consists of nonnegative functions. By Fatou's Lemma, $\int_E f \leq \liminf \int_E f_n$. Let $k \in \mathbb{N}$. Then for $n \geq N_k$, $f_n \leq f$ on A_k and hence $\int_{A_k} f_n \leq \int_{A_k} f$. So $0 \leq \limsup \int_{A_k} f_n \leq \int_{A_k} f$. Since this holds for each $k \in \mathbb{N}$,

$$\limsup \int_E f_n = \limsup \sum_{k=1}^{\infty} \int_{A_k} f_n \leq \sum_{k=1}^{\infty} \limsup \int_{A_k} f_n \leq \sum_{k=1}^{\infty} \int_{A_k} f = \int_E f.$$

Since $\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n$, we have $\lim \int_E f_n = \int_E f$.

Now for the general case. Let $k \in \mathbb{N}$. Then when $x \in A$ and $n \geq N_k$, we have $f_n^+(x) = f_n(x) \geq 0$ and $f_{n+1}^+(x) = f_{n+1}(x) \geq f_n(x) = f_n^+(x)$, as well as $f_n^-(x) = 0$. Thus $\{f_n^+\}$ and $\{f_n^-\}$ both satisfy the previous case. Therefore, $\lim_{n\to\infty} \int_E f_n^+ = \int_E f^+ = \int_E f$ and $\lim_{n\to\infty} \int_E f_n^- = \int_E f^- = 0$. Thus,

$$\lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_E (f_n^+ - f_n^-) = \lim_{n\to\infty} \int_E f_n^+ - \lim_{n\to\infty} \int_E f_n^- = \int_E f.$$

□

Solution:
We claim that $g \equiv 0$. First, some bookkeeping. For each $x, y > 0$ and $n \in \mathbb{N}$, let

$$S_{x,y,n} = \frac{1}{nx^2 + n^3x^{-2}y} = \frac{x^2}{nx^4 + n^3y},$$

so that $f(x,y) = \sum_{n=1}^{\infty} S_{x,y,n}$. Additionally, let $T_{x,y,n} = \frac{1}{n^2x^{-4}y}$. To be thorough, we check that f and g are both well-defined. Let $x, y > 0$. Then there is an $N > 0$ such that when $n \geq N$, $\frac{1}{nx^2} \leq 1$. Thus for $n \geq N$,

$$S_{x,y,n} = \frac{1}{nx^2 + n^3x^{-2}y} = \frac{1}{(nx^2)(1 + n^2x^{-4}y)} \leq \frac{1}{1 + n^2x^{-4}y} < \frac{1}{n^2x^{-4}y} = T_{x,y,n}.$$

Since $\sum_{n=N}^{\infty} T_{x,y,n} = \frac{x^4}{y}\sum_{n=N}^{\infty}\frac{1}{n^2} < \infty$, it follows that $f(x,y) = \sum_{n=1}^{\infty} S_{x,y,n} < \infty$. So f is well-defined. Next, note that for for each $y > 0$ and $n \in \mathbb{N}$, $S_{x,y,n}$ decreases as $x \to \infty$. Hence, the entire sum $f(x,y) = \sum_{n=1}^{\infty} S_{x,y,n}$ decreases as $x \to \infty$. Since $f(x,y)$ is bounded below by 0, it follows that $\lim_{x\to\infty} f(x,y)$ exists as a real number and thus that $g(y)$ is well-defined.

Now to business. Fix $y > 0$. For each $k \in \mathbb{N}$, define $\varphi_k : [1,\infty) \to \mathbb{R}$ by $\varphi_k(t) = S_{k,y,n}$ for $t \in [n, n+1)$. Since for $t \in [n, n+1)$ we have that $\varphi_k(t) = S_{k,y,n}$ decreases to 0 as $k \to \infty$, the sequence $\{\varphi_1 - \varphi_k\}$ is measurewise eventually nonnegative and increasing. Additionally, $(\varphi_1 - \varphi_k) \to \varphi_1$ since $\varphi_k \to 0$. Therefore, by the Measurewise Eventually Monotone Convergence Theorem,

$$\int_1^{\infty} \varphi_1 - \lim_{k\to\infty}\varphi_k = \lim_{k\to\infty}\int_1^{\infty}(\varphi_1 - \varphi_k) = \int_1^{\infty}\varphi_1. \tag{7}$$

But

$$\int_1^{\infty}\varphi_1 = \sum_{n=1}^{\infty}\int_n^{n+1}\varphi_1 = \sum_{n=1}^{\infty}\int_n^{n+1} S_{1,y,n} = \sum_{n=1}^{\infty} S_{1,y,n} = f(x,y) < \infty.$$

So we can cancel both sides of (7) to obtain $\lim_{k\to\infty}\int_1^{\infty}\varphi_k = 0$. Finally,

$$\begin{aligned} g(y) = \lim_{x\to\infty} f(x,y) = \lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} f(k,y) = \lim_{\substack{k\to\infty \\ k\in\mathbb{N}}}\sum_{n=1}^{\infty} S_{k,y,n} &= \lim_{\substack{k\to\infty \\ k\in\mathbb{N}}}\sum_{n=1}^{\infty}\int_n^{n+1} S_{k,y,n} \\ &= \lim_{k\to\infty}\sum_{n=1}^{\infty}\int_n^{n+1}\varphi_k = \lim_{k\to\infty}\int_1^{\infty}\varphi_k = 0. \end{aligned}$$

Since y was arbitrary, $g \equiv 0$ as claimed.

□

Problem 4.5 (Texas A & M, Page 2 Q5)**.** *Let $f : \mathbb{R} \to [0, \infty]$ be a Lebesgue measurable extended real valued function. Define the measure μ by*

$$\mu(E) = \int_E f dx.$$

Show that μ is σ-finite iff $|f(x)| < \infty$ Lebesgue a.e.

Solution. [Andrew Pensoneault]
Attempt:
Before I started, I needed the definition of σ-finite which according to Chapter 17 in Royden, a measure space $(X, \mathscr{M}, \mu)$ is σ-finite if X is the countable union of finite measure sets. In our case, the measure space is $(\mathbb{R}, \mathscr{M}, \mu)$. I realized in the forward direction, showing that $|f(x)| < \infty$ Lebesgue a.e. most likely would involve contradiction. We show somehow assuming the opposite would result in at least one of the finite measure sets which make up $\mathbb{R}$ having infinite measure. In the backwards direction, we would be most likely doing a construction proof involving doubly indexed sequence of sets which cover all of $\mathbb{R}$ in such a way that takes advantage of the fact f is a measurable function (and thus sets in the form $\{x|f(x) < c\}$ are measurable).
Solution:

Assume that μ is σ-finite, thus $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$. Assume for contradiction the set $\tilde{E} = \{x|f(x) = \infty\}$ is of positive measure. Assume for contradiction, $m(\tilde{E} \cap E_k) = 0$ for all $k \in \mathbb{N}$. As $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$, $m(\tilde{E}) = m\big(\bigcup_{k=1}^{\infty}(\tilde{E} \cap E_k)\big) = \sum_{k=1}^{\infty} m(E_k \cap \tilde{E}) = 0$ which is a contradiction. Thus there is at least one E_k such that $m(\tilde{E} \cap E_k) > 0$. Now, as μ is σ-finite, we have $\mu(\tilde{E} \cap E_k) < \mu(E_k) < \infty$. As $f(x)$ is constantly infinite on $\tilde{E} \cap E_k$,

$$\mu(\tilde{E} \cap E_k) = \int_{\tilde{E} \cap E_k} f(x)dx = \infty.$$

Also, we have $\mu(\tilde{E} \cap E_k) < \mu(E_k)$, which implies $\mu(E_k) = \infty$. This is a contradiction as we assume $\mu(E_k) < \infty$. Thus, $|f(x)| < \infty$ Lebesgue a.e.

Assume $|f(x)| < \infty$ Lebesgue a.e. Now, construct the set $E_{i,j} = \{x|f(x) < i\} \cap (0, j)$. As f is a Lebesgue measurable function, this will be a measurable set. Define $V = \{x|f(x) = \infty\}$ and notice $\mu(V) = 0$. By construction, $\mathbb{R} = \bigcup_{i,j=1}^{\infty} E_{i,j} \cup V$. Also, we have

$$\mu(E_{i,j}) = \int_{E_{i,j}} f(x)dx < ij.$$

As $\mathbb{R}$ is the countable collection of finite measure sets, μ is σ-finite. □

Problem 4.6 (Texas A & M, Page 2 Q6)**.** *For A and B, subsets of $\mathbb{R}^2$,define $A + B = \{x + y : x \in A$ and $y \in B\}$.*
(a) Let A and B be compact subsets of $\mathbb{R}^2$. Show that $A + B$ is compact.
(b) Let A and B be closed subsets of $\mathbb{R}^2$. Show that $A + B$ is not necessarily closed.

Solution. [Yanqing Shen]

Attempt:
In order to prove the compactness, we could use some techniques like finite open cover(sometimes together with prove by contradiction) or the equivalent argument of compactness of a metric space(i.e. complete/totally bounded or sequentially compact arguments). In addition, since

we are working on a subset of a complete Euclidean space $\mathbb{R}^2$, we might have some even stronger propositions applying here. But my thoughts was still starting with sequences in our target set $A+B$.

Solution:

Let $\{(x_k, y_k)\}$ be an arbitrary sequence in $A+B$. By the definition of $A+B$, we know that for all k, $\exists (a_k, b_k) \in A, (c_k, d_k) \in B$ such that (x_k, y_k) can be separated into these two points (a_k, b_k) and (c_k, d_k) (i.e. we have $x_k = a_k + c_k$ and $y_k = b_k + d_k$ for all k).

Thus we get two sequence $\{(a_k, b_k)\} \subseteq A$ and $\{(c_k, d_k)\} \subseteq B$ respectively. However, by the compactness of A, we know that $\exists$ a convergent subsequence of $\{(a_k, b_k)\}$ which convergent to a point in $(a, b) \in A$. Denote this subsequence to be $\{(a_{k_i}, b_{k_i})\}$. From this subsequence if we look to the corresponding subsequence in $\{(c_k, d_k)\}$, which can be denoted as $\{(c_{k_i}, d_{k_i})\}$. Because of the compactness of B, this subsequence $\{(c_{k_i}, d_{k_i})\}$ will also have a convergent subsequence $\{(c_{k_{i_j}}, d_{k_{i_j}})\}$, call the limit point (c, d) which must in B.

So next, if we retrospect from $\{(c_{k_{i_j}}, d_{k_{i_j}})\}$ to the corresponding subseqence in $\{(a_{k_i}, b_{k_i})\}$,to get $\{(a_{k_{i_j}}, b_{k_{i_j}})\}$. This $\{(a_{k_{i_j}}, b_{k_{i_j}})\}$ is also a convergent sequences which also convergent to (a, b). Avoid being lengthy, denote $a_{k_{i_j}}$ as a_n. Therefore we have two convergent sequences $\{(a_n, b_n)\} \subseteq A$ and $\{(c_n, d_n)\} \subseteq B$, with $\{(a_n, b_n)\} \to (a, b)$ and $\{(c_n, d_n)\} \to (c, d)$, when $n \to \infty$.

In addition, since $\{(a_n + c_n, b_n + d_n)\} = \{(x_n, y_n)\}$, then $\{(x_n, y_n)\}$ is a subsequence of $\{(x_k, y_k)\}$.

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty}(a_n + c_n) \\ &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} c_n \quad \text{by the fact that both limits exist} \\ &= a + c\end{aligned}$$

Similarly $\lim_{n\to\infty} y_n = b + d$. Therefore $\{(x_n, y_n)\}$ is a convergent sequence and the limit point of this sequence is $(a + b, c + d)$. And

$$(a + b, c + d) = (a, b) + (c, d)$$

with $(a, b) \in A$,$(c, d) \in B$, thus $(a + b, c + d) \in A + B$. We have $\{(x_n, y_n)\}$ converges to a point in $A+B$.

So we just showed that for any sequence in $A+B$, it will have a subsequence that converges to a point in $A+B$ (i.e. $A+B$ is sequentially compact), therefore $A+B$ is compact.

For part (b), let $A = \{x, \frac{1}{x}, x > 0\}, B = \{x, -\frac{1}{x}, x > 0\}$. And A, B are both closed sets since there is no limit point outside them.

Choosing $(a, \frac{1}{a}) \in A$ and $(a, -\frac{1}{a}) \in B$ for some positive a we will get

$$(a, \frac{1}{a}) + (a, -\frac{1}{a}) = (2a, 0) \in A + B, \quad a > 0$$

This implies that the set $\{(x,0), x>0\} \subseteq A+B$. And $(0,0)$ is a limit point of $\{(x,0), x>0\}$, so $(0,0)$ is a limit point of $A+B$. While $(0,0) \notin A+B$, hence $A+B$ is not closed.

□

Problem 4.7 (Andrew Pensoneault, Spring 2001 Q5)**.** *Suppose $\{a_n\}$ is a decreasing sequence of positive numbers and $\sum_{n=0}^{\infty} a_n < \infty$. Show* $\lim_{n\to\infty} na_n = 0$.

Solution.
Attempt:
I started trying to show that there must exist an index such that for all $n > N$, $a_n < \frac{1}{n}$ by contradiction, so assuming there was infinitely many indices such that $a_n \geq \frac{1}{n}$. Then, I converted this sum into an integral, and tried to show that this sum was bounded below by infinity. This was difficult because I was dealing with an arbitrary subsequence.

I next tried to show that a series involving partial sums goes to zero, which could be converted to na_n.
Solution:
Let $S_n = \sum_{i=0}^{n} a_n$ and notice $K_n = S_{2n} - S_n = \sum_{i=n}^{2n} a_i < \sum_{i=n}^{\infty} a_i$, thus we have $\lim_{n\to\infty} K_n = 0$. Since the a_n are decreasing and convergent, we can find an index N such that if $n > N$

$$na_{2n} < \sum_{i=n}^{2n} a_i = K_n < \frac{\epsilon}{2}.$$

Thus, if $n > N$,

$$2na_{2n} < \epsilon.$$

Now, define $R_n = S_{2n+1} - S_n = \sum_{i=n}^{2n+1} a_i < \sum_{i=n}^{\infty} a_i$, so $\lim_{n\to\infty} R_n = 0$. Since the a_n are decreasing and convergent, we can find an index N such that if $n > N$

$$(n+1)a_{2n+1} < \sum_{i=n}^{2n+1} a_i = R_n < \frac{\epsilon}{2}.$$

Thus, we have

$$(2n+1)a_{2n+1} < (2n+2)a_{2n+1} < \epsilon.$$

Since these two subsequence compose all of $\{na_n\}$, we have $\lim_{n\to\infty} na_n = 0$. □

Problem 4.8 (Kaitlin Healy, Fall 2001 Q7)**.** *Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions on X such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu < \infty$. Prove that $\lim_E f_n d\mu = \int_E f d\mu$ for every measurable subset $E \subseteq X$.*

Solution.
Attempt:
I started by writing the integral over E as the integral over X minus the integral over E^c using additivity over domains. I also knew I would need to use the fact that this sequence of functions was nonnegative, which led me to think Fatou's Lemma.

Solution:
Let $E \subseteq X$. Since the sequence $\{f_n\}$ is a nonnegative measurable sequence and $f_n \to f$ pointwise on E a.e., we can apply Fatou's Lemma to see

$$\int_E f \leq \liminf \int_E f_n \leq \limsup \int_E f_n$$

Notice the following by properties of liminfs and limsups:

$$\begin{aligned}\limsup \int_E f_n &= \limsup \left[\int_X f_n - \int_{E^c} f_n\right] \\ &\leq \limsup \int_X f_n + \limsup \int_{E^c} -f_n \\ &= \limsup \int_X f_n - \liminf \int_{E^c} f_n\end{aligned}$$

Again applying Fatou's Lemma along with the problem statements, this gives

$$\begin{aligned}\limsup \int_E f_n &\leq \limsup \int_X f_n - \liminf \int_{E^c} f_n \\ &\leq \int_X f - \int_{E^c} f \\ &= \int_E f\end{aligned}$$

This gives us that $\limsup \int_E f_n$ is both less than or equal to and greater than or equal to $\int_E f$. Thus, the two values are equal. Since $\limsup f_n$ exists, we can conclude that for any $E \subseteq X$,

$$\limsup \int_E f_n = \lim_{n\to\infty} \int_E f_n = \int_E f$$

□

Problem 4.9 (Elaina Aceves, 2000 Q7)**.** *Prove an orthonormal set in a separable Hilbert space is at most countable.*

Solution.
Attempt:
This is similar to Andrew's proof from Kansas Fall 2011 problem 5.
Solution:
Let H be the Hilbert space. BWOC, let O be an uncountable orthonormal set in H. Since H is separable, there exists a countable dense subset of H, call it S. We have that $\forall\, x, y \in O$,

$$||x-y||^2 = < x-y, x-y > = < x,x > -2 < x,y > + < y,y > = < x,x > + < y,y > = 2$$

since O is an orthonormal set. Thus, $||x-y|| = \sqrt{2}$. Therefore, the collection of balls $\{B(x, \sqrt{2}/3)\}_{x\in O}$ (center x, radius $\sqrt{2}/3$) are disjoint. Since S is dense, there exists $y_x \in B(x, \sqrt{2}/3)$ such that $y_x \in S$. Then $\{y_x\}_{x\in O}$ is an uncountable set since O is uncountable, but $\{y_x\}_{x\in O} \subset S$ and S is countable, a contradiction. Hence O must be at most countable. □

Problem 4.10 (W. Tyler Reynolds, Texas A&M, Sep 2002 Q2). *Suppose $(X, \mathscr{M}, \mu)$ is a measure space, μ a σ-finite measure, and $f : X \to [0, \infty]$ is measurable. Suppose that $\int_A f\, d\mu = \mu(A)$ for each measurable set A with $\mu(A) < \infty$. Prove that $f = 1$ a.e.*

Solution.
Attempt:
On the surface this was a fairly straightforward application of some of the principles we've learned about measure and integration. I had to be careful to qualify my assumptions, however. It seems feasible that there exists a measure in which some infinite measure set has no positive-measure, finite-measure subsets. Thus, when my proof called for such subsets to exist, I had to make sure that this would follow from the σ-finite property of the given measure. As an additional note, the continuity of measure in a general measure space had to be checked, and is okay to use by a theorem in Royden.

Lemma 4.11 (The Once Finite Always Finite Lemma). *Let $(X, \mathscr{M}, \mu)$ be a measure space, where μ is σ- finite. If $\mu(E) > 0$, then there is some $A \subset E$ with $0 < \mu(A) < \infty$.*

Proof. Since μ is σ-finite, there is a countable collection $\{A_n\}_{n=1}^{\infty}$ of finite measure sets such that $X = \bigcup_{n=1}^{\infty} A_n$. If $\mu(A_n \cap E) = 0$ for each n, then we would have $\mu(E) \leq \sum_{n=1}^{\infty} \mu(A_n \cap E) = 0$, a contradiction. So there is some n with $\mu(A_n \cap E) > 0$. Since $\mu(A_n) < \infty$, we also have $\mu(A_n \cap E) < \infty$. So letting $A = A_n \cap E \subset E$, we have $0 < \mu(A) < \infty$, as desired. □

Solution:
Let $E = \{x \in X | f(x) \neq 1\}$, $F_1 = \{x \in X | f(x) < 1\}$ and $F_2 = \{x \in X | f(x) > 1\}$. Then $E = F_1 \sqcup F_2$. Suppose for the sake of contradiction that $\mu(E) = \mu(F_1) + \mu(F_2) > 0$. If $\mu(F_1) > 0$, then there must be some $0 \leq a_1 < 1$ such that $\mu\{x \in X | f(x) < a_1\} > 0$, since otherwise $\mu(F_1) = \lim_{a \to 1^-} \mu\{x \in X | f(x) < a\} = 0$ by continuity of measure (Royden Chapter 17 Proposition 2). By the Lemma, we can find some $A_1 \subset \{x \in X | f(x) < a_1\}$ with $0 < \mu(A_1) < \infty$. Note that

$$\int_{A_1} f \leq \int_{A_1} a_1 = a_1 \mu(A_1) < \mu(A_1).$$

This is impossible, so we must have $\mu(F_1) = 0$ and thus $\mu(F_2) > 0$. Thus there must be some $a_2 > 1$ such that $\mu\{x \in X | f(x) > a_2\} > 0$, since otherwise $\mu(F_2) = \lim_{a \to 1^+} \mu\{x \in X | f(x) > a\} = 0$ (again by continuity of measure). By the Lemma, we can find some $A_2 \subset \{x \in X | f(x) > a_2\}$ with $0 < \mu(A_2) < \infty$. Note that

$$\int_{A_2} f \geq \int_{A_2} a_2 = a_2 \mu(A_2) > \mu(A_2).$$

This is another impossibility. We can thus conclude that $\mu(E) = 0$. It follows that $f = 1$ a.e.

□

Problem 4.12 (TAMU, August 2009 Q4)**.** *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Given sets $A_i \in \Sigma$, $i \geq 1$ prove that*

$$\mu(\cap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(\cap_{i=1}^{n} A_i).$$

Give an example to show that this need not hold when $\mu(X) = \infty$.

Solution. [Michael Kratochvil]
Attempt:
The measure of the space being finite reminded me of the proof in Royden for continuity of measure for intersections of sets, so I restructured the sets to make it look like that. The example I provided is one we have definitely seen before and I just simply remembered.
Solution:
Define $E_n = X - \cap_{i=1}^{n} A_i$ so that the E_n's are ascending. Then by continuity of measure, we have

$$\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

But

$$\begin{aligned}\mu(\cup_{n=1}^{\infty} E_n) =& \mu(\cup_{n=1}^{\infty}[X - \cap_{i=1}^{n} A_i]) \\ =& \mu(X - \cap_{i=1}^{n} A_i) \text{ by De Morgan's Identities} \\ =& \mu(X) - \mu(\cap_{i=1}^{\infty} A_i) \text{ by Excision.}\end{aligned}$$

And

$$\begin{aligned}\lim_{n\to\infty} \mu(E_n) =& \mu(X - \cap_{i=1}^{n} A_i) \\ =& \lim_{n\to\infty}[\mu(X) - \mu(\cap_{i=1}^{n} A_i)] \text{ by Excision} \\ =& \mu(X) - \lim_{n\to\infty} \mu(\cap_{i=1}^{n} A_i).\end{aligned}$$

Thus,

$$\begin{aligned}\mu(X) - \mu(\cap_{i=1}^{\infty} A_i) &= \mu(X) - \lim_{n\to\infty} \mu(\cap_{i=1}^{n} A_i) \\ &\Rightarrow \mu(\cap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(\cap_{i=1}^{n} A_i).\end{aligned}$$

As a counterexample, when $\mu(X) \not< \infty$, let $X = \mathbb{R}$, $\mu = \lambda$ and $A_i = [i, \infty)$. Then $\cap_{i=1}^{\infty} A_i = \emptyset$ and $\lambda(\emptyset) = 0$. But $\cap_{i=1}^{n} A_i = A_n$ and $\lambda(A_n) = \infty$, so $\lim_{n\to\infty} \lambda(A_n) = \infty$. □

Problem 4.13 (TAMU, August 2009 Q6)**.** *Let $\ell^2(\mathbb{Z})$ denote the real Hilbert space of square-summable functions on the integers. Let $x_k (\geq 1)$ be a sequence in $\ell^2(\mathbb{Z})$ that converges coordinate-wise to zero, ie, such that $\lim_{k\to\infty} x_k(n) = 0$ for all $n \in \mathbb{Z}$.*

Must x_k converge in norm to 0 as $k \to \infty$? What about if $||x_k||$ is assumed to be bounded?
Must x_k converge weakly to 0 as $k \to \infty$? What about if $||x_k||$ is assumed to be bounded?
Justify your answers by proof or counter-example.

Solution. [Amrei Oswald]
Attempt:
A few definitions are helpful for this problem. For $x, y \in \ell^2(\mathbb{Z})$, the norm of x is $||x|| = \left(\sum_{k\in\mathbb{Z}} (x(k))^2\right)^{1/2}$ and the inner product of x and y is $< x, y >= \sum_{k\in\mathbb{Z}} x(k)y(k)$. A sequence $\{x_i\} \subset \ell^2(\mathbb{Z})$ *converges weakly* to $x \in \ell^2(\mathbb{Z})$ if for every $y \in \ell^2(\mathbb{Z})$, $\lim_{i\to\infty} < x_i, y >=< x, y >$.
Solution:
Define the sequence of functions x_i by

$$x_i(k) = \begin{cases} \frac{1}{k-i} & |i| \neq k \\ 0 & |i| = k \end{cases}.$$

Then we have

$$||x_k||^2 = \sum_{k\in\mathbb{Z}} |x_i(k)|^2 = \sum_{k\in\mathbb{Z}} \frac{1}{(k-i)^2} = \sum_{k=i+1}^{\infty} \frac{1}{(k-i)^2} + \sum_{k=i-1}^{-\infty} \frac{1}{(k-i)^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=-1}^{-\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

The above gives us that $x_i \in \ell^2(\mathbb{Z})$ and $||x_i|| = ||x_j||$ for every $i, j \in \mathbb{N}$. Therefore, $||x_i||$ is bounded. Further, we have

$$\lim_{i\to\infty} x_i(k) = \lim_{i\to\infty} \frac{1}{i-k} = 0.$$

However,

$$\lim_{i\to\infty} ||x_i|| = \lim_{i\to\infty} \left(\sum_{k\in\mathbb{Z}} |x_i(k)|^2\right)^{1/2} = \lim_{i\to\infty} \left(2\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2} = \left(2\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2} > 0.$$

Thus, the x_i are a bounded sequence of functions in $\ell^2(\mathbb{Z})$ such that $\lim_{i\to\infty} x_i(k) = 0$ for every $k \in \mathbb{Z}$ that does not converge to 0 in norm as $i \to \infty$.

Define $y : \mathbb{Z} \to \mathbb{R}$ such that $y(k) = \frac{1}{k}$. Then,

$$\sum_{k\in\mathbb{Z}} (y(k))^2 = \sum_{k\in\mathbb{Z}} \frac{1}{k^2} < \infty \implies y \in \ell^2(\mathbb{Z}).$$

Then, for every $i \in \mathbb{N}$, we have

$$\langle x_i, y\rangle = \sum_{k\in\mathbb{Z}} x_i(k)y(k) = \sum_{k\in\mathbb{Z}} \frac{1}{(k-i)k} = \sum_{k=i+1}^{\infty} \frac{1}{(k-i)k} + \sum_{k=i-1}^{-\infty} \frac{1}{(k-i)k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k(k+i)} + \sum_{k=-1}^{-\infty} \frac{1}{k(k+i)} = \sum_{k=1}^{\infty} \frac{1}{k(k+i)} + \sum_{k=1}^{\infty} \frac{1}{k(k-i)} = \sum_{k=1}^{\infty} \frac{2k}{(k^2-i^2)}$$

$$= \sum_{k=1}^{i-1} \frac{2k}{k^2 - i^2} + \sum_{k=i+1}^{\infty} \frac{2k}{k^2 - i^2} = \sum_{k=1}^{i-1} \frac{2k}{k^2 - i^2} + \sum_{k=1}^{\infty} \frac{2(k+i)}{k(k+2i)} \geq \sum_{k=1}^{i-1} \frac{2k}{k^2 - i^2} + \sum_{k=1}^{\infty} \frac{1}{k}.$$

Here, the last inequality follows from the fact that $\frac{k+i}{k+2i} \geq \frac{1}{2}$ for every $k \in \mathbb{N}$. Since the sum $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the x_i do not converge weakly to 0. □

Problem 4.14 (TAMU, Sep 2009, Q6)**.** *Let λ be Lebesgue measure on $\mathbb{R}$. Is it true that $\lambda(F \backslash int F) = 0$ for every closed set $F \subseteq \mathbb{R}$?*

Solution. [Yanqing Shen]
Attempt:
This problem required us to prove of disprove that a closed set in $\mathbb{R}$ has a measure zero boundary. We already know that the Cantor set C has an empty interior. Thus the boundary of the Cantor set is itself. However, the measure of the Cantor set is 0 on $[0, 1]$. A slight modify is required here.
Solution:
The answer of this argument is NOT true, and below is a counterexample:

Consider the generalized Cantor set, which is constructed in the same manner as the Cantor set.

The only difference we make for the generalized Cantor set is that each of the intervals removed at the n-th deletion stage has length $\alpha \cdot 3^{-n}$, with $\alpha \in (0, 1)$. i.e., every interval we removed from the Cantor set was timed by a constant α in this case.

Similarly as the Cantor set, we have some same properties for this generalized Cantor set, denoted as F.

1. F is a closed set.

   Since $F = \cap F_n$, which F_n denotes as the closed set we get after n-th stage.

2. $\text{int}\, F = \emptyset$.

   BWOC, assume $\exists\, (a, b) \subset F = \cap F_n$, this implies $(a, b) \subset F_n \;\; \forall n \in \mathbb{N}$.

   And F_n is a disjoint union of 2^n closed intervals with same length l_n as

   $$l_n = \frac{1 - \alpha}{2^n} + \frac{1}{3^n},$$

   i.e.,

   $$F_n = \sqcup_{k=1}^{2^n} E_k \;\text{ with }\; l(E_k) = l_n \;\; \forall\, k$$

   In addition, $\lim_{n \to \infty} l_n = 0$.

   However since $(a, b) \subset F_n \;\; \forall n$, then $\exists$ one of these closed intervals $\tilde{E}_n \subset F_n = \sqcup_{k=1}^{2^n} E_k$, s.t.

   $$(a, b) \subset \tilde{E}_n \;\; \forall\, n.$$

Since $l(\tilde{E}_n) = l_n \ \forall n$, and $\lim_{n\to\infty} l_n = 0$, therefore $\exists$ a number N, s.t.

$$l_N = \frac{1-\alpha}{2^N} + \frac{1}{3^N} < b - a = l(a,b)$$

i.e., we can not find a closed interval $\tilde{E}_N \subset F_N$ such that $(a,b) \subset \tilde{E}_N$. Hence this is a contradiction.

Therefore, $\operatorname{int} F = \emptyset$.

3. $\lambda(F) = 1 - \alpha$.

Computer this by consider the set we removed from the $[0,1]$ and computer its measure(length).

Denote $O = [0,1] \setminus F$ and $O = \sqcup O_n$ with O_n is the open set consist by 2^{n-1} open intervals we delete in n-th stages.

Therefore

$$\begin{aligned}
\lambda(F \setminus \operatorname{int} F) = \lambda(F \setminus \emptyset) = \lambda(F) = \lambda([0,1] \setminus O) &= \lambda([0,1]) - \lambda(O) \\
&= 1 - \lambda(\sqcup O_n) \\
&= 1 - \sum \lambda(O_n) \\
&= 1 - \sum l(O_n) \\
&= 1 - \sum \left(\alpha \cdot \tfrac{1}{3} + \cdots + 2^{n-1} \cdot \alpha \cdot \tfrac{1}{3^n} + \cdots\right) \\
&= 1 - \sum_{1}^{\infty} \alpha \cdot \tfrac{2^{n-1}}{3^n} \\
&= 1 - \alpha \cdot \sum_{1}^{\infty} \tfrac{2^{n-1}}{3^n} \\
&= 1 - \left(\alpha \cdot \tfrac{\frac{1}{3}}{1-\frac{2}{3}}\right) \\
&= 1 - \alpha
\end{aligned}$$

And since $\alpha \in (0,1)$, then $\lambda(F) = 1 - \alpha > 0$.

By $1, 2, 3$, we showed that the generalized Cantor set F has the properties that $\lambda(F \setminus \operatorname{int} F) > 0$, and F is a closed set. $\square$

Problem 4.15 (TAMU R, January 2010 Q1)**.** *Is it possible to find uncountably many disjoint measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure?*

Solution. [Sara Reed] No. By way of contradiction, assume the collection of disjoint measurable subsets $\{M\}_{\alpha \in A}$ is uncountable. Note that we can write $\mathbb{R}$ as the union of disjoint intervals of measure 1 in the following way $\mathbb{R} = \cup_{i\in\mathbb{Z}}[n, n+1)$. Let $I_i = [i, i+1)$. Since each M_α has strictly positive measure, there exists $i_\alpha \in \mathbb{Z}$ such that $m(M_\alpha \cap I_{i_\alpha}) > 0$. Since there are a countable number of I_i and an uncountable number of M_α, by the Pigeonhole Principle, there must exist i^* such that there are an uncountable number of $M_{\beta_{\beta\in B}}$ such that $m(M_\beta \cap I_{i^*}) > 0$. Since $\{M_\beta\}$ are disjoint, we know $\{M_\beta \cap I_{i^*}\}$ are disjoint. We also know $\cup_{\beta\in B} M_\beta \cap I_{i^*} \subseteq I_{i^*}$. Therefore, by additivity of measure over disjoint sets, we have

$$m(\bigcup_{\beta\in B} M_\beta \cap I_{i^*}) = \sum_{\beta\in B} m(M_\beta \cap I_{i^*}) \leq m(I_{i^*}) = 1 < \infty.$$

We have shown that $\sum_{\beta\in B} m(M_\beta \cap I_{i^*}) < \infty$. Therefore, it must be the case that B is countable, a contradiction. Therefore, $\{M\}_{\alpha\in A}$ is countable. We conclude that it is not possible to find uncountably many disjoint measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure. □

Problem 4.16 (Texas A& M Real, August 2009 Q3). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonzero elements of $L^2[0,1]$. Prove that there exists $g \in L^2[0,1]$ such that for all $n \geq 1$, we have $\int_0^1 g(x) f_n(x) dx \neq 0$*

Solution. [Jared Grove]
Attempt:
Not really anything. I asked Rolando
Solution:
Gram-Schmidt: When you want to make a system of orthogonal vectors out of a collection of vectors:

$$\begin{aligned} u_k &= v_k - \sum_{j=1}^{k-1} proj_{u_j}(v_k) \\ e_k &= \frac{u_k}{||u_k||} \end{aligned}$$

⇒ Since we don't have much information about the f_n we will consider two cases. The first being that the f_n are othogonal to each other and the second being that they aren't. If they are all orthogonal we know that $\langle f_n, f_m\rangle = 0$ when $m \neq n$, so define

$$g = \sum_{n=1}^{\infty} \frac{f_n}{2^n ||f_n||}$$

Then we will have $\int_0^1 g f_n dx = \langle g, f_n\rangle = \sum_{n=1}^{\infty} \frac{\langle f_m, f_n\rangle}{2^m ||f_m||} = \frac{\langle f_n, f_n\rangle}{2^n ||f_n||} \neq 0$ for all n.

Now we will consider the case where they are not orthogonal. We will proceed by making an orthonormal basis that will span everything that the f_n span. This will happen by following the Gram-Schmidt process outlined above:

$$\begin{aligned}
e_1 &= \frac{f_1}{||f_1||} \\
u_2 &= f_2 - \frac{\langle f_1, f_2\rangle}{||f_1||^2} f_1 = f_2 - \langle f_2, e_1\rangle e_1 \\
e_2 &= \frac{u_2}{||u_2||}, \text{ if } u_2 \neq 0, \text{ else ignore.} \\
u_3 &= f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2 \\
e_3 &= \frac{u_3}{||w_3||}, \text{ if } u_3 \neq 0, \text{ else ignore.} \\
u_k &= f_k - \sum_{j=1}^{k-1} \langle f_k, e_j\rangle e_j \\
e_k &= \frac{u_k}{||u_k||}, \text{ if } u_k \neq 0, \text{ else ignore.}
\end{aligned}$$

Note that we ignore the various e_k when $w_k = 0$. This is because when $w_k = 0$ there was some e_m with $m < k$ such that f_k was othogonal to e_m. This means that f_k is just some linear multiple of f_m and has already been accounted for. Furthermore we have that each $e_n \in L^2[0,1]$ as they are completely defined based $f_n \in L^2[0,1]$. Now we will define

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n$$

where the e_n are all of the e_k that were not ignored. Hence we will have $g \in L^2[0,1]$ as well. Next we will seen that we also have

$$\begin{aligned}
||g||_2^2 &= \int_0^1 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{2^n}\frac{1}{2^m} e_m e_n \\
&= \sum_{n,m=1}^{\infty} \frac{1}{2^{n+m}} \int_0^1 e_n e_m \\
&= \sum_{n,m=1}^{\infty} \frac{1}{2^{n+m}} \delta_{n,m} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \\
&< \infty
\end{aligned}$$

This is because each e_n is orthogonal by construction and $\langle e_n, e_m\rangle = 0$ when $n \neq m$. Since each f_n is not orthogonal to one of the e_k of g we have that (assuming we didn't throw out any of the e_k for simplicity) $\int_0^1 g(x) f_n(x) dx = \langle g, f_n\rangle = \sum_{m=1}^{\infty} \frac{\langle e_m, f_n\rangle}{2^m} = \frac{\langle e_n, f_n\rangle}{2^n} \neq 0$ for all $n \geq 1$. This is because $\langle f_n, e_m \rangle = 0$ for $n \neq m$. For example: $\langle f_1, e_1 \rangle = \frac{\langle f_1, f_1 \rangle}{||f_1||} \neq 0$. But $\langle f_1, e_2 \rangle = \frac{\langle f_1, f_2 \rangle - \frac{\langle f_2, f_1 \rangle}{||f_1||^2}\langle f_1, f_1 \rangle}{||u_2||} = \frac{\langle f_1, f_2 \rangle - \langle f_2, f_1 \rangle}{||u_2||} = 0$. If you want you can probably make an induction argument about this process working, but I leave that as an excercise for the reader. □

Problem 4.17 (W. Tyler Reynolds, Texas A&M, Aug 2010 Q2)**.** *Let E be a subset of $[0,1]$ with positive outer Lebesgue measure, i.e. $m^*(E) > 0$. Show that for each $\alpha \in (0,1)$ there is an interval $I \subset [0,1]$ so that*

$$m^*(E \cap I) \geq \alpha \ell(I).$$

Solution.
Attempt:
My first instinct was to try a constructive proof based around the definition of outer measure. After not seeing anything promising with this approach, I quickly moved to a proof by contradiction. The advantage of this approach was that instead of trying to pull together infinitely many intervals somehow to construct one interval with a particular property, I could obtain a property that would hold for all of the intervals and then use that to find a contradiction about the whole structure of the set being measured. (For better or worse, I also have a much better knack for finding logical errors than for thinking constructively. If anyone comes up a constructive proof, I'd be happy to see it).
Solution:
Let $\alpha \in (0,1)$ and suppose for the sake of contradiction that for each interval $I \subset [0,1]$ we have $m^*(E \cap I) < \alpha \ell(I)$. Notice that since $E \subset [0,1]$, $m^*(E) \leq 1 < \infty$. Let $\epsilon > 0$. By the definition of outer measure, we can find a countable collection $\{I_k\}_{k=1}^{\infty}$ with $E \subset \bigcup_{k=1}^{\infty} I_k$ such that $\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \frac{\epsilon}{\alpha}$. By countable subadditivity of outer measure,

$$m^*(E) \leq \sum_{k=1}^{\infty} m^*(E \cap I_k) \leq \alpha \sum_{k=1}^{\infty} \ell(I_k) < \alpha(m^*(E) + \frac{\epsilon}{\alpha}) = \alpha m^*(E) + \epsilon.$$

Since ϵ was arbitrary, this implies that $m^*(E) \leq \alpha m^*(E)$, a contradiction since $m^*(E) > 0$ and $\alpha \in (0,1)$. It follows that there is some interval $I \subset [0,1]$ with $m^*(E \cap I) \geq \alpha \ell(I)$. □

Problem 4.18 (Kansas, Fall 2013, Q1)**.** *Let X be Banach space. $\{x_n\}$ be a sequence from X that converges weakly to 0. Prove that, the sequence $\{||x_n||\}$ is bounded.*

Solution. [Yanqing Shen]
Attempt:
For this problem, we might need to apply some fundamental results in functional analysis, i.e., *Principle of Uniformly Boundedness* and *Banach-Steinhaus Theorem.* And after talked with Rolando about a few concepts in dual space. We can obtain the proof by the following argument.

Solution:
A few preliminary concepts in this problem:

1. By the definition of $\{x_n\}$ converges weakly to 0.

   We have $\lim_{n\to\infty} f(x_n) = 0, \ \forall f \in X^*$. Note: X^* is the dual space of X as the collection of bounded linear functional on X.

   i.e., f here is defined by $f : X \to F$ and F is some vector space(F could be $\mathbb{R}$ or $\mathbb{C}$).

2. Since X^* is a vector space, therefore we can take its dual again, which is called the double dual of X, say X^{**}.

   And there is a naturally defined continuous linear operator from a normed space to its double dual

   $$\begin{aligned} \varphi : X &\longrightarrow X^{**} \\ x &\longmapsto \varphi_x \end{aligned}$$

   such that $\varphi_x(f) = f(x), \ \forall x \in X, \ \forall f \in X^*$.

   A few remarks below:

   X^{**} can be described as the set $\{\varphi : X^* \longrightarrow F, \text{ with } \varphi \text{ linear functional}\}$,
   The induced map $x \longmapsto \varphi_x$ is injective,
   We regard this map as canonical embedding into the double dual is isometry,
   $||\varphi_x|| = ||x||$ (by *Hahn-Banach Theorem*).

   And we will introduce this map φ in this problem.

3. A consequence of Uniform Boundedness Principle

   Theorem[Banach-Steinhaus]:

   X be a Banach space, Y a normed linear space, and $\{T_n : X \to Y\}$ a sequence of continuous linear operators. Suppose that $\forall x \in X, \ \lim_{n\to\infty} T_n(x)$ exist in Y.

   Then $\{T_n : X \to Y\}$ is uniformly bounded. Furthermore, the operator $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} T_n(x), \ \forall x \in X$ is linear and continuous.

Consider the weakly convergent sequence $\{x_n\}$ in X, and for each x_n, $\exists \varphi_{x_n}$ s.t.

$$\varphi_{x_n}(f) = f(x_n), \ \forall f \in X^*.$$

with $\{\varphi_{x_n}\}$ a sequence of continuous linear operators.
And $\lim_{n\to\infty} \varphi_{x_n}(f) = \lim_{n\to\infty} f(x_n) = 0$.(by the weakly converges argument at very top).
This implies that the $\varphi_{x_n}(f)$ exists for all f in X^*,
therefore by the Banach-Steinhaus Theorem, we have $\{\varphi_{x_n}\}$ is uniformly bounded.
i.e., $||\varphi_{x_n}|| \leq M, \ \forall n$, there is a constant M.
In addition, we have $||\varphi_{x_n}|| = ||x_n||$, therefore $||x_n||$ is bounded for all n.

□

5 UI Urbana-Champaign Quals

Problem 5.1 (Nicholas Camacho, January 2015 Q5)**.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing,*

$$\int_{\mathbb{R}} f' dm = 1, \lim_{x \to -\infty} f(x) = 0, \lim_{x \to \infty} f(x) = 1.$$

Prove that f is absolutely continuous on any interval $[a, b]$.

Solution.
Attempt:
I first thought about going directly by the definition of absolute continuity to start this problem. However, the definition we use does not involve integration, nor limits. So of course, I began thinking about equivalent statements of absolute continuity. In Royden page 124, we have a theorem that says f absolutely continuous on $[a, b] \implies f$ differentiable almost everywhere on (a, b), f' is integrable over $[a, b]$, and $\int_a^b f' = f(b) - f(a)$. Essentially, an absolutely continuous function satisfies the Fundamental Theorem of Calculus. But I'm sure you can already see the problem with trying to use this theorem: I want to prove that f is absolutely continuous, I can't assume it.

But wait, all hope is not lost. On the very next page, we have a theorem which states f is absolutely continuous on $[a, b] \iff$ it is an indefinite integral over $[a, b]$; that is

$$f(x) = f(a) + \int_a^x g, \text{ for some } g \in L^1([a, b]).$$

Evaluating at $x = b$ and letting $g = f'$ yields $\int_a^b f' = f(b) - f(a)$. So lets show this!

... Or I could have turned just ONE MORE PAGE, avoided all this nonsense, and used the Corollary which says: Let f be monotone on $[a, b]$. Then f is absolutely continuous on $[a, b]$ iff $\int_a^b f' = f(b) - f(a)$. So let's just use this.
Solution:
Since f is nondecreasing, it is increasing and we therefore have[1] that

$$\int_a^b f' \leq f(b) - f(a).$$

We show the opposite inequality. For appropriate $x \in \mathbb{R}$, we have by the same reasoning

$$\int_x^a f' \leq f(a) - f(x) \text{ and } \int_b^x f' \leq f(x) - f(b),$$

Then by Additivity over Domains,

$$1 = \int_{\mathbb{R}} f' = \int_{-\infty}^{\infty} f' = \int_{-\infty}^{a} f' + \int_a^b f' + \int_b^{\infty} f' = \lim_{x \to -\infty} \int_x^a f' + \int_a^b f' + \lim_{x \to \infty} \int_b^{\infty} f',$$

[1]Corollary 4, page 113 Royden

and so

$$\begin{aligned}\int_a^b f' &= 1 - \lim_{x\to-\infty}\int_x^a f' + \lim_{x\to\infty}\int_b^\infty f' \\ &\geq 1 + \lim_{x\to-\infty} f(x) - f(a) + \lim_{x\to\infty} f(b) - f(x) \\ &= 1 + 0 - f(a) + f(b) - 1 \\ &= f(b) - f(a).\end{aligned}$$

□

Problem 5.2 (Kaitlin, January 2015 Q6). *Let f_n be a sequence of Lebesgue measurable functions on the interval $[0,1]$. Assume that f_n converges to a function f almost everywhere, and that*

$$\int_{[0,1]} |f_n|^2 d\lambda \leq 1$$

for each n. Prove that f_n converges to f in L^1.

Solution.
Attempt:
To start, I wrote out all of the definitions or theorems I thought I may need to solve this problem.

- f_n **converging to** f **a.e.**: $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1] \setminus A$ where $\lambda(A) = 0$.
- f_n **converging to** f **in** L^1: $\lim_{n\to\infty} ||f - f_n||_1 = 0$ or $\lim_{n\to\infty} f_n = f$ in L^1.
- **Egoroff's Theorem**: Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$ there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } \lambda(E \setminus F) < \varepsilon$$

Since the convergence is pointwise a.e., there is a set E in $[0,1]$ such that $\lambda(E) = 1$ (we are looking at the complement of the measure zero set). Now we tried applying Egoroff's Theorem which gave us a closed set $F \subseteq E$ such that $\lambda(F) = 1 - \varepsilon$ on which f_n converges to f uniformly. Another way that we could write the uniform convergence is $\sup_{x\in F} |f_n(x) -$

$f(x)| \to 0$. Let $M_n = \sup_{x\in F} |f_n(x) - f(x)|$. This all gave us the following:

$$\begin{aligned} ||f_n - f||_1 &= \int_{[0,1]} |f_n - f| \\ &= \int_F |f_n - f| + \int_{F^c} |f_n - f| \\ &\le \int_F M_n + \int_{F^c} |f_n - f| \\ &= M_n(1-\varepsilon) + \int_{F^c} |f_n - f| \\ &\le M_n + \int_{F^c} |f_n - f| \end{aligned}$$

This attempt direction/attempt ended up not working because we could not find a bound for $|f_n - f|$ in F^c.

Solution:

By the statement of the problem, for every n we have

$$\int_{[0,1]} |f_n|^2 \le 1$$

Now, if we raise both sides of this inequality to the half power, we find

$$||f_n||_2 \le 1$$

for every n. This gives us a family of functions in $L^2([0,1])$ that are bounded. By Corollary 2 on page 142, we can conclude that the collection of f_n's is uniformly integrable over $[0,1]$. By definition, this means for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every f_n, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f_n| < \varepsilon$. Notice the following:

$$\int_{[0,1]} |f_n| = \int_0^{\delta} |f_n| + \int_{\delta}^{2\delta} |f_n| + \cdots + \int_{m\delta}^{1} |f_n|$$

Each of these integrals has length δ so we can use the uniform integrability to say

$$\int_{[0,1]} |f_n| \le (m+1)\varepsilon$$

This tells us that each f_n is integrable over $[0,1]$ and thus, $\{f_n\} \in L^1([0,1])$. Now, by the Vitali Convergence Theorem (page 94), we can conclude that f is integrable over $[0,1]$ and $\lim_{n\to\infty} \int_{[0,1]} f_n = \int_{[0,1]} f$. This also implies that $\lim_{n\to\infty} \int_{[0,1]} |f_n| = \int_{[0,1]} |f|$. Since we now know that f is integrable over $[0,1]$, we know $f \in L^1([0,1])$. Applying Theorem 7 (page 148), we conclude that $\{f_n\} \to f$ in $L^1([0,1])$. □

Problem 5.3 (Adam Wood, Fall 2015 Q4)**.** *Let $(X, \mathscr{M}, \lambda)$ be a finite measure space. Fix $p > 0$, and suppose that a sequence E_n of measurable subsets satisfies $\sum_{n=1}^{\infty} (\lambda(E_n))^p < \infty$.*

i. Prove that $\lambda(\limsup(E_n)) = 0$ *provided* $p \leq 1$.

ii. Give a counterexample to the statement in part i, when $p > 1$.

Solution.
Attempt:
Part i. is essentially a generalization of the Borel-Cantelli Lemma and the proof is very similar to that of the Borel-Cantelli Lemma. For part ii., I thought that since $\sum_{i=1}^{n}\left(\frac{1}{n}\right)^p$ converges for $p > 1$, that I should find a sequence of measurable sets each with measure $\frac{1}{n}$. I also thought using something with the generalized Cantor set would be useful.
Solution:
i. Let $p \leq 1$. Recall that $\limsup(E_n) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$. Since X has finite measure, $\bigcap_{k=1}^{\infty} E_k$ has finite measure. Note that $\bigcup_{k=n}^{\infty} E_k$ is a descending sequence of sets. By the continuity of measure and the countable subadditivity of measure,

$$\lambda(\limsup(E_n)) = \lambda\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = \lim_{n\to\infty} \lambda\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty}\sum_{k=n}^{\infty}\lambda(E_k). \tag{8}$$

We must now compute the above limit. Since $\sum_{n=1}^{\infty}(\lambda(E_n))^p < \infty$, for every $\varepsilon > 0$, there exists N so that $\sum_{k=N}^{\infty}(\lambda(E_k))^p < \varepsilon$. We can choose $\varepsilon < 1$. Then, $\lambda(E_k) < 1$ for all k. Since $\lambda(E_k) \leq (\lambda(E_k))^p$ when $\lambda(E_k) < 1$ and $p \geq 1$, we can see that

$$\sum_{k=N}^{\infty}\lambda(E_k) \leq \sum_{k=N}^{\infty}(\lambda(E_k))^p < \varepsilon.$$

Therefore, $\lim_{n\to\infty}\sum_{k=n}^{\infty}\lambda(E_k) = 0$. By (1), $\lambda(\limsup(E_n)) = 0$.

ii. Let $p > 1$. We first quote a (modified version of a) lemma from *Harmonic Analysis*, by Elias Stein:

Let $\{E_n\}$ be a collection of subsets of a fixed compact set with $\sum_{n=1}^{\infty} m(E_n) = \infty$. Then there exists a sequence of translates $F_j = E_j + x_j$ so that

$$\limsup(F_j) = [0, 1]$$

except for a set of measure zero.

Note that $\sum_{n=1}^{\infty}\lambda([0, \frac{1}{n}]) = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$. By the lemma, there exists a sequence of translates F_n of $[0, \frac{1}{n}]$ so that $\limsup(F_n) = [0, 1]$ except for a set of measure zero. That is, there exists a sequence of translates F_n and a set of measure zero, G, so that $\limsup(F_n) \cup G = [0, 1]$. Then,

$$\lambda(\limsup(F_n)) = 1 - \lambda(G) = 1,$$

so that $\lambda(\limsup(E_n)) \neq 0$. Also, $\sum_{n=1}^{\infty}(\lambda(F_n))^p = \sum_{n=1}^{\infty}(\frac{1}{n})^p < \infty$. The collection F_n provides a counterexample to the statement in part i.

Now, for a more concrete counterexample. Let $x \in \mathbb{R} \sim \mathbb{Q}$. Let $E_n = \{\lfloor z \rfloor \mid z \in [0, \frac{1}{n}] + nx\}$. Then, for all n, $\lambda(E_n) = \lambda([0, \frac{1}{n}]) = \frac{1}{n}$. Note that

$$\sum_{n=1}^{\infty}(\lambda(E_n))^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p < \infty$$

since the p-series converges for $p > 1$. A similar argument to that in the above lemma shows that, apart from a set of measure zero, $\limsup(E_n) = [0, 1]$. So,

$$\lambda(\limsup E_n)) = \lambda\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = \lambda([0, 1]) = 1 \neq 0$$

So, this collection of sets provides a counterexample the statement in part i., when $p > 1$. $\square$

Problem 5.4 (Rolando, May 2015 Q3)**.** *Suppose that μ is a measure on X with $\mu(X) < \infty$ and $f_n \in L^(\mu)$ such that $f_n(x) \to f(x)$ pointwise a.e. Further suppose there exists $p > 1$ and a constant C such that*

$$\sup_{n\in\mathbb{N}} \int_X |f_n|^p \leq C.$$

Prove f_n converges to f in $L^1(\mu)$

Solution.
Attempt:
Let's see how we can simplify this problem. Suppose we had uniform convergence instead of pointwise. If $f_n \to f$ uniformly a.e., for a given $\varphi > 0$ there exists an $N \in \mathbb{N}$ so that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for almost every $x \in X$. This is equivalent to saying $\operatorname{ess\,sup}\{|f_n(x) - f(x)|\} := norm f_n - f_\infty < \varphi$. Now if $n \geq N$ it follows

$$\int_X |f_n - f| \leq \int_X \varepsilon < \varepsilon\mu(X), \tag{9}$$

which can be made arbitrarily smaller.

While we may not have uniform continuity, Egoroff's Theorem says we are not too far away from it. **Solution:**
Let $f_n \to f$ pointwise a.e on X and let $\varepsilon > 0$. Then there exists $F \subset X$ closed such that

1. $f_n \to f$ uniformly on F, and
2. $\mu(F^c) < \varepsilon$ (we will revise this estimate, if necessary).

Now partitioning $X = F \sqcup F^c$, we have

$$\int_X |f_n - f| = \int_F |f - f_n| + \int_{F^c} |f_n - f| \tag{10}$$

Now since $f_n \to f$ uniformly on F, Equation (9) implies there exists an $N \in \mathbb{N}$ such that if $n \geq N$, the first integral in Equation (10) has the estimate

$$\int_F |f_n - f| < \varepsilon. \tag{11}$$

Now to estimate the second integral in Equation (10), we use the triangle inequality and Fatou's Lemma to obtain

$$\int_{F^c} |f_n - f| \leq \int_{F^c} |f_n| + \int_{F^c} |f| \leq \int_{F^c} |f_n| + \liminf \int_{F^c} |f_n|. \tag{12}$$

Since everything is it terms of f_n, we estimate only the first part of Equation (12). Let q be the Hölder conjugate of p. From Hölder's Inequality, we have

$$\int_{F^c} |f_n| = \int_{F^c} |f_n| \cdot 1 \leq \|f_n\|_p \|1\|_q \leq C\|1\|_q = C\left(\int_{F_c} 1^q\right)^{1/q} = C\mu(F^c)^{1/q} \tag{13}$$

which means need to assume $\mu(F^c) < \varepsilon^q / C$. Note this changes nothing else. Also notice we have the same bound on $\liminf \int_{F_c} |f_n|$. Combining Equations (11) and (13) with(10), we now obtain that for $\varepsilon > 0$ and $n \geq N$

$$\int_X |f_n - f| < 3\varepsilon,$$

or synonymously, $\|f_n - f\|_1 \to 0$. □

Problem 5.5 (Elaina Aceves, January 2015 Q1)**.** *For each statment, give a counterexample or a short proof/explanation.*
a. If $f'(x) = 0$ a.e. on $\mathbb{R}$, then f is constant on $\mathbb{R}$.
b. If $f'(x) = 0$ a.e. on $\mathbb{R}$ and f is absolutely continuous on $\mathbb{R}$, then f is constant on $\mathbb{R}$.

Solution.
Attempt:
For part (a), I immediately started looking for a counterexample. I first considered the Dirichlet function, but this is not differentiable. Then I realized a step function will do the trick.

For part (b), my intuition was that I would need to prove the statement. However, to use absolute continuity, we need to restrict ourselves to closed intervals in $\mathbb{R}$. Thus, I would prove the statement first on any arbitrary closed interval in $\mathbb{R}$ to prove the statement on all of $\mathbb{R}$ (after excising the set of measure 0 where $f'(x) \neq 0$). Also, to use absolute continuity, I would need a finite disjoint collection of open intervals. Two sections before the definition of absolute continuity in Royden is the Vitali Covering Lemma which gives us an appropriate collection of intervals, but I need a collection of intervals that cover in the sense of Vitali. Thus, I knew I would need to create such a collection using the only property that I had, the fact that $f'(x) = 0$ a.e. on $\mathbb{R}$. This method of reasoning gives the proof that I gave in class, which is correct but very lengthy. However, Rolando drew my attention to another result that says that f is absolute continuous if and only if f is an indefinite integral over the closed interval which gives us the result much faster. That solution is given below.

Solution:

a. Let $f(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases}$ Then $f'(x) = 0$ a.e. on $\mathbb{R}$ and f is not constant on $\mathbb{R}$.

b. We will prove that f is constant on $[a, b]$ for any closed interval in $\mathbb{R}$. We have that f is absolutely continuous on $[a, b]$ if and only if $f(x) = f(a) + \int_a^x f'$ for $x \in (a, b]$ (Theorem 11 on p.125 in Royden). Since $f'(x) = 0$ a.e. on $\mathbb{R}$, we have that $\int_a^x f' = 0$. Hence $f(x) = f(a)$ for all $x \in (a, b]$. Thus f is constant on $[a, b]$ for any closed interval in $\mathbb{R}$, and must be constant on $\mathbb{R}$.

□

Problem 5.6 (UI Urbana-Champaign, August 2015 Q3). *Suppose that f is a continuous function on $\mathbb{R}$, with period 1. Prove that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\theta) = \int_0^1 f(t)dt \tag{\#}$$

for every irrational number $\theta \in \mathbb{R}$.

Solution. [Qing Zou] Before proving the statement, we need to introduce the following claim first.

Claim: If f is continuous, with period 1. Then for every $\varepsilon > 0$, there is a trigonometric polynomial g such that

$$|f(t) - g(t)| < \varepsilon$$

for every real t.

One can find the proof of this claim in Rudin's book "Real and Complex Analysis" (Page 91, Theorem 4.25). We will use the claim later.

Now, let us consider the function $w(t) = e^{2\pi ikt},\ k \in \mathbb{Z}$.

It is clear that $w(t)$ satisfies (#) when $k = 0$.

If $k \neq 0$, then $e^{2\pi ikn\theta} \neq 0$ since θ is irrational. Thus, we can get

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi ikn\theta} = \frac{1}{N} \frac{e^{2\pi ik\theta}(e^{2\pi ikN\theta} - 1)}{e^{2\pi ik\theta} - 1}.$$

This is because the sequence $\{e^{2\pi ikn\theta}\}$ is a geometric sequence with ratio $e^{2\pi ik\theta}$. Therefore, we have

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ikn\theta} \right| &= \frac{1}{N} \cdot \left| e^{2\pi ik\theta} \right| \cdot \left| \frac{e^{2\pi ikN\theta} - 1}{e^{2\pi ik\theta} - 1} \right| \\ &\leq \frac{1}{N} \frac{2}{|e^{2\pi ik\theta} - 1|} \to 0 \qquad (N \to \infty). \end{aligned}$$

So,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ikn\theta}=0.\quad (k\neq 0)$$

Also,

$$\int_0^1 e^{2\pi ikt}dt=\left.\frac{1}{2\pi ik}e^{2\pi ikt}\right|_0^1=0\quad (k\neq 0).$$

To conclude, for all $k\in\mathbb{Z}$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ikn\theta}=\int_0^1 e^{2\pi ikt}dt.$$

Let $g(t)=\sum_{k=-M}^{M}c_k e^{2\pi ikt}$, then by the aforementioned analysis, we can obtain that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}g(n\theta)=\int_0^1 g(t)dt.$$

So, by the claim, we know that $\forall\varepsilon>0$, $\exists g=\sum_{k=-M}^{M}c_k e^{2\pi ikt}$ such that $|f-g|<\varepsilon$. Thus, with $h=f-g$, we have

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N}f(n\theta)-\int_0^1 f(t)dt\right|\\
=&\left|\frac{1}{N}\sum_{n=1}^{N}[h(n\theta)+g(n\theta)]-\int_0^1(h(t)+g(t))dt\right|\\
=&\left|\frac{1}{N}\sum_{n=1}^{N}h(n\theta)+\frac{1}{N}\sum_{n=1}^{N}g(n\theta)-\int_0^1 h(t)dt-\int_0^1 g(t)dt\right|\\
\leq&\left|\frac{1}{N}\sum_{n=1}^{N}h(n\theta)-\int_0^1 h(t)dt\right|+\left|\frac{1}{N}\sum_{n=1}^{N}g(n\theta)-\int_0^1 g(t)dt\right|\\
\leq&\frac{1}{N}\sum_{n=1}^{n}|h(n\theta)|+\int_0^1|h(t)|dt+\left|\frac{1}{N}\sum_{n=1}^{N}g(n\theta)-\int_0^1 g(t)dt\right|\\
\leq&\varepsilon+\varepsilon+\left|\frac{1}{N}\sum_{n=1}^{N}g(n\theta)-\int_0^1 g(t)dt\right|.
\end{aligned}$$

Since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}g(n\theta)=\int_0^1 g(t)dt,$$

then for the ε above, $\exists M > 0$ and when $N > M$, we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} g(n\theta) - \int_0^1 g(t)dt \right| < \varepsilon.$$

Therefore, for $\varepsilon > 0$, $\exists M > 0$ such that when $N > M$, we have

$$\left| \frac{1}{N} \sum_{n=1} N f(n\theta) - \int_0^1 f(t)dt \right| < 3\varepsilon,$$

which means

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\theta) = \int_0^1 f(t)dt.$$

This completes the proof. ☐

Problem 5.7 (Kaitlin Healy, January 2016 Q2)**.** *Let f be a Lebesgue measurable function on the closed interval $[0,1]$. Prove that*

(a) $\lim_{p\to\infty} ||f||_p = ||f||_\infty$

(b) Give a counterexample to show that (a) fails when $[0,1]$ is replaced by $\mathbb{R}$.

Solution.
Attempt:
Kind of figured this out first try.
Solution:
(a) Let $M = ||f||_\infty$. By definition we know $|f(x)| \leq M$ for almost every $x \in [0,1]$. This gives us the following

$$\begin{aligned} ||f||_p &= \left[\int_0^1 |f(x)|^p dx \right]^{1/p} \\ &\leq \left[\int_0^1 M^p dx \right]^{1/p} \\ &= M \cdot \lambda([0,1])^{1/p} \\ &= M \end{aligned}$$

Taking a limit as $p \to \infty$ on both sides of this inequality, we have

$$\lim_{p\to\infty} ||f||_p \leq M$$

Let $\varepsilon > 0$. Since M is the essential supremum of f by definition, we know there exists an $f(x)$ such that $|f(x)| \geq M - \varepsilon$. Define $E = \{x \in [0,1] \mid |f(x)| \geq M - \varepsilon\}$. We know from the previous statement that this set is nonempty and thus $\lambda(E) > 0$. Notice the following

$$\begin{aligned} ||f||_p &= \left[\int_0^1 |f(x)|^p dx\right]^{1/p} \\ &\geq \left[\int_E |f(x)|^p dx\right]^{1/p} \\ &\geq \left[\int_E (M-\varepsilon)^p dx\right]^{1/p} \\ &= (M-\varepsilon)\lambda(E)^{1/p} \end{aligned}$$

Since ε was arbitrary, we can take it to zero and thus, we have

$$||f||_p \geq M\lambda(E)^{1/p}$$

Taking a limit as $p \to \infty$ on each side, we have

$$\lim_{p\to\infty} ||f||_p \geq M$$

Combining this inequality with the one from above, we can conclude

$$\lim_{p\to\infty} ||f||_p = M = ||f||_\infty$$

(b) I think for a counterexample, we can use

$$f = \begin{cases} 1 & x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

This would give us

$$\begin{aligned} \lim_{p\to\infty} ||f||_p &= \lim_{p\to\infty} \left[\int_{\mathbb{R}} |f|^p\right]^{1/p} \\ &= \lim_{p\to\infty} [0]^{1/p} \\ &= \lim_{p\to\infty} 0 \\ &= 0 \end{aligned}$$

However, since $||f||_\infty$ is the essential supremum of f over $\mathbb{R}$, we have $||f||_\infty = 1$. Since $1 \neq 0$, we have our contradiction. □

Problem 5.8 (Andrew Pensoneault, May 2016, Q6)**.** *TO FINISH*

Attempt:
TO FINISH
Solution:
TO FINISH

Problem 5.9 (UI Urbana-Champaign, Fall 2016 Q4).
Fix $1 \le p < \infty$.
i) Assume that f *is absolutely continuous on every compact interval, and* $f' \in L^p(\mathbb{R}, m)$. *Prove that*

$$\sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|^p < \infty.$$

ii) Prove or give a counterexample: The statement above remains valid if we instead assume that f *is continuous, of bounded variation on every compact interval, and* $f' \in L^p(\mathbb{R}, m)$.

Solution. [Alex Bates]
Attempt:
i) At first, I had no clue how to do this problem. Until I looked at Theorem 10 of §6.5 of Royden (pg. 124) and had Equation 14 below. But I kept trying to estimate:

$$\sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|^p = \sum_{n\in\mathbb{Z}} \left| \int_n^{n+1} f' \right|^p \le \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f'| \right)^p, \qquad \text{(Equation 14)}$$

and got stuck. I could be done if I could move the p inside the integral; that's where Rolando pointed my attention to Jensen's Inequality. Since x^p is convex (on $(0, \infty)$), Jensen's Inequality allows me to move the p inside the integral. However, I got stuck again when I realized that I needed (by Jensen's Inequality) a function that was not only convex over $(0, \infty)$ but over $(-\infty, \infty) = \mathbb{R}$. I was able to define a better function (see my solution below) and the proof went through.

ii) My first instinct was to use f being continuous and of bounded variation to prove that it was absolutely continuous. However, this is false. It is impossible to prove the claim given; it is another instance where the Cantor(-Lebesgue) function comes in handy.
Solution:
i) First, define $\phi : \mathbb{R} \to \mathbb{R}$ by:

$$\phi(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^p & \text{if } x > 0. \end{cases}$$

Fix $n \in \mathbb{Z}$. By hypothesis, since the closed and bounded interval $[n, n+1]$ is compact, f is absolutely continous on $[n, n+1]$. By Theorem 10 of §6.5 of Royden (pg. 124), f' is integrable over $[n, n+1]$ and we have:

$$\int_n^{n+1} f' = f(n+1) - f(n) < \infty. \qquad (14)$$

Note that, by definition, f' being integrable over $[n, n+1]$ means that $|f'|$ is integrable over $[n, n+1]$. Also, $\chi_{[n,n+1]} \cdot |f'|^p \le |f'|^p$ on $\mathbb{R}$ and so by monotonicity of integration,

$$\int_n^{n+1} |f'|^p = \int_{\mathbb{R}} \chi_{[n,n+1]} \cdot |f'|^p \le \int_{\mathbb{R}} |f'|^p < \infty, \qquad (f' \in L^p(\mathbb{R}, m))$$

which tells us that $|f'|^p$ is integrable on $[n, n+1]$. (We need this for our application of Jensen's Inequality later.)

Now, define $\phi : \mathbb{R} \to \mathbb{R}$ by:

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^p & \text{if } x > 0, \end{cases}$$

and observe that $|f'|^p = \phi(|f'|)$. Since ϕ can be shown to be convex, by application of Jensen's Inequality of §6.6 of Royden (pg. 133) we see that:

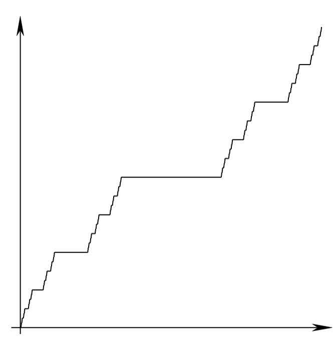
$$\left(\int_n^{n+1} |f'|\right)^p = \phi\left(\int_n^{n+1} |f'|\right) \leq \int_n^{n+1} \phi(|f'|) = \int_n^{n+1} |f'|^p. \tag{15}$$

But now:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|^p &= \sum_{n\in\mathbb{Z}} \left|\int_n^{n+1} f'\right|^p && \text{(Equation 14)} \\ &\leq \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f'|\right)^p \\ &\leq \sum_{n\in\mathbb{Z}} \int_n^{n+1} |f'|^p && \text{(Equation 15)} \\ &= \int_{\mathbb{R}} |f'|^p \\ &< \infty. && (f' \in L^p(\mathbb{R}, m)) \end{aligned}$$

△

ii) Consider the Cantor(-Lebesgue) function $\phi : [0,1] \to [0,1]$ whose graph is given below:



Observe that the Cantor function is constant on $\mathscr{O} := [0,1] \setminus \mathbf{C}$, the relative complement of the Cantor Set $\mathbf{C}$ in $[0,1]$, and $m(\mathbf{C}) = 0$. Furthermore, $\phi(1) = 1$ and $\phi(0) = 0$ so that $\phi(1) - \phi(0) = 1$. Now define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \phi(x - \lfloor x \rfloor) + \lfloor x \rfloor$, and observe that for any $n \in \mathbb{Z}$, $[n, n+1) \ni x \overset{\phi}{\longmapsto} \phi(x-n) + n$. We'll show briefly that this function is continuous and of bounded variation on every compact interval and $f' \in L^p(\mathbb{R}, m)$ so that it satisfies the hypotheses of the claim but not the conclusion.

First thing's first: continuity. Since f mimics ϕ's behavior on $[0,1)$ on every interval $[n, n+1)$, where $n \in \mathbb{Z}$, it can be shown that f agrees on the endpoints of each interval, i.e., for any $n \in \mathbb{Z}$, $\lim_{x \to n^-} f(x) = f(n)$.

Second, to bounded variation! Let $[a,b] \subset \mathbb{R}$ be compact with $a < b$.[2] Set $c := \lfloor a \rfloor$ and $d := \lceil b \rceil$. Then $[a,b] \subset [c,d]$, where $c, d \in \mathbb{Z}$. But then:

$$\begin{aligned} TV(f|_{[a,b]}) &\leq TV(f|_{[c,d]}) && (\text{"Linearity" of } TV) \\ &= f(d) - f(c) && (\text{Monotonicity of } f) \\ &= d - c \\ &< \infty, \end{aligned}$$

so that f is of bounded variation on $[a,b]$, hence for any compact interval.

Third: the L^p condition. Observe that ϕ is constant a.e. on $[0,1]$ and has derivative $\phi'(x) = 0$ for $x \in \mathscr{O} = [0,1] \setminus \mathbf{C}$. Then: $\int_{[0,1]} |\phi'|^p dm = \int_{\mathscr{O}} |\phi'|^p dm + \int_{\mathbf{C}} |\phi'|^p dm = \int_{\mathscr{O}} |\phi'|^p dm = \int_{\mathscr{O}} |0|^p dm = \int_{\mathscr{O}} 0 dm = 0$. Hence,

$$\begin{aligned} \int_{\mathbb{R}} |f'|^p dm &= \sum_{n \in \mathbb{Z}} \int_{[n,n+1]} |f'|^p dm \\ &= \sum_{n \in \mathbb{Z}} \int_{[0,1]+n} |(\phi(x-n) + n)'|^p dm \\ &= \sum_{n \in \mathbb{Z}} \int_{[0,1]+n} |\phi'(x-n)|^p dm \\ &= \sum_{n \in \mathbb{Z}} \int_{[0,1]} |\phi'(x)|^p dm \\ &= \sum_{n \in \mathbb{Z}} 0 \\ &= 0, \end{aligned}$$

which implies that $f' \in L^p(\mathbb{R}, m)$. But lastly,

$$\sum_{n \in \mathbb{Z}} |f(n+1) - f(n)|^p = \sum_{n \in \mathbb{Z}} |n + 1 - n|^p = \sum_{n \in \mathbb{Z}} 1^p = \infty.$$

□

Problem 5.10 (Nicholas Camacho, May 2017 Q1)**.** *Let $(X, \mathscr{M}, \mu)$ be a finite measure space (i.e., $\mu(X) < \infty$), let $\alpha > 0$, and let $(A_n)_{n \in \mathbb{N}} \subseteq \mathscr{M}$ be such that $\mu(A_n) \geq \alpha$ for each $n \in \mathbb{N}$. Put*

$$A := \{x \in X : \exists^\infty n \ x \in A_n\},$$

[2] We're assuming the interval $[a,b]$ is nondegerate, otherwise $TV(f|_{[a,b]}) = TV(f|_{\{a\}}) = 0$, which is no fun.

where $\exists^\infty$ means "for infinitely many $n \in \mathbb{N}$".

(a) Show that $A \in \mathscr{M}$.

(b) Prove that $\mu(A) \geq \alpha$.

(c) Give an example of a measure space $(X, \mathscr{M}, \mu)$ with $\mu(X) = \infty$ and a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}$ such that $\mu(A_n) \geq 1$ for each $n \in \mathbb{N}$, But $A = \emptyset$

Proof. If $x \in A$ then no matter how large an N we pick, we can always find a set in $(A_n)_{n\in\mathbb{N}}$ of index larger than N containing x. Formally,

$$A = \{x \in X : \forall n \in \mathbb{N}, \exists k \geq n \ s.t. \ x \in A_k\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Since $\mathscr{M}$ is a σ-algebra, we have $A \in \mathscr{M}$. This proves (a).

Notice that $\bigcup_{k=n}^{\infty} A_k \supseteq \bigcup_{k=n+1}^{\infty} A_k$ for all $n \in \mathbb{N}$ and hence $\{\bigcup_{k=n}^{\infty} A_k\}_{n\in\mathbb{N}}$ is a descending sequence of sets in $\mathscr{M}$ such that $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) < \infty$, since X is a finite measure space. Hence by the Continuity of Measure,

$$\mu(A) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \overset{(*)}{\geq} \lim_{n\to\infty} \mu(A_n) \geq \alpha,$$

where $(*)$ follows from the fact that $A_n \subseteq \bigcup_{k=n}^{\infty} A_k$ for all n, and the Monotonicity of Measure. This proves (b).

For (c), consider $X = \mathbb{R}$ and $\mu = \lambda$, Lebesgue Measure. The sets $A_n = [n-1, n]$ for $n \in \mathbb{N}$ all have measure 1 and the set A is empty. Indeed, if $x \in \mathbb{R}$, then x is an element of at most 2 sets among $(A_n)_{n\in\mathbb{N}}$. □

Problem 5.11 (Kaitlin Healy, May 2017 Q2)**.** *Compute*

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+e^{-kx}} \frac{1}{1+x^2} dx$$

Justify your computation.

Solution.
Attempt:
At first, I didn't think I could assume measurability or integrability. To try to get around this, I tried breaking $\mathbb{R}$ into intervals of the form $(n, n+1]$. This however proved to be more difficult as now I would have to take limits through infinite sums by using additivity over domains. Once I heard that I could assume measurability and integrability, I knew I would just have to apply some theorem that would pass the limit under the integral.
Solution:
Let $f_k = \frac{1}{1+e^{-kx}} \frac{1}{1+x^2}$ for all k. Notice that as $k \to \infty$, f_k converges to $f = \frac{1}{1+x^2}$ pointwise on $\mathbb{R}$. We can assume from the problem that the sequence $\{f_k\}$ is measurable on $\mathbb{R}$. For every

value of k, we can see that $|f_k| \leq 1$. Let $g = 1$. Since g is integrable over $\mathbb{R}$ and dominates $\{f_k\}$ for all k, we apply Lebesgue dominated convergence theorem. This tells us that f is integrable over $\mathbb{R}$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k = \int_{\mathbb{R}} f$$

Taking the integral of f, we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{1+x^2} dx &= \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \\
&= \lim_{t\to\infty} \int_{-t}^{t} \frac{1}{1+x^2} dx \\
&= \lim_{t\to\infty} \arctan(x)\Big|_{-t}^{t} \\
&= \lim_{t\to\infty} \arctan(t) - \arctan(-t) \\
&= \lim_{t\to\infty} 2\arctan(t) \\
&= 2\left(\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

Therefore, we conclude that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+e^{-kx}} \frac{1}{1+x^2} dx = \pi$$

□

Problem 5.12 (Andrew Pensoneault, May 2017, Q4)**.** *Let $\{f_n\} \subset L^1(X, \mathscr{M}, \mu)$. Assume $\{f_n\}$ converges in measure to f, that $\{f_n\}$ is uniformly absolutely continuous in L_1 and that $\{f_n\}$ vanishes uniformly at ∞ in L^1. Prove that $\{f_n\}$ converges to f in L^1.*

Definition: *Let $(x, \mathscr{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on it. We call $\{f_n\}$ Uniformly Absolutely Continuous in L^1 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $A \in \mathscr{M}$, if $\mu(A) \leq \delta$, then $\int_A |f_n|\, d\mu < \epsilon$ for all $n \in \mathbb{N}$. We say $\{f_n\}$ Vanishes Uniformly at ∞ in L^1 if for all $\epsilon > 0$, there exists a measurable $A \subset X$ where $m(A) < \infty$ such that for all $n \in \mathbb{N}$, $\int_{A^c} |f_n| d\mu < \epsilon$.*

Attempt:
There were a lot of assumptions I was given, and ultimately, I saw we needed to find way to make

$$\int_X |f - f_n| d\mu < \epsilon.$$

Uniform vanishing gives us the ability to bound the behavior on the complement of some finite set (and thus most of X if X is infinite measure). There was a relevant corollary in the book (Corollary 5, Chapter 5.2), that allows us (once we show tight over A and uniform integrability) to remove the remaining portion, and this follows almost immediately from uniformly absolutely continuity in L^1.

Solution:
Fix $\epsilon > 0$, and by the definition of uniformly vanishing at ∞ in L^1, let $A \subset X$ such that

$$\int_A^c |f_n| d\mu < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$. By Riesz Theorem, we can find a subsequence f_{n_k} which converges pointwise almost everywhere on A^c, thus by Fatou's Lemma,

$$\begin{aligned}
\int_X |f_n - f| d\mu &= \int_A^c |f_n - f| d\mu + \int_A |f_n - f| d\mu \\
&\leq \int_{A^c} |f_n| d\mu + \int_{A^c} |f| d\mu + \int_A |f_n - f| d\mu \\
&\leq \frac{\epsilon}{3} + \lim_{k\to\infty} \int_{A^c} |f_{n_k}| d\mu + \int_A g_n d\mu \\
&\leq \frac{2\epsilon}{3} + \int_A g_n d\mu
\end{aligned}$$

defining $g_n = |f_n - f|$. Notice $g_n \to 0$ in measure, and thus if we can show on A, $\{g_n\}$ is uniformly integrable and tight, we can find an $N \in \mathbb{N}$ such that if $n > N$

$$\int_A g_n d\mu < \frac{\epsilon}{3}.$$

So, from uniformly absolutely continuity in L^1, we can find a $\delta > 0$ (since $\mu(A) = M < \infty$), such that for some $R \subset A$, $M - \delta \leq m(R) \leq M$ (so $\mu(A \sim R) < \delta$) and then we have (using Riesz and Fatou's lemma to pass to a convergent subsequence)

$$\int_{A\sim R} g_n d\mu = \int_{A\sim R} |f_n - f| d\mu \leq \int_{A\sim R} |f_n| + \lim_{k\to\infty} \int_{A\sim R} |f_{n_k}| d\mu \leq \epsilon.$$

Thus we have proven g_n is tight over A. Now, to show uniform integrability, let $\delta > 0$ correspond to the ϵ challenge in uniformly absolutely continuity in L^1, then let $m(B) < \delta$, we have

$$\int_B g_n d\mu = \int_B |f_n - f| d\mu \leq \int_B |f_n| + \lim_{k\to\infty} \int_B |f_{n_k}| d\mu \leq \epsilon.$$

Thus we have shown uniform integrability. Thus by corollary 5, we can find an index N such that

$$\int_A g_n d\mu \leq \frac{\epsilon}{3},$$

and thus if we pick $n > N$,

$$\int_X |f_n - f| d\mu < \epsilon$$

so we have shown convergence in L^1.

6 IU Bloomington

Problem 6.1 (Indiana Real Analysis, August 2007)**.** *Define a sequence* $\{a_n\}$ *by setting* $a_1 = \frac{1}{2}$ *and* $a_{n+1} = \sqrt{1 - a_n}$ *for* $n \geq 2$*. Does the sequence* a_n *converge? If so, what is the limit?*

Solution. [Meghan Malachi]
Attempt:
Initially, I wanted to show that the sequence is bounded and monotone so that I could apply the Monotone Convergence Theorem to conclude that the sequence is convergent. After inspection, the sequence turned out to not be monotone, however, so decomposing the sequence into two convergent monotone sequences with the same limit was the way to go.

Solution:
First we show that $\{a_n\}$ is bounded.
Recall: a sequence $\{a_n\}$ is bounded provided there exists a $c \geq 0$ such that $|a_n| \leq c$ for all $n \in \mathbb{N}$. We know that $a_n > 0$ for all n, otherwise our sequence would be undefined for such n. We will now show that $a_n < 1$.

Observe that $a_1 = \frac{1}{2} < 1$.
Now assume that $|a_n| < 1$. Then we have:

$$
\begin{aligned}
a_{n+1} &= \sqrt{1 - a_n} && (16)\\
\Rightarrow |a_{n+1}| &= |\sqrt{1 - a_n}| && (17)\\
&< |1| = 1 && (18)
\end{aligned}
$$

Therefore, $\{a_n\}$ is bounded so that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$.

Now we show that $\{a_n\}$ is not monotone.
Observe that $a_1 = \frac{1}{2} < \frac{1}{\sqrt{2}} = a_2$. Now suppose that $a_{n+1} > a_n$. Then we see that

$$
\begin{aligned}
a_{n+1} > a_n &\Leftrightarrow a_{n+1}^2 > a_n^2 && (19)\\
&\Leftrightarrow 1 - a_n > 1 - a_{n-1} && (20)\\
&\Leftrightarrow a_{n-1} > a_n && (21)
\end{aligned}
$$

Therefore, our sequence is not monotone and actually "looks" something like:

$$a_1 < a_2 > a_3 < a_4 > ...$$

So we will take two subsequences of $\{a_n\}$, the terms of odd indices and the terms of even indices.

Let $b_n = a_{2n}$ for all $n \geq 1$, and let $c_n = a_{2n-1}$ for all $n \geq 1$. Since $a_{2n} = \sqrt{1 - a_{2n-1}} = \sqrt{1 - \sqrt{1 - a_{2n-2}}}$, we have that $b_n = a_{2n} = \sqrt{1 - \sqrt{1 - b_{n-1}}}$.

Observe that $b_2 < b_1$. Now assume that $b_n < b_{n-1}$. We can see that:

$$0 < b_{n+1} < b_n < 1 \Leftrightarrow 0 < b_{n+1}^2 < b_n^2 < 1 \tag{22}$$
$$\Leftrightarrow 1 - b_{n-1} < 1 - b_n \tag{23}$$
$$\Leftrightarrow b_n < b_{n-1}. \tag{24}$$

So $b_{n+1} < b_n \Leftrightarrow b_n < b_{n-1}$. Therefore, $\{b_n\}$ is strictly decreasing. Similarly, $\{c_n\}$ is strictly increasing.

So by the Monotone Convergence Theorem, b_n and c_n are convergent, so $\{b_n\}$ converges to b and $\{c_n\}$ converges to c for some $b, c \in [0, 1]$.

Since $b_n = \sqrt{1 - \sqrt{1 - b_{n-1}}}$, we actually have that $b_n = \sqrt{1 - c_n}$. Therefore, taking limits of both sides we have that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt{1 - c_n}$$

Then for large enough N, we have that $b = \sqrt{1 - c}$, and similarly (by definition of $\{c_n\}$) we'll have that $c = \sqrt{1 - b}$. Therefore, we've established the following relationship between b and c :

$$b = 1 - c^2 \tag{25}$$
$$c = 1 - b^2 \tag{26}$$

By plugging in $1 - c^2$ for b in the second equation, and solving for c, we have:

$$c = 1 - (1 - c^2)^2 \tag{27}$$
$$\Rightarrow 2c^2 - c^4 - c = 0 \tag{28}$$
$$\Rightarrow c(c - 1)(c^2 + c - 1) = 0 \tag{29}$$
$$\Rightarrow c \in \{0, 1, \frac{-1 + \sqrt{5}}{2}\} \tag{30}$$

By plugging in $1 - b^2$ for c in the second equation, and solving for b, we have:

$$
\begin{aligned}
b &= 1-(1-b^2)^2 && (31)\\
&\Rightarrow 2b^2-b^4-b=0 && (32)\\
&\Rightarrow b(b-1)(b^2+b-1)=0 && (33)\\
&\Rightarrow b\in\{0,1,\frac{-1+\sqrt{5}}{2}\} && (34)
\end{aligned}
$$

However, $\{b_n\}$ converges to b, but this sequence is monotone decreasing, so $b \neq 1$. Therefore,

$$b \in \{0, \tfrac{-1+\sqrt{5}}{2}\}$$

And since $\{c_n\}$ converges to c and is monotone increasing, $c \neq 0$. Therefore,

$$c \in \{0, \tfrac{-1+\sqrt{5}}{2}\}$$

In order to show that $b = c = \frac{-1+\sqrt{5}}{2}$, we will find a positive number α such that $b_n > \alpha$ for all n so that $b \neq 0$.

Observe that $b_1 > \frac{-1+\sqrt{5}}{2}$. Now assume that $b_n > \frac{-1+\sqrt{5}}{2}$. Let $\beta = \frac{-1+\sqrt{5}}{2}$. Then

$$
\begin{aligned}
b_{n+1} = 1-\sqrt{1-b_n} &> \beta^2 && (35)\\
&\Leftrightarrow 1-\beta^2 > \sqrt{1-b_n} && (36)\\
&\Leftrightarrow (1-\beta^2)^2 > 1-b_n && (37)\\
&\Leftrightarrow b_n > 1-(1-\beta^2)^2 \geq \beta && (38)\\
& && (39)
\end{aligned}
$$

We know have that $b_n > \beta$ for all n, so $b = c = \frac{-1+\sqrt{5}}{2}$.
Therefore, the sequence $\{a_n\}$ converges, and it converges to $\frac{-1+\sqrt{5}}{2}$.

□

Problem 6.2 (Indiana, August 2007 Q1)**.** *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is differentiable at all points $(x,y) \in \mathbb{R}^2$ except $(0,0)$. Show that f is not differentiable at $(0,0)$.*

Solution. [Michael Kratochvil]
Attempt:
I first attempted this problem by looking at Royden's definition of differentiablity and noticed

an immediate problem: we would be dividing something that is real-valued by something that is vector-valued. UH OH!!! To rectify this, I realized there would need to be a different definition of differentiability in order to even attempt this problem. I appealed to the Internet and eventually baby Rudin's chapter on differentiability for the answer. Even better, within that chapter, I found the Theorem needed to do the first part of the problem. For the second part, I remembered from complex that we considered all kinds of paths to zero to disprove differentiability, so picking a couple intuitive one, the result was straightforward.
Solution:

First, recall the following definitions/theorem (found in Principles of Mathematical Analysis (baby Rudin)):

Definition (Rudin P.212): Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, and $x \in E$. If there exists a linear transformation A of $\mathbb{R}^n$ into $\mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0,$$

then we say that f is *differentiable at* x, and we write

$$f'(x) = A.$$

If f is differentiable at every $x \in E$, we say that f is *differentiable in* E.

Definition (Rudin P. 215): Consider f that maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Let $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_n\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. The *components of* f are the real functions $f_1, \dots, f_m$ defined by

$$f(x) = \sum_{i=1}^{m} f_i(x) u_i, \ \ x \in E,$$

or, equivalently, by $f_i(x) = f(x)\dot{u}_i, 1 \le i \le m$. For $x \in E, 1 \le i \le m, 1 \le j \le n$, we define

$$(D_j f_i)(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t},$$

provided the limit exists. $D_j f_i$ is called a *partial derivative*.

Theorem (Rudin P. 219): Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

Now we can proceed to the proof of this problem.

Proof: First note that for $(x, y) \neq (0,0)$ we see that f has the following partial derivatives:

$$D_x f(x,y) = \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2+y^2)^2}$$
$$D_y f(x,y) = \frac{y^4 + 3x^2y^2 - 2x^3y}{(x^2+y^2)^2}.$$

which is clearly continuous for $(x, y) \neq (0,0)$.
Notice that for all $x \neq 0$ $D_x(x,0) = 1$ and for all $y \neq 0$ $D_y(0,y) = 1$. So by the definition of partial derivative $D_x(0,0) = D_y(0,0) = 1$, so that the partial derivatives of f exist at $(0,0)$. However, it is not the case, that f is differentiable at $(0,0)$. To show this, let $A = [1\ 1]$. Then for $h = (t,0)$ we have

$$\lim_{h\to 0} \frac{|f(0+t,0) - f(0,0) - Ah|}{|h|} = \lim_{t\to 0} \frac{|t-0-t|}{|t|} = \lim_{t\to 0} \frac{0}{|t|} = 0.$$

Setting $h = (0,t)$ we obtain the same result. But looking at the path $h = (t,t)$ we obtain

$$\lim_{h\to 0} \frac{|f(0+t,0+t) - f(0,0) - Ah|}{|h|} = \lim_{t\to 0} \frac{|t-0-2t|}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}} \neq 0.$$

Thus f is not differentiable at $(0,0)$. □

Problem 6.3 (Indiana Real, Jan 2008 Q1). *Give an example a function $f : [0,\infty) \to \mathbb{R}$ that satisfies the three conditions:*

i. $f(x) \geq 0$ *for all* $x \geq 0$,

ii.For every $M > 0, sup_{x>M} f(x) = \infty$,

iii.$\int_0^\infty f(x)dx < \infty$.

or prove that no such f exists.

Solution. [Noah Kaufmann]
Attempt:
My first thought was that having a nonnegative, unbounded function with a finite integral shouldn't work out. But then I realized this was Analysis, so my intuition had to be wrong. I noticed that there was no mention of continuity, so that we could get all sorts of crazy functions. However, you can actually find a continuous function which works for this, and even C^∞ functions that work.
Solution:

Define f(x) as follows:

$$f(x) = \begin{Bmatrix} x & \text{for } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{Bmatrix}$$

So with this function f it is very easy to see that it satisfies the three requirements:

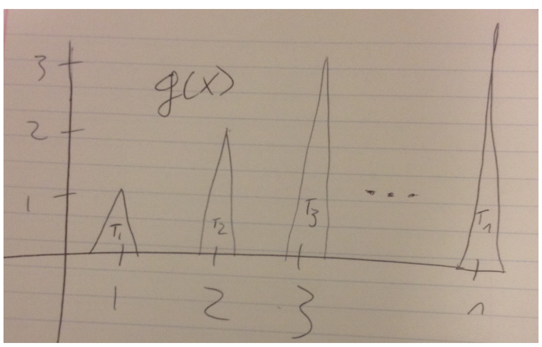
i. Clearly $f(x) \geq 0$ for all x.

ii. f(x) is unbounded, so this condition is also satisfied.

iii. Since $f = 0$ almost everywhere, the integral of f (both Riemann and Lebesgue, as the problem does not specify) is 0, and $0 < \infty$.

Therefore f is the desired function.

For an example of a continuous function which satisfies the above properties, consider the function defined by the following graph:



For this g, the value is 0 outside of the triangles, and each triangle T_n has height n and area $\frac{1}{2^n}$.

So g is nonnegative, unbounded, and the integral of g is the sum $\sum_0^\infty Area(T_n) = 1 < \infty$, since $Area(T_n) = \frac{1}{2^n}$.

Therefore g is a continuous function which satisfies the three original conditions. It is also possible to construct a C^∞ function by considering a function similar to g with bump functions instead of triangles.

□

Problem 6.4 (Indiana, January 2008 Q3). *Let S be a closed, nonempty, convex subset of $\mathbb{R}^n$. Given any point p in $\mathbb{R}^n - S$, let*

$$m = \inf_{q \in S} \|p - q\|$$

where $\| \bullet \|$ is the Euclidean norm. Prove that there exists exactly one point $q \in S$ that achieves this infimum.

Solution. [Shawn]

First we need to show that m is attained. Since S^c is open and $p \in S^c$, $\exists \delta > 0$ such that $B(p, \delta) \in S^c$ so $m > \delta > 0$. Choosing any $q \in S$ gives that $m \leq \|p - q\| < \infty$, so we are taking the infimum over a set of positive finite numbers. Thus m exists and is finite. Now choosing any $q \in S$, since S is closed, if $\|p - q\| = \epsilon$, we have that $S \cap B(p, \epsilon + 1)$ is closed and bounded, hence compact. We have shown elsewhere that the distance function $\inf_{q \in S} \|p - q\|$ is continuous, and we know that a continuous function attains its minimum on a compact set. Thus there exists at least one point $q \in S \cap B(p, \epsilon + 1)$ at which the infimum m is attained.

It remains to show that q is unique, so assume to the contrary that $\exists q_1, q_2$ with $q_1 \neq q_2$ such that $\|p - q_1\| = \|p - q_2\| = m$. Then for $q_3 := \frac{q_1 + q_2}{2}$, we have an isosceles triangle $\overline{p, q_1, q_2}$ containing a right triangle $\overline{p, q_2, q_3}$. Note that, since S is convex, $q_3 \in S$. By the Euclidean Pythagorean Theorem (since we are using the Euclidean norm) we have that $\|p - q_3\|^2 + \|q_3 - q_2\|^2 = \|p - q_2\|^2$. Since these are all positive numbers, $\|p - q_3\|^2 < \|p - q_2\|^2$ and thus $\|p - q_3\| < \|p - q_2\|$, which contradicts that q_2 is the infimum of such distances.

□

Problem 6.5 (Indiana, January 2008 Q6)**.** *Let f be a continuous function on $[0, \infty)$ such that $0 \leq f(x) \leq Cx^{-1-\rho}$ for all $x > 0$ and for some constants $C, \rho > 0$. Let $f_k(x) = kf(kx)$.*

(i) Show that $\lim_{k \to \infty} f_k(x) = 0$ for any $x > 0$ and that the convergence is uniform on $[r, \infty)$ for any $r > 0$.

(ii) Show that f_k does not converge to zero uniformly on $(0, \infty)$, unless f is identically 0.

Solution. [Amrei Oswald]

(i) From the definition of f_k we have

$$0 \leq f(x) \leq \frac{C}{x^{1+\rho}} \implies 0 \leq kf(kx) \leq \frac{kC}{(kx)^{1+\rho}} \implies 0 \leq f_k(x) \leq \frac{C}{k^\rho (x)^{1+\rho}}.$$

Since $\rho > 0$, taking the limit as $k \to \infty$ on the right hand side above gives us

$$0 \leq \lim_{k \to \infty} f_k(x) \leq \lim_{k \to \infty} \frac{C}{k^\rho (x)^{1+\rho}} = 0 \implies \lim_{k \to \infty} f_k(x) = 0.$$

Note that since $1 + \rho > 1$, we have that $r^{1+\rho} \leq x^{1+\rho}$ for $x \in [r, \infty)$. Therefore, on $[r, \infty)$ we have

$$0 \leq f_k(x) \leq \frac{C}{k^\rho (x)^{1+\rho}} \leq \frac{C}{k^\rho (r)^{1+\rho}}$$

$$\implies 0 \le \lim_{k\to\infty} f_k(x) \le \lim_{k\to\infty} \frac{C}{k^\rho (r)^{1+\rho}} = 0.$$

Since the right hand term $\frac{C}{k^\rho (r)^{1+\rho}}$ above does not depend on x, f_k converges uniformly to 0 on $[r, \infty)$.

(ii) Say that f_k converges to zero uniformly on $(0, \infty)$. Then, for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|kf(kx)| = |f_k(x)| < \epsilon \text{ for every } k > N, x \in (0, \infty)$$

$$\implies |f(kx)| < \frac{\epsilon}{k} \text{ for every } k > N, x \in (0, \infty).$$

Since $(k \cdot 0, k \cdot \infty) = (0, \infty)$, this gives us that

$$|f(x)| \le \frac{\epsilon}{k} \text{ for every } k > N, x \in (0, \infty) \implies f(x) = 0.$$

□

Problem 6.6 (Indiana, January 2008 Q8)**.** *Let $f : [0, 1] \to \mathscr{R}$ be a differentiable function such that $|f'(x)| \leqslant M$ for every $x \in (0, 1)$*
Show that for any positive integer n,

$$| \int_0^1 f(x)\, dx \; - \; \tfrac{1}{n} \sum_{k=1}^n f(\tfrac{k-1}{n}) \,| \leqslant \tfrac{M}{n}$$

Solution. [Violet Tiema] Because $f : [0, 1] \to \mathscr{R}$ is a differentiable function, we can see that f is continuous on $[0, 1]$ and therefore f is Riemann Intergrable and thus we can conclude that

$$\int_0^1 f(x)\, dx \; = \; \lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^n f(\tfrac{k-1}{n})$$

Notice that the interval $[\frac{1}{n}, \frac{n-1}{n}]$ is a partition of $(0, 1)$ and thus we have

$$\int_0^1 f(x)\, dx \; = \; \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \; f(x)\, dx$$

$$
\begin{aligned}
|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f(\frac{k-1}{n})\,| &= |\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx - \frac{1}{n}\sum_{k=0}^{n-1} f(\frac{k-1}{n})\,| \\
&= |\sum_{k=0}^{n-1}\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx - \frac{1}{n} f(\frac{k-1}{n})\,\}| \\
&= |\sum_{k=0}^{n-1}\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\frac{k}{n})\,dx\,\}| \\
&= |\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} \{f(x) - f(\frac{k}{n})\,\}\,dx| \\
&\le \sum_{k=0}^{n-1} |\int_{\frac{k}{n}}^{\frac{k+1}{n}} \{\,f(x)\,dx - f(\frac{k}{n})\,\}\,dx| \\
&\le \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |\{\,f(x) - f(\frac{k}{n})|\,|dx|
\end{aligned}
$$

Recall the Mean Value Theorem that states that

$$|\tfrac{f(a)-f(b)}{a-b}| \le |f'(c)| \le M$$

Therefore

$$|f(a) - f(b)| \le M\,|a-b|$$

So for every $x \in [\frac{k}{n}, \frac{k+1}{n}]$, we see that $|f(x) - f(\frac{k}{n})| \le M\,|x - \frac{k}{n}| = M \cdot \frac{1}{n}$
Therefore,

$$
\begin{aligned}
\sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |\{\,f(x) - f(\frac{k}{n})|\,|dx| &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} M\,|x - \frac{k}{n}|\,dx && (40) \\
&= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \{M \cdot \frac{1}{n}\}\,dx && (41) \\
&\le M \cdot \frac{1}{n} \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} dx && (42) \\
&\le M \cdot \frac{1}{n} && (43) \\
&\le \frac{M}{n} && (44)
\end{aligned}
$$

□

Problem 6.7 (W. Tyler Reynolds, Indiana, January 2013 Q3). *Determine all real x for which the following series converges:*

$$\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k.$$

You may use the fact that

$$\lim_{k\to\infty} \frac{k!}{\sqrt{2\pi k}(k/e)^k} = 1.$$

Solution.

Attempt:

My first approach was to try to compute $\limsup \sqrt[k]{a_k}$ or $\lim |a_k/a_{k+1}|$ directly to find the radius of convergence. I assumed that the fact given in the problem statement (called Stirling's approximation) would show up somehow after taking limits. After a finite sequence of failed efforts, I realized that Stirling's approximation was the first thing I should be using. For the formal proof, I had to start with the limits we wanted to end up with, then show that these were equal to the lim sup we wanted to take. Once the radius of convergence was found, the convergence on the boundary could be determined by more applications of Stirling's formula, along with old convergence tests from Calculus II.

Solution:

By L'Hôpital's Rule,

$$\lim_{k\to\infty} \frac{\ln 2\pi k}{2k} = \lim_{k\to\infty} \frac{1}{2k} = 0.$$

Therefore,

$$\lim_{k\to\infty} \sqrt[2k]{2\pi k} = e^0 = 1.$$

Since $\lim\limits_{k\to\infty} k! = \lim\limits_{k\to\infty} \sqrt{2\pi k}(k/e)^k$, we have $\lim\limits_{k\to\infty} \sqrt[k]{k!} = \lim\limits_{k\to\infty} \sqrt[2k]{2\pi k}(k/e)$. Therefore,

$$e = \lim_{k\to\infty} \frac{k}{(k/e)(\sqrt[2k]{2\pi k})} = \lim_{k\to\infty} \frac{k}{\sqrt[k]{k!}} = \lim_{k\to\infty} \sqrt[k]{\frac{k^k}{k!}}.$$

So by definition, the radius of convergence of the series $\sum\limits_{k=1}^{\infty} \frac{k^k}{k!} x^k$ is $1/e$.

Note that the series

$$\sum_{k=1}^{\infty} \frac{k^k}{\sqrt{2\pi k}(k/e)^k} (1/e)^k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi k}}$$

diverges by the p-series test (with $p = 1/2$). On the other hand, the series

$$\sum_{k=1}^{\infty} \frac{k^k}{\sqrt{2\pi k}(k/e)^k} (-1/e)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{2\pi k}}$$

converges by the alternating series test, since $\frac{1}{\sqrt{2\pi k}}$ decreases to 0 in the limit. Since $\lim_{k\to\infty} k! = \lim_{k\to\infty} \sqrt{2\pi k}(k/e)^k$, it follows that the series $\sum_{k=1}^{\infty} \frac{k^k}{k!}(1/e)^k$ diverges, while the series $\sum_{k=1}^{\infty} \frac{k^k}{k!}(-1/e)^k$ converges.

In conclusion, the series $\sum_{k=1}^{\infty} \frac{k^k}{k!}x^k$ converges precisely when $x \in [-1/e, 1/e)$. □

Problem 6.8 (W. Tyler Reynolds, Indiana, January 2013 Q4)**.**
(a) Prove that for all $a \in R$,

$$\left|\sum_{n=1}^{\infty} \frac{a}{n^2+a^2}\right| < \frac{\pi}{2}.$$

(b) Determine the least upper bound of the set of numbers

$$\left\{\left|\sum_{n=1}^{\infty} \frac{a}{n^2+a^2}\right| : a \in \mathbb{R}\right\}.$$

Solution.
Attempt:
My first try at this problem involved trying to simply get good bounds on the given series by arguments along the lines of straight inequalities, without bringing in any extra machinery. Having failed this, I eventually found a way to relate this problem to integration. The key to using integrals as bounds is the monotonicity of the series' terms. For an alternate approach and a fun digression that relates to complex analysis, you can do a bit of research and try to figure out why I can factually assert that $\sum_{k=1}^{\infty} \frac{a}{n^2+a^2} = \frac{\pi a \coth(\pi a) - 1}{2a}$.
Solution:
(a) If $a = 0$, we are done. Suppose $a > 0$. Since $\frac{a}{x^2+a^2}$ decreases with increasing $x \in (0, \infty)$, we have

$$\sum_{n=1}^{\infty} \frac{a}{n^2+a^2} < \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{a}{x^2+a^2}\,dx = \int_0^{\infty} \frac{a}{x^2+a^2}\,dx = \lim_{x\to\infty} \tan^{-1}(x/a) - \tan^{-1}(0)$$
$$= \frac{\pi}{2}.$$

It follows that if $a < 0$,

$$\left|\sum_{n=1}^{\infty} \frac{a}{n^2+a^2}\right| \leq \sum_{n=1}^{\infty} \frac{|a|}{n^2+|a|^2} < \frac{\pi}{2}.$$

(b) Again since $\dfrac{a}{x^2+a^2}$ decreases with increasing $x \in (0,\infty)$, we have

$$\sum_{n=1}^{\infty} \frac{a}{n^2+a^2} > \sum_{n=1}^{\infty} \int_n^{n+1} \frac{a}{x^2+a^2}\,dx = \int_1^{\infty} \frac{a}{x^2+a^2} = \lim_{x\to\infty} \tan^{-1}(x/a) - \tan^{-1}(1/a)$$
$$= \frac{\pi}{2} - \tan^{-1}(1/a).$$

Thus
$$\lim_{a\to\infty} \sum_{n=1}^{\infty} \frac{a}{n^2+a^2} \geq \lim_{a\to\infty} \left(\frac{\pi}{2} - \tan^{-1}(1/a)\right) = \frac{\pi}{2}.$$

Since $\sum_{n=1}^{\infty} \dfrac{a}{n^2+a^2} < \dfrac{\pi}{2}$ for all a, it follows that the sup $\left\{\left|\sum_{n=1}^{\infty} \dfrac{a}{n^2+a^2}\right| : a \in \mathbb{R}\right\} = \dfrac{\pi}{2}$. □

Problem 6.9 (W. Tyler Reynolds, Indiana, January 2013 Q5)**.** *Let $f(x)$ be continuous in the interval $I = (0,1)$. Define*

$$D_+ f(x_0) = \lim_{h\to 0^+} \inf \frac{f(x_0+h) - f(x_0)}{h}.$$

Put
$$S = \{x \in I : D_+ f(x) < 0\}.$$

Suppose that the set $f(I \setminus S)$ does not contain any non-empty open intervals. Prove that $f(x)$ is non-increasing on I.

Solution.
Attempt:
There wasn't any prior attempt here; we saw similar problems on differentiation during our first semester. In the immortal words of Yoda: "Do or do not; there is no try." Realistically, I followed my intuition and it rewarded me, fickle thing that it is.
Solution:
If f is constant, then $I = S$ and $f(I \setminus S) = \emptyset$ does not contain any non-empty open intervals, but $f(x)$ is increasing on I in the non-strict sense. We therefore must set out to prove that $f(x)$ is never strictly increasing on I.

Suppose for the sake of contradiction that $f(x)$ is strictly increasing on some open interval $(a,b) \subset I$. Then for $x \in (a,b)$, $\dfrac{f(x+h) - f(x)}{h} > 0$ for $0 < h < 1-b$, and hence $D_+ f(x) \geq 0$. So $(a,b) \subset I \setminus S$. Since f is strictly increasing and the continuous image of a connected set is connected, we have $f(a,b) = (f(a), f(b)) \subset f(I \setminus S)$. This is a contradiction. It follows that $f(x)$ is never strictly increasing on I. □

Problem 6.10 (Indiana Aug 2015, Problem 4)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with f' uniformly continuous. Suppose* $\lim_{x\to\infty} f(x) = L$. *Does* $\lim_{x\to\infty} f'(x)$ *exist?*

Solution. [Noah Kaufmann]
Attempt:

Intuitively, we know that if this limit exists, it must be equal to 0. In general, the limit of f' need not exist, but the condition of uniform continuity gives us exactly what we need to show that the limit does exist.
Solution:
We proceed by contradiction. Suppose $\lim_{x\to\infty} f(x)$ does not converge to 0. Looking at the definition a limit converging, we have that there exists an $\epsilon > 0$ and a sequence x_n such that $\lim_{n\to\infty} x_n = \infty$, and $|f'(x_n)| > \epsilon$ for all n. Now we make a couple of simplifying assumptions. We can assume L = 0, and we can assume that $f'(x_n) > \epsilon$ by picking a subsequence of x_n and renaming it, by considering f - L and -f respectively.

Now let δ be as in the definition of uniform continuity for f'. Then for any x $\in$ $[x_n - \frac{\delta}{2}, x_n + \frac{\delta}{2}], f'(x) > \frac{\epsilon}{2}$ for every n.

Since f converges to 0 at infinity, we can choose any sequence y_n which goes to infinity, and f(y_n) will go to 0. Choose y_n to contain only points of the form $x_n - \frac{\delta}{2}$ and $x_n + \frac{\delta}{2}$, ordered so that $x_n - \frac{\delta}{2}$ appears directly before $x_n + \frac{\delta}{2}$ for every n, and these terms show up in the same order as the original x_n's. Note that this is valid because the x_n's go to infinity.

Now, since f(y_n) goes to 0, we can choose an N so that $|f(y_n)| < \frac{\delta\epsilon}{4}$ for every n $\geq$ N. Then choose an m $\geq$ N such that $y_m = x_k - \frac{\delta}{2}$. This means that $y_{m+1} = x_k + \frac{\delta}{2}$, and we should have that both y_m and y_{m+1} are less than $\frac{\delta\epsilon}{4}$. However, since $f' > \frac{\epsilon}{2}$ on the interval (of length δ) between y_m and y_{m+1}, we have (using the fundamental theorem of calculus) that $y_{m+1} > y_m + \frac{\delta\epsilon}{2}$. Since $|y_m| < \frac{\delta\epsilon}{4}$, this forces $y_{m+1} > \frac{\delta\epsilon}{4}$. This contradicts the definition of convergence, so our assumption that f' did not converge to 0 must be false. Therefore $\lim_{x\to\infty} f'(x)$ exists and is equal to 0.

Note: This proof gets kind of flooded with $\epsilon's$ and $\delta's$ at the end. The main idea is that if f' does not go to 0, there is some subsequence that stays positive. Then uniform continuity gives that there must be a sequence of intervals with f' positive, so that f cannot go to 0. □

Problem 6.11 (Indiana, August 2015 Q5)**.** *Let $E \subset \mathbb{R}$ such that any countable closed cover of E contains a finite subcover. Show that E is a finite set of points.*

Solution. [Shawn]
Attempt:
BWOC, assume that E has a countably infinite subset $\{x_n\}_{\mathbb{N}}$. It suffices to show that we can construct a countable closed cover of E with no finite subcover. We begin with the countable collection of point sets $\{x_n\}_{\mathbb{N}}$. To this collection we add all sets of the form $[x_\alpha + \frac{1}{n}, x_\beta - \frac{1}{n}]$ for $n \in \mathbb{N}$ where x_α and x_β are any two elements of $\{x_n\}_{\mathbb{N}}$ which have no other element of $\{x_n\}_{\mathbb{N}}$ between them. Finally we include the sets $(-\infty, \inf\{x_n\}_{\mathbb{N}}]$ and $[\sup\{x_n\}_{\mathbb{N}}, \infty)$. The

countable union of these closed sets covers all of $\mathbb{R}$, so it certainly covers E. However, the only sets in this cover of E that contain the points $\{x_n\}_{\mathbb{N}}$ are the point sets $\{x_n\}_{\mathbb{N}}$ themselves (except perhaps for the infinite sets if the sup or inf of E are elements of $\{x_n\}_{\mathbb{N}}$). Thus no finite subcollection of this cover can cover the points $\{x_n\}_{\mathbb{N}}$ in E, so no finite subcover exists. The issue with this is that $\{x_n\}_{\mathbb{N}}$ could be dense so that the intervals between the points can't be formed.

Solution:
Assume E is not finite but still every countable closed cover contains a finite subcover. If E were countable with elements x_n for $n \in \mathbb{N}$, then $\{x_n\}_{\mathbb{N}}$ is a closed cover with no finite subcover, so E must be uncountable. An uncountable subset of $\mathbb{R}$ must be uncountable on some closed interval (since otherwise $E = \cup E_n$ for $n \in \mathbb{Z}$ where $E_n = [n, n+1] \cap E$, so E would be a countable union of countable sets). WLOG assume E is uncountable on $[0, 1]$. Define the first element of a closed cover by $E_0 = (-\infty, 0] \cup [1, \infty)$. Now express $[0, 1]$ as the union $[0, a] \cup [a, 1]$ for some $a \in (0, 1)$. We can choose a so that one of the intervals has uncountably many elements of E while the other contains at least one element of E. Let U_1 be the set with uncountably many elements of E and call the other E_1. We then repeat the process on U_1, letting E_2 contain at least one element of E while U_2 contains uncountably many. Continuing in this way, $\{E_n\}_{\mathbb{N}} \cup E_0$ gives a countable closed cover of E. However, since the elements of the cover intersect in only countably many points (their endpoints) while every set in $\{E_n\}_{\mathbb{N}}$ contains points in E, no finite subcollection can cover E. By this contradiction E can not be uncountable, and since we know it can't be countably infinite, it must be finite.

□

Problem 6.12 (Indiana Real, August 2016 Q4)**.** *Using only the definitions of continuity and (sequential) compactness, prove that if $K \subset \mathbb{R}$ is (sequentially) compact and $f : K \to \mathbb{R}$ is continuous, then f is uniformly continuous, that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

Continuous Function: A real valued function f is continuous iff when $\{x_n\} \to x$, then $\{f(x_n)\} \to f(x)$ for every x in the domain of f.
Sequenctial Compactness: A metric space is sequentially compact provided every sequence X has a subsequence that converges to a point in X.

Solution. [Jared Grove]
Attempt:
My first attempt was to show it directly, but I ran into some problems. Take $x, y \in K$, since K is compact there exists some sequences $\{x_n\}$ and $\{y_n\}$ that converge to x and y respectively. Because f is continuous let $\epsilon > 0$ then there exists $\delta > 0$ such that when $|x_n - x| < \delta$ and $|y_n - y| < \delta$ then $|f(x) - f(x_n)| < \epsilon$ and $|f(y) - f(y_n)| < \epsilon$. Then look at $|f(x) - f(y)| = |f(x) - f(x_n) + f(x_n) - f(y) + f(y_n) - f(y_n)| \leq |f(x) - f(x_n)| + |f(y_n) - f(y)| + |f(x_n) - f(y_n)| < 2\epsilon + |f(x_n) - f(y_n)|$. I didn't know how to deal with that last

term so after beating my head against a wall trying to make this work Alex pointed me in a different direction.
Solution:
Assume for contradiction that we don't have uniform continuity. That is for some $\epsilon > 0$ for every $\delta > 0$ there exists $x_\delta, y_\delta \in K$ such that when $|x_\delta - y_\delta| < \delta$, $|f(x) - f(y)| \geq \epsilon$. Let $\{x_n\}, \{y_n\}$ be sequences of these points and $\delta_n = \frac{1}{n}$. Thus when $|x_n - y_n| < \frac{1}{n}$ then $|f(x_n) - f(y_n)| \geq \epsilon$. Since K is sequentially compact there exists subsequences of $\{x_n\}$ and $\{y_n\}$ that converge to some $x, y \in K$. We will denote the subsequences $\{x_k\}$ and $\{y_k\}$. Now look at:

$$|y_k - x| = |y_k - x_k + x_k - x| \leq |y_k - x_k| + |x_k - x| < \frac{1}{k} + \delta$$

Thus $\{y_k\}$ converges to x as well. Essentially the two sequences are converging to the same point. From here I will present two ways of approachng this

1 Since $|x_k - y_k| < \frac{1}{k}$, we have that $\lim_{k\to\infty} |x_m - y_m| = |x - y| \leq 0$. Thus $x = y$. Since $|x - y| < \delta$ for any δ, then $|f(x) - f(y)| \geq \epsilon$, but as $x = y$, $f(x) - f(y) = 0$ and we have a contradiction. Thus f must be uniformly continuous.

2

$$|f(y_k - f(x)| \leq |f(y_m) - f(x_m)| + |f(x_m) - f(x)| < 2\epsilon'$$

Because f is continuous and both $\{y_k\}$ and $\{x_k\}$ converge to x. Either way contradiction and we have uniform continuity.

□

Problem 6.13 (Indiana, August 2016 Q5)**.** *Show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, then the set of limit points of $\{x_n\}$ is connected, that is, either empty, a single point, or an interval.*

Solution. [Michael Kratochvil]
Attempt:
The heart of the problem was pretty much immediate: either there are at least two limit points or there are zero or one, so that it boiled down to just proving an arbitrary point between any two limit points is also a limit point. Early attempts tried to make use of convergent subsequences and convexity, but that "quickly" led nowhere. What helped the most was drawing a picture of the interval between the two limit points and realizing that the sequence would ultimately be oscillating back and forth between the two limit points with very small steps. That intuition led to the following proof.
Solution:

Suppose the set of limit points has at least two distinct elements, say x and y with $x < y$. Let $z \in (x, y)$ and let $\epsilon \in (0, \frac{x+y}{2})$. We want to show that there is an $x_n \in B(z, \epsilon)$. Note that there is an $N \in \mathbb{N}$ such that $|x_{n+1} - x_n| < 2\epsilon$ for all $n \geq N$. Define

$$\delta_1 = \min\{|x - x_1|, |x - x_2|, \ldots, |x - x_N|, \epsilon, \frac{y-x}{2}\},$$
$$\delta_2 = \min\{|y - x_1|, |y - x_2|, \ldots, |y - x_N|, \epsilon, \frac{y-x}{2}\},$$

and let $m, k \geq N$ such that $x_m \in B(x, \delta_1)$, $x_k \in B(x, \delta_2)$ (without loss of generality assume $m < k$). I claim $B(z, \epsilon) \cap \{x_n\}_{n=m}^k \neq \emptyset$. To show this assume $x_m \notin B(z, \epsilon)$ (if this is the case, the proof is trivially complete). Let x_l be the *last* element of $\{x_n\}_{n=m}^k$ such that $x_n \leq z - \epsilon$ (this must occur since by construction $x_m < z - \epsilon < x_k$). Then since $x_{l+1} - x_l < 2\epsilon$, we have $x_{l+1} < x_l + 2\epsilon \leq z + \epsilon$. Further, $x_{l+1} > z - \epsilon$ by assumption. Thus $x_l \in B(z, \epsilon)$. Since ϵ was arbitrarily chosen, we have that z is a limit point. This shows that all real numbers between two distinct limits points are themselves limit points, showing that the set of limit points is an interval. □

Problem 6.14 (Nicholas Camacho, Jan 2017 Q9)**.** *A continuously differentiable function $f : [0, 1] \to [0, 1]$ has the properties*

(a) $f(0) = f(1) = 0$

(b) $f'(x)$ is a non-increasing function of x.

Prove that the arclength of the graph of f does not exceed 3.

Proof.
Attempt: I first assumed that arclength and Total Variation are the same thing, but they are not. It turns out that they are the same only when f is absolutely continuous. So, instead I used the Calculus definition of arclength and the associated Riemann Sum.

Solution: By (a), the Mean Value Theorem gives us $c \in (0, 1)$ such that $f'(c) = 0$. Since f' is non-increasing (i.e., f' is decreasing) then

$$f'(x) = \begin{cases} \geq 0 & \text{if } x \in [0, c] \\ \leq 0 & \text{if } x \in [c, 1] \end{cases},$$

or in other words, $f(x)$ is increasing on $[0, c]$ and decreasing on $[c, 1]$. So if $P = \{x_0, \ldots, x_n\}$ is any partition of $[0, 1]$, let $P_c = \{x_0, \ldots, x_k = c, \ldots x_n\}$. Then $V(f, P) \leq V(f, P_c)$, and

$$\begin{aligned} V(f, P_c) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{k} f(x_i) - f(x_{i-1}) + \sum_{i=k+1}^{n} f(x_{i-1}) - f(x_i) \\ &= -f(0) + f(c) + f(c) - f(1) \\ &= 2f(c) \\ &\leq 2 \end{aligned}$$

So if $||P|| = 1/n$, (i.e., the distance between consecutive points in the partition is $1/n$). Then by the Mean Value Theorem, we can pick $x_i^* \in (x_{i-1}, x_i)$ such that $f'(x_i^*) = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = n(f(x_i) - f(x_{i-1}))$. Now, here's an obnoxious string of equalities/inequalities:

$$\begin{aligned}
\sum_{i=1}^{n}(x_i - x_{i-1})\sqrt{1+(f'(x_i^*))^2} &= \sum_{i=1}^{n} \frac{1}{n}\sqrt{1+n^2(f(x_i)-f(x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{\frac{1}{n^2} + (f(x_i)-f(x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{|x_i - x_{i-1}|^2 + (f(x_i)-f(x_{i-1}))^2} \\
&\leq \sum_{i=1}^{n} |x_i - x_{i-1}| + (f(x_i)-f(x_{i-1})) \\
&= \sum_{i=1}^{n} \frac{1}{n} + (f(x_i)-f(x_{i-1})) \\
&= 1 + \sum_{i=1}^{n}(f(x_i)-f(x_{i-1})) \\
&\leq 3
\end{aligned}$$

Recall that the arclength of f is

$$\int_0^1 \sqrt{1+(f'(x))^2}\ dx,$$

and so by definition of the Riemann integral, we have

$$\int_0^1 \sqrt{1+f'(x)^2}\ dx = \lim_{n\to\infty}\sum_{i=1}^{n}(x_i - x_{i-1})\sqrt{1+(f'(x_i^*))^2} \leq 3.$$

□

Problem 6.15 (Indiana, August 2016 Q9)**.** *Define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$d(x,y) = \frac{||x-y||}{||x||^2 + ||y||^2 + 1}$$

where $||x||^2 = x_1^2 + \cdots + x_n^2$. Let $A \subset \mathbb{R}^n$ be such that there exists $\epsilon > 0$ so that if $a, b \in A$ with $a \neq b$, then $d(a,b) \geq \epsilon$. Show that A is finite.

Solution. [Sara Reed] We will consider two cases: $A = \emptyset$ and $A \neq \emptyset$. If $A = \emptyset$, A is finite. Now, consider $A \neq \emptyset$. Let $a \in A$. Now, consider $A' = \{x \in \mathbb{R}^n : x + a \in A\}$. Note that

$0 \in A'$. If we can show A' is finite, then we can conclude A is finite. Note that we still have the property that there exists $\epsilon > 0$ so that if $c, d \in A'$ with $c \neq d$, then $d(c, d) \geq \epsilon$. Consider $x \in \mathbb{R}^n$ such that $||x|| > \frac{1}{\epsilon}$. Then we know

$$d(x, 0) = \frac{||x||}{||x||^2 + 1} < \frac{||x||}{||x||^2} = \frac{1}{||x||} < \epsilon.$$

Therefore, $x \notin A'$ when $||x|| > \frac{1}{\epsilon}$. So $A' \subset [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ and therefore A' is bounded. Now we want to show A' is closed. Let a^* be a limit point of A'. By way of contradiction, assume $a^* \notin A'$. By definition of a limit point, there exists $x \in A'$ such that $\delta = d(a^*, x) \leq \epsilon$. Similarly, there exists $y \in A'$ such that $\sigma = d(a^*, y) \leq \frac{\delta}{2}$. Again, there exists $z \in A'$ such that $d(a^*, z) \leq \frac{\sigma}{2} < \frac{\delta}{2}$. Note that $y \neq z$. Then

$$\begin{aligned} d(y, z) &\leq d(y, a^*) + d(a^*, z) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \\ &\leq \epsilon. \end{aligned}$$

We have reached a contradiction since y, z cannot both be in A' since $d(y, z) < \epsilon$. Therefore, $a^* \in A'$ and A' is closed. Since A' is closed and bounded, we have that A' is compact. Therefore, A' is sequentially compact. Finally, by way of contradiction, assume A is not finite. Take a sequence $\{x_n\}_n \subset A'$. Since A' is sequentially compact, there exists a subsequence $\{x_{n_k}\}_k$ that converges to $x^* \in A'$. By the definition of A', we know $B(x^*, \frac{\epsilon}{2}) \cap A' = \emptyset$. This is a contradiction of $x*$ being a limit point of $\{x_{n_k}\}_k$. Therefore, we conclude A' is finite which implies A is finite as desired. □

Problem 6.16 (Elaina Aceves, Jan 2017 Q7)**.** *Let $f_n(x)$ and $f(x)$ be continuous functions on $[0, 1]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0, 1]$.*
(a) Can we conclude that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$?
(b) If in addition we assume $|f_n(x)| \leq 2017$ for all n and for all $x \in [0, 1]$, can we conclude that $\lim_{n\to\infty} \int_0^1 f_n(x))dx = \int_0^1 f(x)dx$?

Solution.
Attempt:
For part (a), I immediately started looking for a counterexample and for part (b), the Bounded Convergence Theorem sprang to mind. Part (b) is actually a special case of BCT and a counterexample for part (a) is given in the textbook immediately before the BCT.
Solution:
(a) Define $f_n : [0, 1] \to \mathbb{R}$ such that

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, x \geq 2/n \\ n & \text{if } x = 1/n \\ \text{linear} & \text{if } x \in [0, 1/n] \text{ or } x \in [1/n, 2/n] \end{cases}$$

To clarify, $f_n(x)$ is the line that joins 0 and $1/n$ for $x \in [0, 1/n]$ and similarly, $f_n(x)$ is the line that joins $1/n$ and $2/n$ for $x \in [1/n, 2/n]$ which makes a 'triangle' of height n and width $2/n$. Then f_n is continuous for all n and for all $x \in [0, 1]$. Also,

$$\begin{aligned} \int_0^1 f_n(x)dx &= \int_0^{2/n} f_n(x)dx + \int_{2/n}^1 0\,dx \\ &= \frac{(2/n)(n)}{2} + 0 = 1 \end{aligned}$$

for all n/ Also, $f_n \to f$ where $f \equiv 0$ and we know that $\int_0^1 0\,dx = 0$. Thus,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1 \neq 0 = \int_0^1 f(x)dx$$

(b) Recall that every continuous, real-valued function defined on a measurable subset of $\mathbb{R}$ is measurable. Then by the Bounded Convergence Theorem (p. 78) with $E = [0, 1]$ and $M = 2017$, we have the result. □

Problem 6.17 (Indiana, January 2017 Q2). *Prove that the sequence*

$$a_1 = 1,\ \ a_2 = \sqrt{7},\ \ a_3 = \sqrt{7\sqrt{7}},\ \ a_4 = \sqrt{7\sqrt{7\sqrt{7}}},\ \ a_5 = \sqrt{7\sqrt{7\sqrt{7\sqrt{7}}}}...$$

converges, and find its limit.

Solution. [Rajinda Wickrama]

Proof. Observe that we can write the above sequence recursively as given below

$$a_1 = 1, \quad a_{n+1} = \sqrt{7a_n}\ \ \forall n \in \mathbb{N}$$

Now, let us prove that $\{a_n\}$ is increasing. This can be done inductively.
When $n = 1$, observe that $a_2 = \sqrt{7} > 1 = a_1$. Now suppose that for some $k \in \mathbb{N}$, $a_k \geq a_{k-1}$. Then, $a_{k+1} = \sqrt{7a_k} \geq \sqrt{7a_{k-1}} = a_k$. Therefore $a_{n+1} \geq a_n\ \forall n \in \mathbb{N}$.

Similarly, we can show that this sequence is bounded above by 7. When $n = 1$, $a_1 < 7$. Now suppose $a_k \leq 7$ for some $k \in \mathbb{N}$. Then, $a_{k+1} = \sqrt{7a_k} \leq \sqrt{7 \cdot 7} = 7 \implies a_n \leq 7\ \forall n \in \mathbb{N}$. Therefore, $\{a_n\}$ converges.

Now suppose $\lim\limits_{n\to\infty} a_n = L$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7a_n} = \sqrt{7L} \implies L^2 = 7L \implies L = 0 \text{ or } 7$$

Since the sequence is always positive and increasing $L = 7$.

□

□

Problem 6.18 (Indiana, August 201, 1.). *Let $f(x)$ be a continuous function on $(0,1]$ and*

$$\liminf_{x\to 0^+} f(x) = \alpha, \quad \limsup_{x\to 0^+} f(x) = \beta.$$

Prove that for any $\xi \in [\alpha, \beta]$, there exists $\{x_n \in (0,1] \mid n = 1, 2, \ldots\}$ such that

$$\lim_{n\to\infty} f(x_n) = \xi.$$

Solution. [Adam Wood]

Proof. Let $\xi \in [\alpha, \beta]$. Since $\liminf_{x\to 0^+} f(x) = \alpha$, $\lim_{x\to 0^+} (\inf\{f(t) \mid 0 < t < x\}) = \alpha$. That is, for every $n \in \mathbb{N}$, there exists $\delta_1 > 0$ so that if $0 < x < \delta_n$,

$$|\inf\{f(t) \mid 0 < t < x\} - \alpha| < \frac{1}{2n}.$$

Similarly, since $\limsup_{x\to 0^+} f(x) = \beta$, $\lim_{x\to 0^+} (\sup\{f(t) \mid 0 < t < x\}) = \beta$. That is, for every $n \in \mathbb{NN}$, there exists $\delta_2 > 0$ so that if $0 < x < \delta_2$,

$$|\sup\{f(t) \mid 0 < t < x\} - \beta| < \frac{1}{2n}.$$

Let $\delta_n = \min\{\delta_1, \delta_2\}$, suppose $x < \delta_n$, let $a_n = \inf\{f(t) \mid 0 < t < x\}$, and let $b_n = \sup\{f(t) \mid 0 < t < x\}$. By definition of the infimum and the supremum, for every $n \in \mathbb{N}$, there exists $t_1, t_2 \in (0, x)$ so that $f(t_1) - a_n < \frac{1}{2n}$ and $b_n - f(t_2) < \frac{1}{2n}$. Then,

$$|f(t_1) - \alpha| = |f(t_1) - a_n + a_n - \alpha| \leq |f(t_1) - a_n| + |a_n - \alpha| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

and

$$|f(t_2) - \beta| = |f(t_2) - b_n + b_n - \beta| \leq |f(t_2) - b_n| + |b_n - \beta| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

Note that $f(t_1) < f(t_2)$ since $f(t_1)$ is close to the infimum and $f(t_2)$ is close to the supremum. Let $t_* = \min\{t_1, t_2\}$ and let $t^* = \max\{t_1, t_2\}$. Choose $c \in [f(t_1), f(t_2)]$ so that $|c - \xi| < \frac{1}{n}$. We can make such a choice since $\xi \in [\alpha, \beta]$ and since $f(t_1)$ and $f(t_2)$ are at most $\frac{1}{n}$ away from α and β, respectively. Then, since f is continuous on $(0,1]$, f is in particular continuous on $[t_*, t^*]$. By the intermediate value theorem, since $c \in [f(t_1), f(t_2)]$, there exists $x_n \in [t_*, t^*]$ so that $f(x_n) = c$. That is, $|f(x_n) - \xi| < \frac{1}{n}$. Letting $n \to \infty$, we have that

$$\lim_{n\to\infty} f(x_n) = \xi.$$

□

□

7 Additional Practice

7.1 Group Work I

Problem 7.1 (Group Work 1 Real, Number 1)**.** *Let $\ell^2(\mathbb{N})$ be the collection of all square sumable sequences in $\mathbb{R}$. Prove that the closed unit ball is not compact.*

Solution. [Jared Grove, Meghan Malachi, Alex Bates]
Attempt:
Some of the ideas we had were to make a sequence which wouldn't converge (the right way we just didn't come up with one) or show that it is infinitely dimensional (this would contradict Riesz's Theorem p261). In order to do that you need to show that there are two norms on this space which are not equivalent.
Solution:
First note that $\ell^2(\mathbb{N})$ is a normed linear space with the norm: $||\{a_n\}||_2 = \sum_{n=1}^{\infty} |a_n|^2$. The unit ball in this space is $B = \{\{x_n\} : \sum_{n=1}^{\infty} |x_n|^2 \leq 1\}$. Let's assume Ab Absurdo that B is compact. Then we can also say that B is sequentially compact and every sequence in B must have a subsequence that converges to a point in B. Consider the sequences:

$$\begin{aligned} a_1 &= (1,0,0,0,0,0,0,\ldots) \\ a_2 &= (0,1,0,0,0,0,0,\ldots) \\ a_3 &= (0,0,1,0,0,0,0,\ldots) \\ &\vdots \end{aligned}$$

where for $\{a_n\}$ the n-th element in the sequence will be 1 while every other element in 0. Clearly $\{a_n\} \in B$ for all $n \in \mathbb{N}$ as $\sum_{k=1}^{\infty} |a_{n_k}|^2 = 1$ (Here a_{n_k} denotes the kth element in the sequence a_n). However, $\lim_{n\to\infty}\{a_n\}$ does not exist.

If it did exist then there exists some $f \in B$ such that $\lim_{n\to\infty} ||f - a_n||_2 = 0$. If there exists some N s.t. $||f - a_N||_2 = 0$, then $f_N = 1$ and $f_k = 0$ for all $k \neq N$. However, $||f - a_n||_2 = |f_n - a_{n_n}|^2 + |f_N - a_{n_N}|^2 = |0-1|^2 + |1-0|^2 = 2 \neq 0$ for all $n \neq N$. Thus $\lim_{n\to\infty} ||f - a_n||_2 = 2$ if $||f - a_N||_2 = 0$. If $||f - a_n||_2 \neq 0$ for all $n \in \mathbb{N}$ then $\sum_{k=1} |f_k - a_{n_k}|^2 \neq 0$ while $\lim_{n\to\infty} \sum_{k=1} |f_k - a_{n_k}|^2 = \lim_{n\to\infty} |f_n - a_{n_n}|^2 + \sum_{k=1,k\neq n} |f_k|^2 = 0$. In order for this to be possible we would need to have that $f_n \to 1 = a_{n_n}$ and $f_n \to 0$ as $n \to \infty$. Since it can't do both at once the limit must not exist.

Now consider M an infinite subset of $\mathbb{N}$. Define $\{a_m\}$ the same as $\{a_n\}$ so that $\{a_m\}$ is any subsequence of $\{a_n\}$. Clearly $\lim_{m\to\infty}\{a_m\}$ does not exists for any $M \subset \mathbb{N}$ by the same arguments as above. Thus the sequence of $\{\{a_n\}\}_{n=1}^{\infty}$ has no convergent subsequence and is therefore not sequentially compact and also not compact.

□

Problem 7.2 (Group Work I, Real, Number 3)**.** *Suppose $\{f_\alpha\}_{\alpha\in A}$ is a family of continuous real-valued functions such that $\sup_{\alpha\in A} |f_\alpha(x)|$ is finite for each $x \in \mathbb{R}$. Prove there exists a non-empty open interval $I \subset \mathbb{R}$ such that $\sup_{x\in I} \sup_{\alpha\in A} |f_{\alpha\in A}| < \infty$.*

Solution. [Michael Kratochvil, Amrei Oswald, Yanqing Shen]
Attempt:
We initially tried to prove this by contradiction. We assumed that no such interval exists and then defined $I_n = (x' - \frac{1}{n}, x' + \frac{1}{n})$ for some $x' \in \mathbb{R}$. Then we have

$$\sup_{x \in I_n} \sup_{\alpha \in A} |f_\alpha(x)| = \infty \ \forall\ n \in \mathbb{N}$$

$$\implies \infty = \lim_{n \to \infty} \sup_{x \in I_n} \sup_{\alpha \in A} |f_\alpha(x)| = \sup_{\alpha \in A} |f_\alpha(x)|.$$

This would be a contradiction. However, it turns out that the equality on the right above is not correct. We actually only have

$$\lim_{n \to \infty} \sup_{x \in I_n} \sup_{\alpha \in A} |f_\alpha(x)| \geq \sup_{\alpha \in A} |f_\alpha(x)|.$$

Rolando suggested using the Baire Category Theorem instead.
Solution:
Define $g(x) = \sup_{\alpha \in A} |f_\alpha(x)|$ and $E_n = \{x \in \mathbb{R} | g(x) \leq n\}$. Note that

$$E_n = \bigcap_{\alpha \in A} \{x \in \mathbb{R} | f_\alpha(x) \leq n\} = \bigcap_{\alpha \in A} f_\alpha^{-1}([-n, n]).$$

Since f_α is continuous for every $\alpha \in A$, the pre-image of the closed set $[-n, n]$ under f_α is closed. As such, each E_n is an intersection of closed sets and is therefore closed.

Further, since $g(x)$ is finite for each $x \in \mathbb{R}$, $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$. By Corollary 4 (Royden, p 212), at least one of the E_n's has a nonempty interior, say E_N. Then $\text{int}E_N$ is a nonempty open set $\implies$ $\text{int}E_N = \bigcup I_k$ where $\{I_k\}$ is a collection of nonempty, open intervals. Choose $I_1 = I$. Then we have

$$\sup_{x \in I} \sup_{\alpha \in A} |f_{\alpha \in A}| = \sup_{x \in I} g(x) \leq N < \infty.$$

□

Problem 7.3 (Group Work I Real, Number 5)**.** *Let $\{f_n\}_{n \in \mathbb{N}}, f : E \to \mathbb{R}$ be a family of measurable real-valued functions which converge to a measurable function f in measure. Prove the existence of a subsequence f_{n_k} which converges to f pointwise almost everywhere. Provide an example of a sequence of measurable functions which convergence in measure but not pointwise almost everywhere.*

Solution. [Kaitlin Healy, Sara Reed, W. Tyler Reynolds, Adam Wood]
Attempt:
The key idea here was to pick a subsequence so that some collection of sets would have a finite sum of measures, allowing us to use the Borel-Cantelli Lemma and thus obtain a property implicit in those sets (in this case, pointwise convergence) which holds almost everywhere. I can't recall if we had any other promising ideas before arriving at this one; but once we were there, it was just a matter of fleshing it out. This proof can also be found in Royden (Section 5.2 Theorem 4, pages 100-101).

Solution:
By definition of convergence in measure, for each $\eta > 0$ we have $\lim_{n\to\infty} \mu(\{x \in E : |f(x) - f_n(x)| > \eta\}) = 0$. For each k, let $E_k = \{x \in E : |f(x) - f_{n_k}(x)| > 1/k\}$. We may pick a subsequence $\{n_k\}$ so that $m(\{x \in E : |f(x) - f_{n_k}(x)| > 1/k\}) < 1/2^k$ for each k. For each k, let $E_k = \{x \in E : |f(x) - f_{n_k}(x)| > 1/k\}$. By the countable subadditivity of measure,

$$m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$

By the Borel-Cantelli Lemma, there is a set $E_0 \subset E$ with $m(E \setminus E_0) = 0$ such that each $x \in E_0$ belongs to only finitely many of the E_k's. Given $x \in E_0$ and $\epsilon > 0$, let K be such that $1/K < \epsilon$ and $x \notin E_k$ for $k \geq K$. Then for $k \geq K$,

$$|f(x) - f_{n_k}(x)| \leq \frac{1}{K} < \epsilon.$$

Hence $f_{n_k} \to f$ pointwise on E_0, and thus pointwise a.e. on E.

For an example of a sequence of measurable functions which converge pointwise in measure but not pointwise a.e., let $E = [0,1]$ and consider the sets $A_1 = [0,1]$, $A_2 = [0,1/2]$, $A_3 = [1/2,1]$, $A_4 = [0,1/3]$, $A_5 = [1/3,2/3]$, $A_6 = [2/3,1]$, and so on. For each k, let $f_k = \chi_{A_k}$. Given $\eta > 0$, we can find a K so that $1/k < \eta$ for $k \geq K$; since $m(A_k) < 1/k < \eta$ for $k \geq K$, it follows that $f_k \to 0$ in measure on $\mathbb{R}$. However, for every $x \in [0,1]$, x belongs to infinitely many of the A_k's and thus $f_k(x) = 1$ for infinitely many k. Thus f_k does not converge to f pointwise on $[0,1]$.

□

7.2 Group Work II

7.3 Practice Exam I

7.4 Practice Exam II

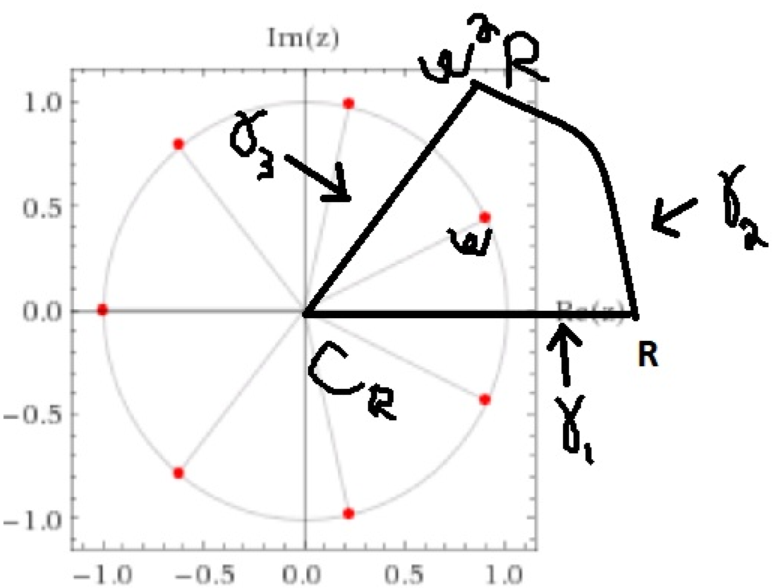
Part II
Complex Analysis

8 Kansas Qual

Problem 8.1 (Kansas Spring, 2004 Q3). *Compute* $\int_0^\infty \frac{dx}{1+x^7}$.
Hint: For (large) $R > 1$, *use the boundary of the circular sector*

$$C_R = \{re^{i\theta} : 0 < r < R, 0 < \theta < \frac{2\pi}{7}\}.$$

Solution. [Michael Kratochvil]
Attempt:
We have seen similar problems in the form of $\int_0^\infty \frac{dx}{1+x^n}$ for $n \in \mathbb{N} - \{1\}$ for various values of n in class, and the procedure for this one is the same as those. The curve we need to integrate on is the "pizza slice." The only part of evaluation that I had difficulty with was the part that maps from the top of the curve back to the origin in a straight line fashion. Once I realized that I could just work in polar coordinates and integrate with respect to the size of the radius, the rest was pretty straightforward. I should also note that this problem generalizes for any $n \in \mathbb{N} - \{1\}$. The proof is identical to what I have below, just replace 7 with n, basically.
Solution:
Let $\gamma_1, \gamma_2, \gamma_3$ be defined from the picture below.



Define $f(z) = \frac{1}{1+z^7}$ and $\omega = e^{i\pi/7}$. Then we have

$$\int_{\partial C_R} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz \tag{45}$$

$$= \int_0^R \frac{dx}{1+x^7} + \int_0^{\frac{2\pi}{7}} \frac{Re^{i\theta}}{1+R^7e^{7i\theta}}d\theta + \int_R^0 \frac{\omega^2 dy}{1+(\omega^2 y)^7}. \tag{46}$$

Now the last term in (2) becomes

$$\int_R^0 \frac{\omega^2 dy}{1+(\omega^2 y)^7} = -\omega^2 \int_0^R \frac{dy}{1+y^7}, \tag{47}$$

since $\omega^14 = e^{i2\pi} = 1$.
Now, since f has a simple pole at $\omega \in C_R$, the Residue Theorem gives

$$\int_{\partial C_R} f(z)dz = 2\pi i \text{Res}(f(z);\omega). \tag{48}$$

But since the pole at ω is order 1, we have

$$\begin{aligned} \text{Res}(f(z);\omega) &= \lim_{z\to\omega} f(z)(z-\omega) \\ &= \lim_{z\to\omega} \frac{z-\omega}{1+z^7} \\ &= \lim_{z\to\omega} \frac{1}{7z^6} \text{ (by L'Hopital's Rule)} \\ &= \frac{1}{7\omega^6} \\ &= -\frac{\omega}{7}, \end{aligned}$$

since $\omega^6 = \omega^7\omega^{-1} = -\omega^{-1}$. Finally, by the triangle inequality we have

$$\left| \int_{\gamma_2} f(z)dz \right| \leq \int_{\gamma_2} \frac{|dz|}{|1-R^7|} \leq \frac{2\pi|R|}{7|1-R^7|} \to 0 \tag{49}$$

as $R \to \infty$. Thus, combining (2), (3), and (4) and letting $R \to \infty$, we get

$$\begin{aligned} &(1-\omega^2)\int_0^\infty \frac{dx}{1+x^7} = -\frac{2\pi i\omega}{7} \\ &\Rightarrow \int_0^\infty \frac{dx}{1+x^7} = -\frac{2\pi i \omega}{7(1-\omega^2} = \frac{\pi}{7}\frac{2i}{\omega-\omega^{-1}} = \frac{\pi}{7}\frac{2i}{e^{\frac{i\pi}{7}} - e^{\frac{-i\pi}{7}}} = \frac{\pi}{7\sin(\pi/7)}. \end{aligned}$$

□

Problem 8.2 (Kansas, Spring 2004, Q4)**.** *Let* $f : [0, \infty) \to \mathbb{C}$ *be a Lebesgue measurable function. Assume there exist real numbers* $a, k > 0$ *such that*

$$|f(x)| \leq ae^{-kx}, \forall\, x \geq 0.$$

Consider the half-plane $H = \{z \in \mathbb{C} | Imz > k\}$*.*

(i) Prove that, for every $z \in H$*, the function* $g : [0, \infty) \to \mathbb{C}, t \mapsto e^{itz} f(t)$ *is Lebesgue integrable.*

(ii) Prove that the function $F : H \to \mathbb{C}$*,* $z \mapsto \int_0^\infty e^{itz} f(t)dt$ *is holomorphic.*

Solution. [Amrei Oswald]
Attempt:
My first step was to figure out how to handle Lebesgue integrals of complex valued function. For a complex valued function $f(x + iy) = u(x, y) + iv(x, y)$ the Lebesgue integral is defined as

$$\int f(x + iy) = \int u(x, y) + i \int v(x, y),$$

and f is integrable if and only if $\int |f| < \infty$. From here, the problem is quite straightforward.

Also, there is likely a typo in the statement of the problem. Letting $H = \{z \in \mathbb{C} | \text{Im}z > -k\}$ still allows us to bound the integral of $|g|$ by $a \int_0^\infty e^{-t}$ giving us a bound of a rather than $a/2k$.
Solution:

(i) Note that g is Lebesgue integrable if and only if $|g|$ is Lebesgue integrable. Fix $z \in H$. Since $\text{Im}z > k$, we have

$$\int_0^\infty |g| = \int_0^\infty \left|e^{itz} f(t)\right| \leq \int_0^\infty \left|e^{itz}\right| \left|ae^{-kt}\right| = a \int_0^\infty e^{-t(\text{Im}z+k)}$$
$$\leq a \int_0^\infty e^{-2kt} = a \left(\lim_{t\to\infty} \frac{e^{-2kt}}{-2k} - \frac{1}{-2k} \right) = \frac{a}{2k}.$$

Therefore, $|g|$ is Lebesgue integrable $\implies$ g is Lebesgue integrable.

(ii) Let T be any triangular path in H. Then, from the bound we found for $g(t)$ above, we have

$$\left|\int_T F dz\right| \leq \int_T |F|\, dz = \int_T \left|\int_0^\infty e^{itz} f(t)dt\right| dz = \int_T \left|\int_0^\infty g(t)dt\right| dz$$
$$\leq \int_T \int_0^\infty |g(t)|\, dtdz \leq \int_T \frac{a}{2k} dz = 0.$$

The last equality follows since $\frac{a}{2k}$ is a constant function and therefore entire, so its integral over any closed path is 0.

The above shows that $\left|\int_T F dz\right| = 0 \implies \int_T F dz = 0$ over any triangular path T. By Morera's Theorem, F is holomorphic.

□

Problem 8.3. *[Shrey Sanadhya, Spring 2004, Q8] Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with the property: $f(z + m + mi) = f(z), \forall z \in \mathbb{C}, m, n \in \mathbb{Z}$. Prove that f is constant.*

Proof. Let $I = [0, 1]^2$ be a unit square in $\mathbb{C}$. Since I is compact set and f is continuous $f|_I$ is bounded (say $f|_I \leq M$)
Given $f(z + m + mi) = f(z), \forall z \in \mathbb{C}, m, n \in \mathbb{Z}$
We can write $z = x + iy$ and $x = \lfloor x \rfloor + r$ and $y = \lfloor y \rfloor + s$ for some $r, s \in I$
Thus $|f(z)| = |f(x + iy)| = |f(r + is + \lfloor x \rfloor + i\lfloor y \rfloor)| = |f(r + is)| \leq M$
Thus $f(z)$ is bounded entire function. Hence constant by Liouville theorem

□

Problem 8.4 (Kansas, Jan 2007 Q6)**.** *Evaluate*

$$\int_0^\infty \frac{\log(x)}{x^4 + 1} dx$$

Solution. [Rajinda Wickrama]

Define a branch of the logarithm such that $\ell(z) = \log|z| + i\theta$ where $\theta \in [-\pi/2, 3\pi/2]$. Define

$$f(z) = \frac{\ell(z)}{z^4 + 1}.$$

Let $0 < r < R$ and let γ be the same curve as in Example 2.7 (Conway 115). Then f has two simple poles, $e^{i\pi/4}$ and $e^{3i\pi/4}$ inside γ. By the residue theorem;

$$\int_\gamma f = 2\pi i \ (\text{Res}(f, e^{i\pi/4}) + \text{Res}(f, e^{3i\pi/4}))...(1)$$

$$\text{Res}(f, e^{i\pi/4}) = \lim_{z \to e^{i\pi/4}} (z - e^{i\pi/4}) f(z) = \frac{i\pi/4}{(e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{5i\pi/4})(e^{i\pi/4} - e^{7i\pi/4})}$$

After simplifying we get that

$$\text{Res}(f, e^{i\pi/4}) = \tfrac{\pi}{\sqrt{2}} \left(\tfrac{1}{16} - \tfrac{i}{16} \right)$$

Similarly,

$$\text{Res}(f, e^{3i\pi/4}) = \tfrac{\pi}{\sqrt{2}} \left(\tfrac{3}{16} + \tfrac{3i}{16} \right)$$

Therefore, by applying these values to (1) we get that

$$\int_\gamma f = \int_\gamma \frac{\ell(z)}{z^4 + 1} dz = \frac{\sqrt{2}\pi^2 i}{4} - \frac{\sqrt{2}\pi^2}{8}...(2)$$

Further, by parametrizing all the curves we get the following;

$$\int_\gamma \frac{\ell(z)}{z^4+1}dz = \int_r^R \frac{\log x}{1+x^4}dx + iR\int_0^\pi \frac{\log R + i\theta}{1+R^4e^{4i\theta}}e^{4i\theta}d\theta + ir\int_\pi^0 \frac{\log r + i\theta}{1+r^4e^{4i\theta}}e^{4i\theta}d\theta + \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^4}dx ...(3)$$

Also,

$$\int_r^R \frac{\log x}{1+x^4}dx + \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^4}dx = 2\int_r^R \frac{\log x}{1+x^4}dx + i\pi\int_r^R \frac{1}{1+x^4}dx$$

Now let's prove that

$$\int_0^\infty \frac{1}{1+x^4}dx = \frac{\pi}{2\sqrt{2}} ...(4)$$

In order to do this we use the path γ in Example 2.5 (Conway: pg 113/114). First define a function

$$g(z) = \frac{1}{1+z^4}$$

Observe that, as $R \to \infty$

$$\left|\int_{\gamma_R} \frac{1}{z^4+1}dz\right| \leq \int_{\gamma_R} \frac{1}{|z^4+1|}|dz| = \frac{\pi R}{|R^4-1|} \to 0$$

Also, g has two simple poles, $e^{i\pi/4}$ and $e^{3i\pi/4}$ inside γ. By the residue theorem,

$$\int_\gamma \frac{1}{z^4+1}dz = 2\pi i(Res(g, e^{i\pi/4}) + Res(g, e^{3i\pi/4})) = \frac{\pi}{\sqrt{2}} = \int_{\gamma_R} \frac{1}{z^4+1}dz + \int_{-R}^R \frac{1}{x^4+1}dx$$

As $R \to \infty$,

$$\int_{-\infty}^\infty \frac{1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$$

Since $\frac{1}{x^4+1}$ is symmetric about the y axis,

$$\int_0^\infty \frac{1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}$$

Now let us show that $iR\int_0^\pi \frac{\log R + i\theta}{1+R^4e^{4i\theta}}e^{4i\theta}d\theta \to 0$ as $R \to \infty$. So, as $R \to \infty$, observe that,

$$\left|iR\int_0^\pi \frac{\log R + i\theta}{1+R^4e^{4i\theta}}e^{4i\theta}d\theta\right| \leq R\int_0^\pi \frac{|\log R| + \theta}{|1-R^4|}d\theta = \frac{R|\log R|\pi}{|1-R^4|} + \frac{R\pi^2}{2|1-R^4|} \to 0$$

Similarly, as $r \to 0^+$, $\left| ir \int_0^\pi \frac{\log r + i\theta}{1+r^4 e^{4i\theta}} e^{4i\theta} d\theta \right| \to 0$.

Therefore, as $R \to \infty$, $r \to 0^+$ in equation (3),and also using results from (2) and (4), we get that

$$\int_0^\infty \frac{\log x}{1+x^4} dx = \frac{-\pi^2}{8\sqrt{2}}$$

□

Problem 8.5 (Elaina Aceves, January 2007 Q8). *Find a conformal map from the strip $\{z : 0 < Re(z) < 1\}$ onto the half disk $\{z : Im(z) > 0, 0 < |z| < 1\}$.*

Solution.
Attempt:
My first idea was to use Mobius transformations to get from the vertical strip to the half disk. So, I began with using the map $z \to iz$ to get the vertical strip to the horizontal strip and then using the exponential to get the right half plane. Then I can use the map $z \to \frac{z-1}{z+1}$ to get from the right half plane to the unit disk. Unfortunately, I did not know of any conformal maps from the unit disk to the half disk. This is when I turned to the list of maps that Rolando provided. There is a map that is from half of a horizontal strip to the half disk, but I wasn't sure how to map into half of a horizontal strip. However, there was a map from the half disk onto a horizontal strip of width π. The inverse of that map (which we can use because conformal maps are bijective) is what I used in the solution.
Solution:
We will define three maps to create a conformal map from the vertical strip $\{z : 0 < Re(z) < 1\}$ to the half disk $\{z : Im(z) > 0, 0 < |z| < 1\}$. Let f_1 be the map that sends $z \to iz$. This will take the vertical strip to the horizontal strip $\{z : 0 < Im(z) < 1\}$. Let f_2 be the map that sends $z \to \pi z$, which will send the horizontal strip $\{z : 0 < Im(z) < 1\}$ to the horizontal strip $\{z : 0 < Im(z) < \pi\}$. Finally, let f_3 be the map $z \to \log(\frac{z-1}{z+1})$ which maps the half disk $\{z : Im(z) > 0, 0 < |z| < 1\}$ onto the horizontal strip $\{z : 0 < Im(z) < \pi\}$. Then if we let $f = f_3^{-1} \circ f_2 \circ f_1$, f is a conformal map from the vertical strip to the half disk. □

Problem 8.6 (Kansas, August 2008 Q2). *(a) Let M and R be positive numbers and f be a holomorphic function in $R\mathbb{D}$ and bounded by M. Show that $|f(w) - f(0)| \leq 2MR^{-1}|w|$ $w \in R\mathbb{D}$. Hint: apply Schwarz's Lemma to an appropriate function.*
(b) If F is holomorphic and bounded in $\mathbb{C}$, use (a) to infer (Liouville's Theorem) that F is constant.

Solution. [Sara Reed] Note the following will be used in the proof:

- Schwarz's Lemma (p. 130): Let $\mathbb{D} = \{z : |z| < 1\}$ and suppose f is analytic on $\mathbb{D}$ with
  (a) $|f(z)| \leq 1$ for z in $\mathbb{D}$
  (b) f(0)=0.

Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z in the disk $\mathbb{D}$. Moreover if $|f'(0)| = 1$ or if $|f(z)| = z$ for some $z \neq 0$ then there is a constant c, $|c| = 1$, such that $f(w) = cw$ for all w in $\mathbb{D}$.

(a):

Proof. Define $g(z) := f(Rz) - f(0)$ where $z \in \mathbb{D}$ and therefore $w = Rz \in R\mathbb{D}$. It follows

$$|g(z)| = |f(Rz) - f(0)| \leq |f(Rz)| + |f(0)| \leq 2M.$$

Thus $g : \mathbb{D} \to 2M\mathbb{D}$ and g is analytic. Define $h(z) := \frac{g(z)}{2M}$. Thus, $h : \mathbb{D} \to \mathbb{D}$, h is analytic and

$$h(0) = \frac{g(0)}{2M} = \frac{f(0) - f(0)}{2M} = 0.$$

Therefore, by Schwarz's Lemma,

$$\begin{aligned}
|h(z)| &\leq |z| \\
\left|\frac{g(z)}{2M}\right| &\leq |z| \\
|f(Rz) - f(0)| &\leq 2M|z| \\
|f(w) - f(0)| &\leq 2MR^{-1}|w|
\end{aligned}$$

for $w = Rz \in R\mathbb{D}$ as desired. □

(b):

Proof. Let $F : \mathbb{C} \to \mathbb{C}$ be holomorphic and $|F(z)| \leq M$ for all $z \in \mathbb{C}$. Let $w \in \mathbb{C}$. Choose $R > 0$ such that $w \in R\mathbb{D}$. By part a: $|F(w) - F(0)| \leq 2MR^{-1}|w|$ for $w \in R\mathbb{D}$. Let $\mathbb{R} \to \infty$. Then we have $|F(w) - F(0)| \leq 0$ which implies $|F(w) - F(0)| = 0$. Thus, $F(w) = F(0)$. This conclusion holds for all $w \in \mathbb{C}$. So $F(w) = F(0)$ for all $w \in \mathbb{C}$. Therefore, we conclude F is constant. □

□

Problem 8.7 (Kansas-Aug 2008 Q4: Shrey Sanadhya)**.** *Calculate* $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + \alpha^2)}$

Proof. Recall that $\cos x = \text{Re}\,(e^{ix})$, so $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + \alpha^2)} = \text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+\alpha^2)}\right)$.

So let $f(z) = \dfrac{e^{iz}}{(z^2 + \alpha^2)} = \dfrac{e^{iz}}{(z - i\alpha)(z + i\alpha)}$, and notice that this function has poles of order 1 at $i\alpha$ and $-i\alpha$. Now consider a semicircle contour of radius R centered at the origin. Only one pole $i\alpha$ lie in the contour: Using the Residue Theorem, we know that $\int_\gamma f(z) = 2\pi i \mathrm{Res}_{i\alpha} f$.

$$\begin{aligned}
\mathrm{Res}_{i\alpha} f &= \lim_{z \to i\alpha} \left[(z - i\alpha) \frac{e^{iz}}{(z - i\alpha)(z + i\alpha)} \right] \\
&= \lim_{z \to i\alpha} \left[\frac{e^{iz}}{(z + i\alpha)} \right] \\
&= -\frac{i}{2\alpha e^{\alpha}}
\end{aligned}$$

Now we have that $\int_\gamma f(z) = \int_{-R}^{R} f(z)\ dz + \int_{C_R} f(z)\ dz = 2\pi i \mathrm{Res}_{i\alpha} f = \frac{\pi}{\alpha e^{\alpha}}$.

Parameterizing C_R as Re^{it} for $t \in [0, \pi]$, we bound $\displaystyle\int_{C_R} f(z)$ and show that this integral goes to 0 as $R \to \infty$:

$$\left|\int_{C_R} \frac{e^{iz}}{(z^2+\alpha^2)}\right| \leq \int_0^\pi \left| iR\frac{e^{iRe^{it}}}{(R^2e^{2it}+\alpha^2)}\right| dt \tag{50}$$

$$\leq \int_0^\pi \left| iR\frac{e^{-R}}{(R^2-\alpha^2)}\right| dt \tag{51}$$

The inequality in line (2) above comes from the fact that t only takes values between 0 and π, so at "best", the numerator will be $e^iR(i)$ and the denominator will be $-R^2+\alpha^2$ (which has the same norm as $R^2-\alpha^2$). Now as $R \to \infty$, we see that $\int_{C_R} f(z) \to 0$.

Returning to the residue theorem, we have that $\displaystyle\int_\gamma f(z) = \int_{-\infty}^{\infty} f(z)dz + \int_{C_R} f(z) = \int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} \frac{\cos z}{(z^2+\alpha^2)} + i\int_{-\infty}^{\infty}\frac{\sin x}{(z^2+\alpha^2)} = \frac{\pi}{\alpha e^\alpha}$.

Since the real and imaginary parts above must be equal, we get that $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x}{(z^2+\alpha^2)} = 0$ and $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x}{(z^2+4)^2} = \frac{\pi}{\alpha e^\alpha}$. □

Problem 8.8 (Kansas, August 2008 Q5)**.** *Let K be a compact set of an open connected set $\Omega \subset \mathbb{C}$. Let $u : \Omega \to \mathbb{C}$ be harmonic, $c \in \mathbb{R}$ and $u \leq c$ on $\Omega - K$. Prove that $u \leq c$ on Ω.*

Solution. [Rajinda Wickrama]
There is an entire chapter on harmonic function in Conway (Chapter X) and I have used some results from this chapter in my proof.

Proof. By the maximum modulus theorem for harmonic functions (Conway: 1.6-page 253) u attains its maximum on the boundary of K. Suppose the maximum is attained at $a \in \partial K$. Then

$$\forall z \in K, u(z) \leq u(a) \tag{52}$$

Observe that a is a limit point of $\Omega - K$. Let $\{z_n\}$ be a sequence in $\Omega - K$ that converges to a. Therefore,

$$u(a) = u\Big(\lim_{n\to\infty} z_n\Big) = \lim_{n\to\infty} u(z_n) \leq c \tag{53}$$

By combining the results in (1) and (2) we get that $u \leq c$ on Ω.

□

□

Problem 8.9 (Kaitlin Healy, Fall 2011 Q4)**.** *Use the Residue Theorem to integrate the rational function*

$$R(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$

over $\mathbb{R}$.

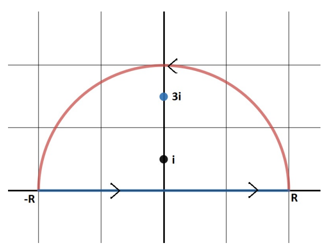
Solution.
Attempt:
Got this on the first attempt!
Solution:
We want to calculate

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$$

Notice that $R(z)$ can be rewritten as

$$R(z) = \frac{z^2 - z + 2}{(z + 3i)(z - 3i)(z + i)(z - i)}$$

This tells us that $R(z)$ has poles at $i, -i, 3i, -3i$ each with multiplicity one. Thus, we will use the following contour to integrate over:



Call this path $\sigma = \alpha \cup \gamma$, where α is the part of the curve on the real axis and γ is the curve $\gamma = Re^{i\theta}$ for $\theta \in [0, \pi]$ and $R > 3$. This gives us

$$\int_{\sigma} R(z)dz = \int_{\alpha} R(x)dx + \int_{\gamma} R(z)dz$$

Let's first look at $\int_{\sigma} R(z)dz$. We know from the residue theorem that $\int_{\sigma} R(z)dz = 2\pi i(\text{Res}(R(z), i)+$

Res$(R(z), 3i)$). We can calculate each of those residues in the following way:

$$\begin{aligned}
\text{Res}(R(z), i) &= g(i) \\
&= (z-i)R(z) \\
&= \frac{i^2 - i + 2}{(i+3i)(i-3i)(i+i)} \\
&= \frac{1-i}{16i} \\
&= \frac{3-3i}{48i} \\
\text{Res}(R(z), 3i) &= h(i) \\
&= (z-3i)R(z) \\
&= \frac{(3i)^2 - 3i + 2}{(3i+3i)(3i+i)(3i-i)} \\
&= \frac{-7-3i}{-48i} \\
&= \frac{7+3i}{48i}
\end{aligned}$$

This gives us

$$\int_\sigma R(z)dz = 2\pi i\left(\frac{3-3i}{48i} + \frac{7+3i}{48i}\right) = 2\pi i\left(\frac{5}{24i}\right) = \frac{5\pi}{12}$$

Now we must work with the integrals over each individual part of our path. Notice that the integral over α is the integral we want to calculate as $R \to \infty$. Thus, we only need to work with the integral over γ. Notice the following:

$$\begin{aligned}
\int_\gamma R(z)dz &= \int_\gamma \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}dz \\
&= \int_\gamma \frac{z^2 - z + 2}{(z^2+9)(z^2+1)}dz \\
&= \int_0^\pi \frac{R^2e^{2i\theta} - Re^{i\theta} + 2}{(R^2e^{2i\theta} + 9)(R^2e^2i\theta + 1)} iRe^{i\theta}d\theta
\end{aligned}$$

Taking an absolute value, we have

$$\begin{aligned}\left|\int_\gamma R(z)dz\right| &= \left|\int_0^\pi \frac{R^2e^{2i\theta}-Re^{i\theta}+2}{(R^2e^{2i\theta}+9)(R^2e^2i\theta+1)}iRe^{i\theta}d\theta\right| \\ &\le \int_0^\pi \frac{|R^2e^{2i\theta}-Re^{i\theta}+2|}{|R^2e^{2i\theta}+9||R^2e^2i\theta+1|}|iRe^{i\theta}||d\theta| \\ &= R\int_0^\pi \frac{|R^2e^{2i\theta}-Re^{i\theta}+2|}{|R^2e^{2i\theta}+9||R^2e^2i\theta+1|}d\theta \\ &\le R\int_0^\pi \frac{|R^2e^{2i\theta}|+|Re^{i\theta}|+2}{(R^2-9)(R^2-1)}d\theta \\ &= R\int_0^\pi \frac{R^2+R+2}{(R^2-9)(R^2-1)}d\theta \\ &= \pi R\frac{R^2+R+2}{(R^2-9)(R^2-1)}\end{aligned}$$

As $R \to \infty$, this will go to zero. Thus, the integral over γ will go to zero as $R \to \infty$. Therefore, combining all of our parts, we have

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$$

□

Problem 8.10 (Kansas, Spring 2013 Q3)**.** *For some* $\alpha > 0$*,* $S =: \{re^{i\theta} : r > 0, 0 < \theta < \alpha\}$*.* $f \in H(S)$ *is bounded. Show that*

$$\lim_{r\to\infty} f'(re^{i\theta}) = 0$$

for each $0 < \theta < \alpha$*.*

Solution. [Qing Zou]

Define $f_r(z) = f(rz)$ for $r > 0$. Since f is holomorphic on S and bounded, we know that there exists $M > 0$ such that $|f| \le M$. Then f_r is also holomorphic on S and $|f_r| \le M$. Let $\theta \in (0, \alpha)$. Then there exists $R > 0$ such that $B(e^{i\theta}, R) \subset S$ and on $B(e^{i\theta}, R)$, f is analytic and $|f_r| \le M$. So by Cauchy's Estimate,

$$|f_r'(e^{i\theta})| \le \frac{M}{R}.$$

Since

$$f_r'(z) = [f(rz)]' = r \cdot f'(rz).$$

Then

$$|f_r'(e^{i\theta})| = |r \cdot f'(re^{i\theta})| \le \frac{M}{R},$$

which implies

$$|f'(re^{i\theta})| \leq \frac{M}{rR} \to 0, (r \to \infty).$$

Therefore,

$$\lim_{r\to\infty} f'(re^{i\theta}) = 0$$

for each $0 < \theta < \alpha$.

□

Problem 8.11 (Kansas, Spring 2015 Q7). *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire, non-constant function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.*

Solution. [Amrei Oswald]
Say that $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Then there exists a $z_0 \in \mathbb{C}$ and an $\epsilon > 0$ such that $f(\mathbb{C}) \cap B(z_0, \epsilon) = \emptyset$. Then if $f_1(z) = f(z) - z_0$, f_1 is entire and $f_1(\mathbb{C}) \cap B(0, \epsilon) = \emptyset$. Let $f_2(z) = \frac{1}{\epsilon} f_1(z)$. Then f_2 is entire and $f_2(\mathbb{C}) \cap B(0, 1) = \emptyset$.
Finally, if $f_3(z) = 1/f_2(z)$, f_3 is entire and $|f_3(z)| \leq 1$ for every $z \in \mathbb{C}$. Therefore, by Liouville's Theorem, f_3 is constant, say $f_3 \equiv c$ for $c \in \mathbb{C}$. Then we have,

$$f_3(z) = \frac{\epsilon}{(f(z) - z_0)} = c \implies f(z) = \frac{\epsilon}{c} + z_0,$$

and f is constant. However, this contradicts our assumption about f. Therefore, we cannot have that $f(\mathbb{C})$ is not dense in $\mathbb{C}$. □

Problem 8.12 (Adam Wood, Fall 2013, 4.). *Given $\varepsilon > 0$, find a compact set $K \subset [0, 1]$ that contains no rationals and satisfies $m(K) > 1 - \varepsilon$.*

Solution.
Attempt:
None.
Solution:

Proof. Let $\varepsilon > 0$. Note that a compact set contained in $[0, 1]$ is equivalent to a closed set contained in $[0, 1]$ since the compact sets in $\mathbb{R}$ are the closed and bounded sets. Let $I = [0, 1]$ and consider $I \sim (\mathbb{Q} \cap I)$. Since $m(\mathbb{Q}) = 0$, $m(\mathbb{Q} \cap I) = 0$, so $m(I \sim (\mathbb{Q} \cap I)) = m(I) - m(\mathbb{Q} \cap I) = 1$. By the approximation properties of Lebesgue measure, since $I \sim (\mathbb{Q} \cap I)$ is measurable, there exists a closed set $K \subseteq I \sim (\mathbb{Q} \cap I)$ so that $m((I \sim (\mathbb{Q} \cap I)) \sim K) < \varepsilon$. That is, $m(K) > 1 - \varepsilon$. □

□

Problem 8.13 (Elaina Aceves, Spring 2009 Q1). *P is a polynomial of degree at most $n \in \mathbb{N}$ and $\sup\{|P(u) : u \in \mathbb{T}\} = 1$. Show that $|P(z)| \leq |z|^n$ for all $z \in \mathbb{C} \setminus \mathbb{D}$ and determine for what P equality holds.*
Note that $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Solution.
Attempt:
Since we want to show that $|P(z)| \leq |z|^n$, it made sense to consider the function $f(z) = P(z)/z^n$ and show that $|P(z)/z^n| \leq 1$. To show this the Maximum Modulus Theorem can be applied. However, I could not find a bound for $|f(z)|$ for all of $z \in \mathbb{C} \setminus \mathbb{D}$. Thus, I needed to introduce the sets $G_R = \{z \in \mathbb{C} : 1 \leq |z| \leq R\}$ for $R > 1$ which allow me to find a bound and as $R \to \infty$, we have that $G_R \to \mathbb{C} \setminus \mathbb{D}$.
Solution:
Define $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(z) = P(z)/z^n$. Then f is analytic on $\mathbb{C} - \{0\}$. Notice that if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, then

$$f(z) = \frac{P(z)}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n.$$

By the Maximum Modulus Theorem, since $|P(z)| \leq 1$ on $\mathbb{T}$ by assumption, we have that $|P(z)| \leq 1$ on $\mathbb{D}$. By Cauchy's Estimate, since $|P(z)| \leq 1$ on $\mathbb{D}$, we have that $|P^{(m)}(0)| \leq \frac{m! \cdot 1}{1^m} = m!$ for all $0 \leq m \leq n$. Then

$$|a_m| = \frac{1}{m!} |P^{(m)}(0)| \leq \frac{m!}{m!} = 1 \text{ for all } 0 \leq m \leq n.$$

Consider the set $G_R = \{z \in \mathbb{C} : 1 \leq |z| \leq R\}$. The boundary of G_R is $\mathbb{T} \cup \{z \in \mathbb{C} : |z| = R\}$. On $\mathbb{T}$, we have that $|z| = 1$ and $|P(z)| \leq 1$ by the Maximum Modulus Theorem, so

$$|f(z)| = \frac{|P(z)|}{|z|^n} = |P(z)| \leq 1.$$

So we have a bound for that section of the boundary. On $\{z \in \mathbb{C} : |z| = R\}$, via the triangle inequality and $|a_m| \leq 1$ for all $0 \leq m \leq n$, we obtain

$$|f(z)| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + |a_n| \leq \frac{1}{R^n} + \frac{1}{R^{n-1}} + \cdots + 1 \quad (*)$$

Hence on the boundary of G_R, we have the bound $|f(z)| \leq \frac{1}{R^n} + \frac{1}{R^{n-1}} + \cdots + 1$. Then by the Maximum Modulus Theorem, we have that $|f(z)| \leq \frac{1}{R^n} + \frac{1}{R^{n-1}} + \cdots + 1$ for all $z \in G_R$. If we let $R \to \infty$, we have that $|f(z)| \leq 1$ for all $z \in \mathbb{C} \setminus \mathbb{D}$. Hence $|P(z)| \leq |z|^n$ for all $z \in \mathbb{C} \setminus \mathbb{D}$.

Now, we need to determine for what P equality holds. If $|P(z)| = |z|^n$, then $|f(z)| = 1$ and inequality $(*)$ becomes equality throughout when we let $R \to \infty$. This forces $|a_m| = 1$ for all $0 \leq m \leq n$. Thus, $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ where $|a_m| = 1$ for all $0 \leq m \leq n$. $\square$

Problem 1. Kansas, Spring 2013 Q1
(i) Show that each $f \in H(\mathscr{D})$ satisfies

$$f(z) - f(0) = \int_0^1 z f'(tz)\, dt$$

for every $z \in \mathscr{D}$

Hint: Is there a derivative with respect to t present?

(ii) If $F \in H(\Omega)$ and $F' = 0$, show that F is constant.

Hint: Fixing $z_0 \in \Omega$, use (i) to show that $\{z \in \Omega : F(z) = F(z_0)\}$ is open.

Proof. [Violet Tiema]

Notice that $\frac{d}{dt} f(tz) = zf'(tz)$. Therefore,

$$\begin{aligned}\int_0^1 z\, f'(tz)\, dt &= \int_0^1 \frac{d}{dt} f(tz)\, dt \\ &= f(tz)\Big|_0^1 \\ &= f(z) - f(0)\end{aligned}$$

Now for (ii). Since F is analytic in Ω . Pick $z_0 \in \Omega$ and since Ω is open, $\exists \epsilon > 0$ such that $B(z_0, \frac{\epsilon}{2}) \subset \Omega$. Since F is analytic on $B(z_0, \frac{\epsilon}{2})$ then we can write F(z) as

$$F(z) = \sum a_n (z - z_0)^n \; where \; a_n = \frac{F^n(z_0)}{n!}$$

But since $F' = 0$, It then follows that $a_n = 0 \;\; , \forall n \geq 1$. Therefore, $F(z) = F(z_0) \;\; , \forall z \in B(z_0, \frac{\epsilon}{2})$. Now define a function $g : \Omega \to \mathscr{C}$ such that $g(z) = F(z) - F(z_0)$ is analytic. Then $g^n(z_0) = 0 \;\; , \forall n \geq 0$ Thus $g \equiv 0$ by theorem 3.7 page 78 (Conway) Therefore $F(z) - F(z_0) \equiv 0$ which implies $F(z) = F(z_0) \;\; , \forall z \in \Omega$. Since Ω is open, by the identity theorem we can conclude that F is constant.

□

9 Texas A & M Quals

Problem 9.1 (Texas A& M Complex, August 2009 Q8). *Find the function $w(z)$ that maps $\Omega = \{Im(z) > 0\} \sim [0, i]$ conformally into $\mathbb{D}$ which satisfies $w(\frac{5i}{4}) = 0$ and $w(i) = -i$.*

Solution. [Jared Grove]
Attempt:
Nothing of value, just know how z^2 and $\sqrt{z}$ work and it is much more obvious.
Solution:
We will define the following functions:

$$\begin{aligned} f_1(z) &= z^2 \\ f_2(z) &= z + 1 \\ f_3(z) &= \sqrt{z} = e^{\frac{1}{2}\log(z)} \\ f_4(z) &= \frac{z - \frac{3i}{4}}{z + \frac{3i}{4}} \\ f_5(z) &= e^{i\frac{\pi}{2}} \end{aligned}$$

for f_3 we will say $log(z) = ln|z| + i \arg(z)$ using the branch cut obtained by removing the positive reals. Then say

$$w(z) = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$$

this will be the function we want. To get an idea of what is happening

$$\begin{aligned} f_1(\Omega) = \mathbb{C} \sim [-1, \infty] = \Omega_1, &\quad f_1(\tfrac{5i}{4}) = \tfrac{-25}{16}, \quad f_1(i) = -1 \\ f_2(\Omega_2) = \mathbb{C} \sim [0, \infty] = \Omega_3, &\quad f_2(\tfrac{-25}{16}) = \tfrac{-9}{16}, \quad f_2(-1) = 0 \\ f_3(\Omega_3) = \{z : Im(z) > 0\} = \Omega_4, &\quad f_3(\tfrac{-9}{16}) = \tfrac{3i}{4}, \quad f_3(0) = 0 \\ f_4(\Omega_4) = \mathbb{D} = \Omega_5, &\quad f_4(\tfrac{3i}{4}) = 0, \quad f_3(0) = -1 \\ f_5(\Omega_5) = \mathbb{D}, &\quad f_5(0) = 0, \quad f_3(-1) = -i \end{aligned}$$

Notice Ω_2 is the Complex plane minus the real line and the bit from 0 to -1, Ω_3 is the Complex plane minus the positive reals, and Ω_4 is the upper half plane (the positive reals got mapped to the entire real line).

□

Problem 9.2 (TAMU, January 2010 Q6). *Let f be a function holomorphic in the unit disk $\mathbb{D}$ and continuous in the closure $\overline{\mathbb{D}}$.*
a) Show that if $\mathscr{R}f = 0$ on $\partial\mathbb{D}$ then f is a constant.
b) Show that the previous statement is false if $\partial\mathbb{D}$ is replaced with a proper subarc of $\partial\mathbb{D}$.

Solution. [Michael Kratochvil]
Attempt:

Each part had completely different approaches. The forthcoming lemma was shown to me by Alex Bates, and the proof of the first part immediately followed. The second part requires extra knowledge of transformations, as well as some algebraic manipulation.
Solution:
Lemma: Let g be analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ such that g does not vanish on $\mathbb{D}$ and $|g| \equiv 1$ on $\partial\mathbb{D}$. Then g is constant.

Proof: The Maximium Modulus Theorem (MMT) immediately tells us that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Define $h(z) = \frac{1}{g(z)}$. Then h is analytic on $\mathbb{D}$ (since $g(z) \neq 0$) and h is continuous on $\overline{\mathbb{D}}$. Further $|h(z)| = 1/|g(z)| = 1$ on $\partial\mathbb{D}$ which implies $|h(z)| \leq 1$ on $\mathbb{D}$ (by MMT). This implies $|g(z)| \geq 1$ on $\mathbb{D}$ and therefore $|g(z)| = 1$. So g maps the unit disk to the unit circle. But a quick exercise with angles and conformal maps (one we did for homework) tells us that g must be constant.

(a) Setting $p(z) = e^{f(z)}$ we immediately have that $p(z) \neq 0$ on $\mathbb{D}$ and $|p(z)| \equiv 1$ on $\partial\mathbb{D}$. So by the lemma above, f is constant.
(b) For Γ a proper subarc of $\partial\mathbb{D}$ and Γ_H the lower half circle, let $\varphi : \mathbb{D} \to \mathbb{D}$ be a Mobius transformation with $\varphi(\Gamma) = \Gamma_H$. The function

$$\zeta(w) = \left(\frac{1+w}{1-w}\right)^2$$

maps the upper half disk onto the upper half plane. Further

$$\eta(\zeta) = \frac{\zeta - i}{\zeta + i}$$

maps the upper half plane onto the unit disk. Thus for $\phi = \eta \circ \zeta$

$$\phi(w) = \frac{(1+w)^2 - i(1-w)^2}{(1+w)^2 + i(1-w)^2}$$

maps the upper half unit disk onto $\mathbb{D}$. In particular ϕ maps the interval $[-1, 1]$ onto Γ_H. Thus

$$i(\phi^{-1} \circ \varphi)(z)$$

maps Γ to $[-i, i]$. satisfying the counterexample. □

Problem 9.3 (TAMU, January 2010 Q7)**.** *Let entire functions f and g satisfy $e^f + e^g \equiv 1$. Prove that then both are constants.*

Solution. [Amrei Oswald]
Attempt:
I tried bounding various combinations of f and g with other entire functions and showing that the image of either f or g wasn't dense in $\mathbb{C}$, but I couldn't get any of these approaches

to work. Rolando pointed me to Picard's Little Theorem, which states that if f is a non-constant entire function on $\mathbb{C}$, then image of f is either all of $\mathbb{C}$ or $\mathbb{C}$ minus a single point.
Solution:
Since f is entire, e^f is entire. Note that if $f(z) = 0$ for some $z \in \mathbb{C}$, we have $e^0 + e^{g(z)} = 1 \implies e^{g(z)} = 0$ which is a contradiction. Therefore, $f(z) \neq 0$ for any $z \in \mathbb{C}$ which means that $e^{f(z)} \neq 1$ for any $z \in \mathbb{C}$. Since $e^{f(z)} \neq 0$ for any $z \in \mathbb{C}$, e^f is an entire function whose range omits at least two points in the complex plane. By Picard's Little Theorem, we must have that $e^{f(z)}$ is constant which implies that f is constant. Then, $e^g = 1 - e^f$ is constant, so that g is also constant. □

Problem 9.4 (TAMU C, January 2010 Q8)**.** *Find a general formula for all functions $w(z)$ that map the domain $\Omega = \{|z| < 1\} \sim [1/2, 1]$ conformally onto the domain $\{|Im(z)| < 1\}$.*

Solution. [Sara Reed] We will construct a sequence of conformal maps and define $w(z)$ to be the composition of these maps. We begin with $\Omega = \{|z| < 1\} \sim [1/2, 1]$.

1. We will map $\Omega = \{|z| < 1\} \sim [1/2, 1]$ to $\{|z| < 1\} \sim [0, 1]$ with the following mobius transformation:
$$\varphi_{\frac{1}{2}}(a) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}.$$

2. We will map $\{|z| < 1\} \sim [0, 1]$ to $\mathbb{H} \sim \{\text{Re}(z) = 0 \text{ and } \text{Im}(z) \geq 1\}$ with the inverse of the map that sends the upper half plane to the disk:
$$T_1(z) = \frac{i(z+1)}{-z+1}.$$
Note that $[0, 1] \to \{\text{Re}(z) = 0 \& \text{Im} \geq 1\}$.

3. We will map $\mathbb{H} \sim \{\text{Re}(z) = 0 \text{ and } \text{Im}(z) \geq 1\}$ to $\mathbb{C} \sim \{-1 \leq \text{Re}(z) \leq 0 \text{ and } \text{Im}(z) = 0\}$ by
$$T_2(z) = z^2.$$
Note that $\{\text{Re}(z) = 0 \text{ and } \text{Im} \geq 1\}$ gets mapped to $\{-1 \leq \text{Re}(z) \leq 0 \text{ and } \text{Im}(z) = 0\}$.

4. We will map $\mathbb{C} \sim \{-1 \leq \text{Re}(z) \leq 0 \text{ and } \text{Im}(z) = 0\}$ to $\mathbb{C} \sim \{\text{Re}(z) \leq 0 \text{ and } \text{Im}(z) = 0\}$ by
$$T_3(z) = \frac{z}{z+1}$$
.

5. Let the negative real axis be our branch cut. Choose the branch of the logarithm on this branch cut.
$$T_4(z) = \log(z)$$
will map $\mathbb{C} \sim \{\text{Re}(z) \leq 0 \text{ and } \text{Im}(z) = 0\}$ to $\{0 < \text{Im}(z) < 2\pi i\}$.

6. We will use the following dilation:

$$T_5(z) = \frac{z}{\pi}$$

and then the translation

$$T_6(z) = z - i$$

to finally map to $\{|\text{Im}(z)| < 1\}$.

Therefore, our map will be the following:

$$f(z) = (T_6 \circ T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1 \circ \varphi_{\frac{1}{2}})(z).$$

□

Problem 9.5 (TAMU, August 2011 Q4). *Suppose f is a continuous function on $\{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disk, and f is holomorphic on the open unit disk. Prove that if $f(z)$ is real when $|z| = 1$, then f is a constant function.*

Solution. [Rajinda Wickrama]

Proof. Let $f = u + iv$. Since $f : \mathbb{D} \to \mathbb{C}$ is analytic, u and v and harmonic conjugates. Observe that $v(z) = 0$ when $|z| = 1$ since $f(z)$ is real when $|z| = 1$. By the maximum modulus principle for harmonic functions (Pg 253), v attains its maximum on the boundary of $\mathbb{D}$. However, since $v(z) = 0$ when $|z| = 1$, we get that $v \equiv 0$ on $\{z \in \mathbb{C} : |z| \leq 1\}$. Therefore, f is real on $\{z \in \mathbb{C} : |z| \leq 1\}$.

Finally, we need to show that f is constant. Since f is analytic on the unit disk, u, v satisfy the Cauchy-Riemann equations. Hence,

$$u_x = v_y = 0 \text{ and } u_y = -v_x = 0 \implies u \text{ is constant on } \mathbb{D}$$

Therefore, f is constant on $\mathbb{D}$. Since, f is continuous on $\{z \in \mathbb{C} : |z| \leq 1\}$, f is constant on $\{z \in \mathbb{C} : |z| \leq 1\}$.

□

□

Problem 9.6 (Texas A& M Complex, January 2013, 3.). *Suppose that $a_0 > a_1 > \cdots > a_N > 0$. Prove that $\sum_{n=0}^{N} a_n z^n \neq 0$ when $|z| \leq 1$.*

Problem 9.7 (Elaina Aceves, Aug 2011 Q6). *Use the residue theorem to prove that $\int_0^\infty \frac{x^2}{1+x^5} dx = \frac{\pi/5}{\sin(2\pi/5)}$.*

Solution.
Attempt:
I adopted Michael's approach from his solution (Kansas Sp 04 Q3) where he used the contour formed by the line from the origin to R along the x-axis, the arc from R to $\omega^2 R$ where ω is $e^{\pi i/5}$, and the line from $\omega^2 R$ back to the origin. If we use this 'pizza slice', then only ω is contained in the contour to help simplify calculations of the integral.
Solution:
Let $f(z) = \frac{z^2}{1+z^5}$ and $\omega = e^{\pi i/5}$. Let γ_1 be the line from the origin to R along the x-axis, let γ_2 be the arc from R to $\omega^2 R$, and let γ_3 be the line from $\omega^2 R$ back to the origin. Let $C_R = \gamma_1 + \gamma_2 + \gamma_3$ oriented counterclockwise. Michael provides a picture of this contour in Kansas Sp 04 Q3. For γ_2, we have the parametrization $z \to Re^{i\theta}$ where $dz = Rie^{i\theta}$ and γ_3, we have the parametrization $z \to \omega^2 y$ where $dz = \omega^2 dy$. Then we have that

$$\begin{aligned}\int_{\partial C_R} f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz \\ &= \int_0^R \frac{x^2}{1+x^5}dx + \int_0^{2\pi/5} \frac{(Re^{i\theta})^2 Rie^{i\theta}}{1+(Re^{i\theta})^5}d\theta + \int_R^0 \frac{(\omega^2 y)^2\omega^2}{1+(\omega^2 y)^5}dy \\ &= \int_0^R \frac{x^2}{1+x^5}dx + i\int_0^{2\pi/5} \frac{R^3 e^{3i\theta}}{1+R^5 e^{5i\theta}}d\theta - \omega^6 \int_0^R \frac{y^2}{1+\omega^{10}y^5}dy \quad (*)\end{aligned}$$

We can further simplify the third integral in $(*)$.

$$\begin{aligned}-\omega^6 \int_0^R \frac{y^2}{1+\omega^{10}y^5}dy &= \omega \int_0^R \frac{y^2}{1+\omega^{10}y^5}dy \text{ since } \omega^5 = -1 \\ &= \omega \int_0^R \frac{y^2}{1+y^5}dy \text{ since } \omega^{10} = (\omega^5)^2 = (-1)^2 = 1\end{aligned}$$

We will show that the second integral from $(*)$ vanishes as $R \to \infty$.

$$\begin{aligned}\left| i\int_0^{2\pi/5} \frac{R^3 e^{3i\theta}}{1+R^5 e^{5i\theta}}d\theta \right| &\leq \int_0^{2\pi/5} \frac{|R^3 e^{3i\theta}|}{|1+R^5 e^{5i\theta}|}|d\theta| \\ &\leq \int_0^{2\pi/5} \frac{|R^3|}{|1-R^5|}|d\theta| \\ &\leq \frac{2\pi |R|^3}{5|1-R^5|} \to 0 \text{ as } R \to \infty\end{aligned}$$

By the Residue Theorem since f has a simple pole at $\omega \in C_R$, we have that

$$\int_{C_R} f(z)dz = 2\pi i Res(f, \omega)$$

where

$$\begin{aligned} Res(f,\omega) &= \lim_{z\to\omega} f(z)(z-\omega) = \lim_{z\to\omega} \frac{z^2(z-\omega)}{1+z^5} = \lim_{z\to\omega} \frac{z^3 - z^2\omega}{1+z^5} \\ &= \lim_{z\to\omega} \frac{3z^2 - 2z\omega}{5z^4} \text{ by L'Hospital's Rule} \\ &= \frac{\omega^2}{5\omega^4} = \frac{1}{5\omega^2} \end{aligned}$$

Thus, after combining all of our results and letting $R \to \infty$, we obtain

$$(1+\omega)\int_0^\infty \frac{x^2}{1+x^5}dx = 2\pi i\left(\frac{1}{5\omega^2}\right)$$

Finally,

$$\int_0^\infty \frac{x^2}{1+x^5}dx = 2\pi i\left(\frac{1}{5\omega^2}\right)\left(\frac{1}{1+\omega}\right) = \frac{\pi}{5}\left(\frac{2i}{\omega^2+\omega^3}\right) = \frac{\pi}{5}\left(\frac{2i}{\omega^2+\omega^{-2}}\right) = \frac{\pi/5}{\sin(2\pi/5)}$$

□

Problem 9.8 (Andrew Pensoneault, May 2012, Q4)**.** *Find*

$$\int_{-\infty}^\infty \frac{\cos(x)}{(x^2+1)(x^2+4)}dx$$

Attempt:
I initially forgot $\cos(z)$ is infinite as $|r| \to \infty$, so we cannot just use a semicircle construction directly on $\cos(z)$. However, if we split it into $(e^{iz}+e^{-iz})/2$ and split the integral, we can reduce the problem into finding the integral of $\frac{e^{iz}}{(x^2+1)(x^2+4)}$

Solution:

First, notice

$$\begin{aligned} \int_{-\infty}^\infty \frac{\cos(x)}{(x^2+1)(x^2+4)}dx &= \int_{-\infty}^\infty \frac{e^{ix}+e^{-ix}}{2(x^2+1)(x^2+4)}dx \\ &= \int_{-\infty}^\infty \frac{e^{ix}}{2(x^2+1)(x^2+4)}dx + \int_{-\infty}^\infty \frac{e^{-ix}}{2(x^2+1)(x^2+4)}dx \\ &= \int_{-\infty}^\infty \frac{e^{ix}}{2(x^2+1)(x^2+4)}dx - \int_{\infty}^{-\infty} \frac{e^{ix}}{2(x^2+1)(x^2+4)}dx \\ &= \int_{-\infty}^\infty \frac{e^{ix}}{(x^2+1)(x^2+4)}dx \end{aligned}$$

By the Residue theorem (using an increasing counterclockwise half-circle in the upper-halfplane):

$$\int_{-\infty}^\infty \frac{e^{ix}}{(x^2+1)(x^2+4)}dx + \lim_{R\to\infty}\int_0^\pi \frac{iRe^{ix}e^{Rie^{ix}}}{(R^2e^{2ix}+1)(e^{2ix}+4)}dx = 2\pi i(\text{Res}(f,i)+\text{Res}(f,2i))$$

$$\text{Res}(f,i) = \lim_{z\to i} \frac{(z-i)e^{iz}}{(z^2+1)(z^2+4)} = \frac{1}{6ie}$$

$$\text{Res}(f,2i) = \lim_{z\to 2i} \frac{(z-2i)e^{iz}}{(z^2+1)(z^2+4)} = \frac{1}{-12ie^2}$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2+4)}dx + \lim_{R\to\infty}\int_0^{\pi} \frac{iRe^{ix}e^{Rie^{ix}}}{(R^2e^{2ix}+1)(R^2e^{2ix}+4)}dx = \frac{\pi(2e^{-1}-e^{-2})}{6}$$

Now looking at

$$\begin{aligned}
\Big|\int_0^{\pi} \frac{iRe^{ix}e^{Rie^{ix}}}{(R^2e^{2ix}+1)(R^2e^{2ix}+4)}dx\Big| &\le \int_0^{\pi} \Big|\frac{iRe^{ix}e^{Rie^{ix}}}{(R^2e^{2ix}+1)(R^2e^{2ix}+4)}\Big||dx| \\
&\le \int_0^{\pi} \frac{|Re^{Rie^{ix}}|}{|R^2e^{2ix}+1||R^2e^{2ix}+4|}|dx| \\
&\le \int_0^{\pi} \frac{Re^{-\text{Im}(Re^{ix})}}{|R^2-1||R^2-4|}|dx| \\
&\le \pi\frac{Re^{-R}}{|R^2-1||R^2-4|}
\end{aligned}$$

If we take the limit as $R\to\infty$, this

$$\lim_{R\to\infty}\int_0^{\pi} \frac{iRe^{ix}e^{Rie^{ix}}}{(R^2e^{2ix}+1)(R^2e^{2ix}+4)}dx = 0$$

Thus we have

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)(x^2+4)}dx = \frac{\pi(2e^{-1}-e^{-2})}{6}$$

Problem 9.9 (Texas A& M Complex, August 2014 Q6). *Prove that if* $0<|z|<1$ *then*

$$\frac{1}{4}|z| < |1-e^z| < \frac{7}{4}|z|.$$

Solution. [Jared Grove]
Attempt:
I tried to use all sorts of stuff to get the bottom bound. I had $|1-e^z| = \sqrt{1-e^x\cos(y)+e^{2x}}$ and did the calc three find a minimizer ($f_{xx}f_{yy}-f_{xy}^2$), but that didn't work out. I had the hardest time accounting for the min occuring at $z=0$ and not knowing how to work around that when $|z|>0$. In the end Rolando saved the day and reminded me that alternating series are things.

Solution:
We will start with the bound on the right.

$$
\begin{aligned}
|1-e^z| &= |1-\sum_{n=0}^{\infty}\frac{z^k}{k!}| \\
&= |1-(1+\frac{z}{1}+\frac{z^2}{2!}+\cdots)| \\
&= |z\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}| \\
&= |z||\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}| \\
&\leq |z|(\sum_{k=0}^{\infty}\frac{|z|^k}{(k+1)!}) \\
&\leq |z|(\sum_{k=0}^{\infty}\frac{1}{(k+1)!}) \\
&= |z|(1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots) \\
&< |z|(1+\frac{3}{7}+(\frac{3}{7})^2+(\frac{3}{7})^3+\cdots) \\
&= |z|(\sum_{k=0}^{\infty}(\frac{3}{7})^k) \\
&= \frac{7}{4}|z|
\end{aligned}
$$

Notice that $\sum_{k=0}^{\infty}(\frac{3}{7})^k)$ converges because it is a geometric series. If you need convincing of the $<$ inequality in work do out a few terms in both series. You will notice that the $(k+1)!$ gets much larger much quicker and thus the fraction will become much smaller at the end.

Now for the other direction:

$$
\begin{aligned}
|1-e^z| &= |z||\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}| \\
&\geq |z|(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}) \\
&= |z|(1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}\cdots) \\
&\approx |z|(\frac{76}{120}) \\
&> \frac{1}{4}|z|
\end{aligned}
$$

The approximate solution comes from doing adding the terms I already have listed out and realizing that the remaining terms will have negligable impact on making the sum get anywhere near a quarter.

□

Problem 9.10 (Texas A& M Complex, August 2014 Q7)**.** *Prove that the equation*

$$az^3 - z + b = e^{-z}(z+2)$$

has two solutions in the right half-plane $\{\mathscr{R}z > 0\}$ *when* $a > 0$*,* $b > 2$*.*

Solution. [Michael Kratochvil]
Attempt:
This immediately appeared to be an application of Roche's Theorem. Originally I wanted the function to be $az^3 + b$ since it is easy to show that there are two roots with positive real part, but on the right half circle, on the imaginary access the necessary inequality does not always hold. So I picked $az^3 - z + b$ which was a bit of a pain computationally to verify the desired location of roots, but at least the bounding was relatively straightforward.
Solution:
Consider $g(z) = az^3 - z + b$. I claim this function has one negative real-valued root and two roots with positive real part. To prove the first claim, note that for $x \in \mathbb{R}$, $g(x) = ax^3 - ax + b$ has critical points $(\pm\frac{1}{\sqrt{3a}}, b \mp \frac{2}{3\sqrt{3a}})$.. Since a positive local maximum is achieved at a negative critical value, $g(0) = b > 0$ and a local minimum occurs when $x > 0$, this implies that g has only one real negative root.
To prove the second claim, note that if $b < \frac{2}{3\sqrt{3a}}$, the local minimum of g is less than zero, implying that g has two positive real roots. If $b = \frac{2}{3\sqrt{3a}}$, the local minimum of g is zero, implying g has one positive real root with multiplicity 2.
If $b > \frac{2}{3\sqrt{3a}}$, the remaining roots are complex, so letting $z = x + iy$, $x, y \in \mathbb{R}, y \neq 0$ we have

$$g(z) = (ax^3 - 3axy^2 - x + b) + i(3ax^2y - ay^3 - y) = 0$$

so the imaginary and real parts are zero. $ax^3 - 3axy^2 - x + b = 0 \Rightarrow y^2 = \frac{ax^3 - x + b}{3ax}$, so

$$\begin{aligned}\mathscr{I}g(z) = 3ax^2y - a(\frac{ax^3 - x + b}{3ax})y - y = y(8ax^3 - 2x - b) = 0 \\ \Rightarrow 8ax^3 - 2x - b = 0.\end{aligned}$$

But $h(x) = 8ax^3 - 2x - b$ has $h(0) = -b < 0$ and only negative critical points (since $b > \frac{2}{3\sqrt{3a}}$ by assumption), which implies the only real root $x^* > 0$. Thus, the remaining roots of g have real part greater than zero and the claim is proven.

Now defining $f(z) = -(az^3 - z + b) + e^{-z}(z+2)$, fixing $R > 0$, and defining

$$\begin{aligned}\gamma_1 = \{z \in \mathbb{C} | z = Re^{i\theta}, \theta \in (-\pi, \pi)\} \\ \gamma_2 = \{iy | y \in [-R, R]\}.\end{aligned}$$

If we can show that for sufficiently large R that on $\gamma = \gamma_1 \cup \gamma_2$, $|f(z) + g(z)| < |g(z)|$, then by Rouche's Theorem, we are done.
Fixing $R > b, \sqrt{2/a}$, and R larger than the roots of g in modulus, we have on γ_1

$$|f(z) + g(z)| = |e^{-z}(z+2)| \leq |z+2| \leq R+2,$$
$$|g(z)| = |az^3 - z + b| \geq ||az^3| - |-z+b|| \geq aR^3 - R + b.$$

But

$$\begin{aligned} R > \sqrt{\frac{2}{a}} &\Rightarrow R^2 > \frac{2}{a} \\ &\Rightarrow aR^2 > 2 \\ &\Rightarrow aR^2 - 1 > 1 \\ &\Rightarrow aR^3 - R > R \\ &\Rightarrow aR^3 - R + b > R + 2. \end{aligned}$$

Hence, $|f(z) + g(z)| < |g(z)|$ on γ_1.
On $gamma_2$ $z = iy$ so

$$|f(z) + g(z)| = |e^{-iy}(iy+2)| = y^2 + 4.$$

But

$$|g(z)| = |-aiy^3 - iy + b| = (ay^3 + y)^2 + b^2 > y^2 + 4 = |f(z) + g(z)|.$$

Thus the claim holds on γ_2, as well and the proof is complete.

□

Problem 9.11 (TAMU, August 2014 Q9)**.** *Let $f_n : \mathbb{D} \to \mathbb{D}$ be a sequence of holomorphic functions in the unit disk $\mathbb{D}$. Suppose that $f_n(z_0) \to 1$ for some $z_0 \in \mathbb{D}$. Prove that then f_n converges to 1 normally in $\mathbb{D}$.*

Solution. [Amrei Oswald]
Attempt:
Since normal convergence is not defined in our textbook, I had to search the internet for a definition. There are a few different ones, but the one that made the most sense in this context is the following. The sequence $\{f_n\}$ *converges normally* to 1 in $\mathbb{D}$ if $\{f_n\}$ converges to 1 uniformly on any compact subset of $\mathbb{D}$.
Solution:
Note that $|f_n(x)| \leq 1$ for every $n \in \mathbb{N} \implies \{f_n\}$ is bounded on $\mathbb{D}$ and is therefore locally bounded on $\mathbb{D}$. By Montel's Theorem, $\{f_n\}$ is normal. Then, by the Arzela-Ascoli Theorem, the set $\{f_n\}$ is equicontinuous at every $z \in \mathbb{D}$.

Fix $\epsilon > 0$. Since $\{f_n\}$ is equicontinuous at z_0, there exists a $\delta > 0$ such that $d(f_n(z), f_n(z_0)) < \epsilon/2$ for every $n \in \mathbb{N}$ and $z \in B(z_0, \delta)$. Since $f_n(z_0) \to 1$, there exists an $N \in \mathbb{N}$ such that $d(f_n(z_0), 1) < \epsilon/2$ for every $n > N$. Thus, by the triangle inequality, we have

$$d(f_n(z), 1) \leq d(f_n(z), f_n(z_0)) + d(f_n(z_0), 1) < \epsilon \text{ for every } n > N, z \in B(z_0, \delta)$$

$$\implies f_n(z) \to 1 \text{ for every } z \in B(z_0, \delta).$$

Then, by the same argument as above, for every $z \in B(z_0, \delta)$ $\exists\, \delta_z > 0$ such that $f \to 1$ uniformly on $B(z, \delta_z)$. By continuing this procedure, we can cover $\mathbb{D}$ in sets $\{B(z, \delta_z)\}_{z \in \mathbb{D}}$ such that $f \to 1$ uniformly on $B(z, \delta_z)$.

Let $K \subset \mathbb{D}$ be a compact set. Then $K \subset \bigcup_{z \in \mathbb{D}} B(z, \delta_z)$. Since K is compact, there exists a finite subcover $\{B_i\}_{i=1}^m \subset \{B(z, \delta_z)\}$, $m \in \mathbb{Z}^+$.

Let $\eta > 0$. Then, $f_n \to 1$ uniformly on $B_1, ..., B_m \implies$ there exist $N_1, ..., N_m$ such that $d(f_n(z), 1) < \eta$ for $n > N_i, z \in B_i, 1 \le i \le m$. Let $N_0 = \max\{N_1, ..., N_m\}$. Then,

$$d(f_n(z), 1) < \eta \text{ for every } n > N_0, z \in K,$$

and $\{f_n\}$ converges normally to 1 on $\mathbb{D}$.

□

Solution. [Adam Wood]

Attempt:

I initially tried looking at the modulus of the expression and using that a_0 was the largest coefficient to get a bound on the modulus. Then, I tried using Rouche's Theorem. I tried $f =$ the whole sum and $g = a_n z^n$ and I tried $f = a_n z^n$ and $g =$ the whole sum. I also tried using induction, but kept on running into the problem that the triangle inequality didn't give me what I wanted. After looking a some proofs of the fundamental theorem of algebra and techniques for dealing with polynomials, I tried looking $(1-z)f(z)$, which worked. I was trying to use contradiction the whole time, but just needed to look at the "correct" polynomial.

Solution:

Proof. Let $f(z) = \sum_{n=0}^{N} a_n z^n$ and consider the function $g(z) = (1-z)f(z)$. Then,

$$g(z) = a_0 + \sum_{n=1}^{N} (a_i - a_{i-1}) z^i - a_N z^{N+1}.$$

Suppose, for the sake of contradiction, that $f(z_0) = 0$ for some z_0 with $|z_0| < 1$. By definition of $g(z)$, we also have that $g(z_0) = 0$. Therefore,

$$a_0 + \sum_{n=1}^{N} (a_i - a_{i-1}) z_0^i - a_N z_0^{N+1} = 0.$$

Since $a_i < a_{i-1}$ for all $1 \le i \le N$, $a_i - a_{i-1} < 0$. So, the statement above is equivalent to

$$a_0 = \sum_{n=1}^{N} (a_{i-1} - a_i) z_0^i + a_N z_0^{N+1}.$$

Then,

$$|a_0| = \left|\sum_{n=1}^{N}(a_{i-1} - a_i)z_0^i + a_N z_0^{N+1}\right| \le \sum_{n=1}^{N}(a_{i-1}-a_i)|z_0|^i + a_N|z_0|^{N+1} < \sum_{i=1}^{N}(a_{i-1}-a_i)+a_N = a_0,$$

which is a contradiction. Thus, f has no zeros z_0 with $|z_0| < 1$. It remains to show that $f(z_0) \neq 0$ if $|z_0| = 1$. If $f(z_0) = 0$ for some z_0 with $|z_0| = 1$, then by the Maximum Modulus Theorem, f attains its maximum modulus over the unit disk on the boundary of the unit disk. Therefore, $f(z) = 0$ for all z with $|z| \le 1$, which is a contradiction, since f has no zeros within the unit disk. Thus, $f(z) \neq 0$ when $|z| \le 1$. □

□

Problem 9.12 (TAMU, August 2014 Q2). *a) Find and classify all isolated singularities of*

$$f(z) = \frac{z^2(z-\pi)}{\sin^2 z} \qquad \textit{and} \qquad g(z) = (z^2-1)\cos\frac{1}{z-1}.$$

b) Find the residue of f at $z = 2\pi$ and the residue of g at $z = 1$.

Solution. [Sara Reed] We will be using the following Theorems and definitions to classify singularities: Part a: Consider $f(z)$. We know the singularities occur when $\sin^2 z = 0$ which happens when $z = n\pi$ for all $n \in \mathbb{Z}$. When $n = 0$, there is a removable singularity as shown using L'Hospital's Rule:

$$\lim_{z\to 0}\frac{zz^2(z-\pi)}{\sin^2 z} = \lim_{z\to 0}\frac{4z^3 - 3\pi z^2}{2\sin z\cos z} = \lim_{z\to 0}\frac{12z^2 - 6\pi z}{-2\sin^2 z + 2\cos^2 z} = 0.$$

Note that

$$\lim_{z\to n\pi}\left|\frac{z^2(z-\pi)}{\sin^2 z}\right| = \infty$$

for $n \in \mathbb{Z} \sim \{0\}$. Therefore, there is a pole at $z = n\pi$ for $n \in \mathbb{Z} \sim \{0\}$. We may want to take special attention to when $n = 1$ but note that using L'Hospital's Rule, we find:

$$\lim_{z\to\pi}\left|\frac{z^2(z-\pi)}{\sin^2 z}\right| = \lim_{z\to\pi}\left|\frac{3z^2 - 2\pi z}{2\sin z\cos z}\right| = \infty.$$

Now, we look to find the order of the pole. We want to find the smallest m such that $f(z)(z-a)^m$ has a removable singularity at $z = a$. So we want to find m such that $\lim_{z\to n\pi}(z-a)^{m+1}f(z) = 0$ for $n \in \mathbb{Z} \sim \{0\}$. It follows using L'Hospital's Rule:

$$\begin{aligned}
&\lim_{z\to n\pi}\frac{(z-n\pi)^{m+1}z^2(z-\pi)}{\sin^2 z} = \lim_{z\to n\pi}\frac{(z-n\pi)^{m+1}(z^3-\pi z^2)}{\sin^2 z} \\
&= \lim_{z\to n\pi}\frac{(m+1)(z-n\pi)^m(z^3-\pi z^2) + (z-n\pi)^{m+1}(3z^2-2\pi z)}{2\sin z\cos z} \\
&= \lim_{z\to n\pi}\frac{m(m+1)(z-n\pi)^{m-1}(z^3-\pi z^2) + (m+1)(z-n\pi)^m(3z^2-2\pi z)}{-2\sin^2 z + 2\cos^2 z} \\
&\quad + \frac{(m+1)(z-n\pi)^m(3z^2-2\pi z) + (z-n\pi)^{m+1}(6z-2\pi)}{-2\sin^2 z + 2\cos^2 z} \\
&= 0
\end{aligned}$$

which implies that $m-1>0$ and therefore $m=2$. Note that in the case that $n=1$, we would follow a similar argument to show the pole is of order $m=1$.
Now consider $g(z)$. We know a singularity occurs at $z=1$ so we will work to write $g(z)$ as a power series around 1.

$$\begin{aligned}
g(z) =&(z^2-1)\cos\frac{1}{z-1}\\
=&(z+1)(z-1)\Big(1-\frac{1}{2!(z-1)^2}+\frac{1}{4!(z-1)^4}-\frac{1}{6!(z-1)^6}+\cdots\Big)\\
=&(z-1+2)\Big((z-1)-\frac{1}{2!(z-1)}+\frac{1}{4!(z-1)^3}-\frac{1}{6!(z-1)^5}+\cdots\Big)\\
=&(z-1)\Big((z-1)-\frac{1}{2!(z-1)}+\frac{1}{4!(z-1)^3}-\frac{1}{6!(z-1)^5}+\cdots\Big)\\
&+2\Big((z-1)-\frac{1}{2!(z-1)}+\frac{1}{4!(z-1)^3}-\frac{1}{6!(z-1)^5}+\cdots\Big)\\
=&\Big((z-1)^2-\frac{1}{2!}+\frac{1}{4!(z-1)^2}-\frac{1}{6!(z-1)^4}+\cdots\Big)\\
&+\Big(2(z-1)-\frac{1}{(z-1)}+\frac{2}{4!(z-1)^3}-\frac{2}{6!(z-1)^5}+\cdots\Big).
\end{aligned}$$

Since $a_n \neq 0$ for infinitely many negative integers n, $z=1$ is an essential singularity of g.

Part b: We want to find $\text{Res}(f;2\pi)$. From part a, we know 2π is a pole of order 2. Let $g(z)=(z-2\pi)^2 f(z)$. We can simplify using the power expansion of sine around $z=2\pi$ as follows:

$$\begin{aligned}
g(z) &= (z-2\pi)^2 f(z)\\
&= \frac{(z-2\pi)^2 z^2 (z-\pi)}{\Big((z-2\pi)-\frac{(z-2\pi)^3}{3!}+\frac{(z-2\pi)^5}{5!}-\cdots\Big)^2}\\
&= \frac{(z-2\pi)^2 z^2 (z-\pi)}{(z-2\pi)^2\Big(1-\frac{(z-2\pi)^2}{3!}+\frac{(z-2\pi)^4}{5!}-\cdots\Big)^2}\\
&= \frac{z^3-\pi z^2}{\Big(1-\frac{(z-2\pi)^2}{3!}+\frac{(z-2\pi)^4}{5!}-\cdots\Big)^2}.
\end{aligned}$$

Then it follows

$$g'(z) = \frac{\left(1 - \frac{(z-2\pi)^2}{3!} + \frac{(z-2\pi)^4}{5!} - \cdots\right)^2 (3z^2 - 2\pi z)}{\left(1 - \frac{(z-2\pi)^2}{3!} + \frac{(z-2\pi)^4}{5!} - \cdots\right)^4} - \frac{2(z^3 - \pi z^2)\left(1 - \frac{(z-2\pi)^2}{3!} + \frac{(z-2\pi)^4}{5!} - \cdots\right)\left(\frac{2(z-2\pi)}{3!} + \frac{4(z-2\pi)^3}{5!} - \cdots\right)}{\left(1 - \frac{(z-2\pi)^2}{3!} + \frac{(z-2\pi)^4}{5!} - \cdots\right)^4}$$

It follows from Proposition 2.4:

$$\text{Res}(f; 2\pi) = g'(2\pi) = 8\pi^2.$$

Next, we want to find $\text{Res}(g; 1) = a_{-1} = -1$ which can be seen by the expansion of g in part a. □

Problem 9.13 (Texas A& M Complex, January 2015 Q1)**.** *Prove that if $z \in \mathbb{C}$ and $k \in \mathbb{Z}^+$ then*

$$|Im(z^k)| \leq k|Im(z)||z|^{k-1}.$$

Solution. [Jared Grove]
Attempt:
First I tried doing cases and was able to show it worked when $x = 0$ or $y = 0$ for $z = x + iy$, but not for any other cases.
Solution:
Trig Identities:

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a)$$

$\Rightarrow$ We will begin with the polar representation of a complex number $z = re^{it}$ for $0 \leq t \leq 2\pi$. Then looking at the different parts of the inequality:

$$|Im(z^k)| = |Im(r^k e^{ikt})| = r^k|\sin(kt)|$$

$$k|Im(z)||z|^{k-1} = k|r\sin(t)||re^{it}|^{k-1} = kr|\sin(t)|r^{k-1} = kr^k|sin(t)|$$

Since these equations both have r^k in common we only need to show that $|sin(kt)| \leq k|\sin(t)|$.

$$\text{If } k = 1, \ |\sin(t)| \leq |\sin(t)|$$
$$\text{If } k = 2, \ |\sin(2t)| = |2\sin(t)\cos(t)| \leq 2|\sin(t)|$$
$$\text{If } k = 3, \ |\sin(3t)| = |\sin(2t + t)| = |\sin(2t)\cos(t) + \cos(2t)\sin(t)| \leq |\sin(2t)| + |\sin(t)|$$
$$\leq 2|\sin(t)| + |\sin(t)| = 3|\sin(t)|$$

Now we notice the beginning of an induction argument. So assume that $|\sin(kt)| \leq k|\sin(t)|$ up to k and we only need to show it for $k+1$.

For $k+1$, $|\sin((k+1)t)| = |\sin(kt+t)| = |\sin(kt)\cos(t) + \cos(kt)\sin(t)| \leq |\sin(kt)| + |\sin(t)|$
$$\leq k|\sin(t)| + |\sin(t)| = (k+1)|\sin(t)|$$

Since we were able to show that this inequality holds it must be the case that:

$$|Im(z^k)| \leq k|Im(z)||z|^{k-1}$$

□

Problem 9.14 (TAMU, January 2015 Q4)**.** *Prove the following: If f is a holomorphic function that maps the open unit disk into itself, and if z_1 and z_2 are two zeroes of f in the unit disk, then*

$$|f(z)| \leq \left|\frac{(z-z_1)(z-z_2)}{(1-\overline{z}_1 z)(1-\overline{z}_2 z)}\right| \text{ when } |z| < 1.$$

Solution. [Michael Kratochvil]
Attempt:
I immediately realized that f is supposed to be bounded by the product of φ_{z_1} and φ_{z_2} so I tried to solve by bounding by one of these functions first. I tried to directly reapply the proof of Schwarz's lemma, but that did not work. I finally realized that using a composition, I can apply Schwarz's lemma (and not reprove it). From there the rest followed pretty easily.
Solution: Define $h(w) = (f \circ \varphi_{z_1}^{-1})(w)$. Then h is analytic on $\mathbb{D}$, $h(0) = 0$ and $|h(w)| \leq 1$ for all $w \in \mathbb{D}$. So by Schwarz's Lemma, $|h(w)| \leq |w|$ for all w. In particular for $w = \varphi_{z_1}(z)$ for $z \in \mathbb{D}$, we have

$$|h(\varphi_{z_1}(z)| = |f(z)| \leq |\varphi_{z_1}(z)|.$$

Now let $g(z) = \frac{f(z)}{\varphi_{z_1}(z)}$ for $z \neq 0$ and $g(0) = (1-|a|^2)f'(0)$ so that g is analytic on $\mathbb{D}$. Define $m(w) = (g \circ \varphi_{z_2}^{-1})(w)$ for $w \in \mathbb{D}$. Then by similar argument as above we have

$$|g(z)| \leq |\varphi_{z_2}(z)|.$$

It immediately follows that

$$|f(z)| \leq |\varphi_{z_1}(z)||\varphi_{z_2}(z)| = \left|\frac{(z-z_1)(z-z_2)}{(1-\overline{z}_1 z)(1-\overline{z}_2 z)}\right|.$$

□

Problem 9.15 (TAMU, January 2015 Q5)**.** *Suppose f is an entire function such that the product $|Re(f)||Im(f)|$ is bounded. Prove that f must be a constant function.*

Solution. [Amrei Oswald]
Attempt:
This proof is motivated by a picture of the image of f. Say that M is the bound on $|\text{Re}(f)|\,|\text{Im}(f)|$. Then consider the graph of $|\text{Im}\,(f)| \leq \frac{M}{\text{Re}(f)}$ in the imaginary plane or equivalently, the graph of $|y| \leq \frac{M}{|x|}$ in the $x - y$ plane. From here, its fairly straigtforward to identify a ball in the first quadrant that does not intersect the image, shift the center of this ball to the origin and invert to get a bounded entire function.
Solution:
Since $|\text{Re}(f)|\,|\text{Im}(f)|$ is bounded, there exists a $M > 0$ such that $|\text{Re}(f)|\,|\text{Im}(f)| < M$. Let $R > 1$ and $z_0 = (1 + 2R) + (M + 2R)i$. Consider $B = B(z_0; R)$. Say that $x + iy \in B \cap f(\mathbb{C})$. Then, we have

$$|x|\,|y| \geq (1 + 2R - R)(M + 2R - R) = (1 + R)(M + R) > M$$

which is a contradiction. Therefore, $B \cap f(\mathbb{C}) = \emptyset$. Then we have that $|f(z) - z_0| \geq R$ for every $z \in \mathbb{C}$ and the function $g(z) = \frac{1}{f(z)-z_0}$ is entire. Then for every $z \in \mathbb{C}$ we have

$$|g(z)| = \frac{1}{|f(z) - z_0|} \leq \frac{1}{R}.$$

By Liouville's theorem, $g(z)$ is constant. Say that $g(z) = c \in \mathbb{C}$. Then,

$$\frac{1}{f(z) - z_0} = c \implies f(z) = \frac{1}{c} + z_0$$

so f is constant. □

10 UI Urbana-Champaign Quals

Problem 10.1. *Let m be Lebesgue measure on R. Suppose that Integral $\int_R |f|dm = c \in (0,\infty)$ and $\alpha \in (0,1)$*
Show that

$$\lim_{n\to\infty} \int_R nlog(1 + (|f|/n)^\alpha)dm = \infty$$

Proof. We first consider

$$\begin{aligned}
\lim_{n\to\infty} \int_R nlog(1 + (|f|/n)^\alpha)dm = &= \int_R \lim_{n\to\infty} nlog(1 + (|f|/n)^\alpha)dm \\
&= \lim_{n\to\infty} \frac{log(1 + (\frac{|f|}{n})^\alpha)}{\frac{1}{n}} \\
&= \frac{0}{0} \\
&= \lim_{n\to\infty} \frac{\frac{1}{1+(\frac{|f|}{n})^\alpha} \frac{-|f|^\alpha}{n^(\alpha+1}}{\frac{-1}{n^2}} dm \qquad \text{(by L'Hospital's Rule)} \\
&= \lim_{n\to\infty} \frac{n^2|f|^\alpha}{1 + (\frac{|f|}{n})^\alpha (n^\alpha + 1)} \\
&= \frac{\infty}{1} \\
&= \infty
\end{aligned}$$

since $\int_R |f|dm = c \in (0,\infty)$ we have a non-negative sequence of measurable functions and thus we have point-wise convergence. Therefore, by Fatous Lemma

$$\int_R \lim_{n\to\infty} fn = \int_R f = \infty \leqslant \liminf \int_R fn = \lim_{n\to\infty} \int_R fn$$

□

Problem 10.2. *Find the radius of convergence for the following power series*

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{k^3} z^{k^2}$$

Solution. Notice:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^3} z^{k^2} = 2z+\left(\frac{3}{2}\right)^8 z^4+\left(\frac{4}{3}\right)^2 7z^9+\left(\frac{5}{4}\right)^6 5z^{16}+... \quad (54)$$

$$= 2z+0z^2+0z^3+\left(\frac{3}{2}\right)z^4+0z^5+... \quad (55)$$

Therefore, we can write:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^3} z^{k^2} = \sum_{n=1}^{\infty} a_n z^n,$$

where

We know that given a power series $\sum_{k=1}^{\infty} a_n(z-z_0)^n$, its radius of convergence R is defined to be:

$$\tfrac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$$

Therefore, we wish to show that the limit below is equal to 1 so that $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$.

$$\lim_{k\to\infty} \ln\left(\left(1+\tfrac{1}{k}\right)^k\right)$$

Using L'Hopital's Rule, we can compute the limit:

$$\lim_{k\to\infty} \ln\left(\left(1+\frac{1}{k}\right)^k\right) = \lim_{k\to\infty} k\ln\left(1+\frac{1}{k}\right) \quad (56)$$

$$= \lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} \quad (57)$$

$$= 1 \quad (58)$$

Therefore, $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$, so $R=\frac{1}{e}$.

□

Problem 10.3 (UI Urbana-Champaign, January 2016-Q1)**.**
a) Is there a Mobius Transformation w = T(z) that maps i to -i, -1 to -2i and sends real numbers to real numbers?

b) Find the Mobius Transformation $w = T(z)$ *that maps* i *to* ∞*,* $1-i$ *to* i *and* 2 *to* $1-i$*.*

Solution. [Shawn]

a) No, by the Symmetry Principle. If z_1, z_2 and z_3 are distinct points lying on a circle, z and w are said to be symmetric about that circle if $(z, z_1, z_2, z_3) = \overline{(w, z_1, z_2, z_3)}$. The Principle states that points symmetric about a circle are mapped to points symmetric about the image of that circle under a Mobius transformation. Since a line is a circle in $\mathbb{C}$, symmetry about a line means equidistant on a perpendicular. In our case, the hypothetical $T(z)$ would map i and $-i$, which are symmetric about the real line, to points $-i$ and $-2i$, which are not symmetric about the image of the real line (which is itself.)

b) While there are many approaches to this problem, the no-brainer is to use the given conditions to set up and solve a system of equations. We know that a Mobius Transformation has the form $T(z) = \frac{az+b}{cz+d}$ with $ad \neq bc$, and if $T(i) = \infty$, the denominator must take the form $c(z-i)$. We can absorb the constant c into the numerator's coefficients to see that our map must take the form $T(z) = \frac{az+b}{z-i}$. We now have two coefficients to find, and we have two other conditions to use, so we solve the system of equations given by $T(1-i) = i$ and $T(2) = 1-i$. After some careful arithmetic, we can find in this case that $T(z) = \frac{(-5-31)z+12}{2z-2i}$.
□

Problem 10.4 (UI Urbana-Champaign, January 2016-Q3)**.** *Evaluate* $\int_0^\infty \frac{x^{1/2}}{1+x^2} dx$.

Solution. [Noah Kaufmann]
Attempt:
This is exactly what you expect when you see an integral in the Complex section: an application of the Residue Theorem. See Conway Chapter V, Section 2 for details on residues and the Residue Theorem.
Solution:
Since the integral has only nonnegative values of x, no one stops us from making the substitution x = t^2. This changes the integral to

$\int_0^\infty \frac{2t^2}{1+t^4} dt$

Using the fact that the integrand is even lets us replace the above integral with

$\int_{-\infty}^\infty \frac{t^2}{1+t^4} dt$

(Note: Once we have the integral in this form, this is exactly example 2.5 on page 113 of Conway. Feel free to read the solution there instead of here.)

Now we will take the path defined by the upper semicircle centered at 0 of radius R>1, oriented counterclockwise. Call this path γ. Let f(z) = $\frac{z^2}{1+z^4}$

We know from the Residue Theorem that

$\int_\gamma f(z)= 2\pi i \sum_j Res(f; a_j)$

Where the a_j's are the poles of f(z) that lie inside γ.

In this case, the poles of f(z) are exactly the fourth roots of -1, which are $e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$, $e^{\frac{5\pi i}{4}}$, and $e^{\frac{7\pi i}{4}}$. Since we chose γ to be the upper semicircular path, only $e^{\frac{\pi i}{4}}$ and $e^{\frac{3\pi i}{4}}$ are inside of γ.

We have that $Res(f; e^{\frac{\pi i}{4}})$ is given by

$$\lim_{z \to e^{\frac{\pi i}{4}}} (z - e^{\frac{\pi i}{4}}) f(z)$$

And similarly, $Res(f; e^{\frac{3\pi i}{4}})$ is given by

$$\lim_{z \to e^{\frac{3\pi i}{4}}} (z - e^{\frac{3\pi i}{4}}) f(z)$$

Evaluating these limits and summing gives that

$\int_\gamma f(z)= 2\pi i \sum_j Res(f; a_j) = 2\pi i (\frac{-i}{2\sqrt{2}}) = \frac{\pi}{\sqrt{2}}$.

Now we look at $\int_\gamma f(z)$. This integral can be split into two distinct parts, one of which is the integral we are looking for.

$\int_\gamma f(z) = \int_{-R}^{R} \frac{z^2}{1+z^4} dz + \int_0^\pi \frac{R^3 e^{3it}}{1+R^4 e^{4it}} dt$

Because e^{4it} has modulus 1, we have that $|1 + R^4 e^{4it}| > R^4 - 1$ and $|R^3 e^{3it}| \leq R^3$ which gives us the bound

$\int_0^\pi \frac{R^3 e^{3it}}{1+R^4 e^{4it}} dt \leq \frac{\pi R^3}{R^4 - 1}$

This tells us that as we let R go to ∞, $\int_0^\pi \frac{R^3 e^{3it}}{1+R^4 e^{4it}} dt$ goes to 0. Looking at our previous equations, we have that

$\int_{-R}^{R} \frac{z^2}{1+z^4} dz + \int_0^\pi \frac{R^3 e^{3it}}{1+R^4 e^{4it}} dt = \frac{\pi}{\sqrt{2}}$

for any R. Letting R go to ∞ on both sides gives us the final answer, that

$\int_0^\infty \frac{x^{1/2}}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{z^2}{1+z^4} dz = \frac{\pi}{\sqrt{2}}$.

□

Problem 10.5 (UI Urbana-Champaign, January 2016-Q3)**.** *Use residues to evaluate the*

following definite integral:
$$\int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)}.$$

Solution. [Jared Grove]

Attempt:

When making the function for the residue be careful when canceling terms out. Notice that:
$$g(z) = (z-(\frac{-i}{2})f(z) = (z+\frac{i}{2})\frac{1}{2(z+\frac{i}{2})(z+2i)} = \frac{1}{2(z+2i)}.$$

I missed a small part on this:
$$g(z) = (z-(\frac{-i}{2})f(z) = (z+\frac{i}{2})\frac{1}{(2z+i)(z+2i)} \neq \frac{1}{(z+2i)}.$$

Using the second will give a final answer of $\frac{4\pi}{3}$, which while close is still wrong.

Solution:

First we will define some important stuff: $z = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$ and $\gamma = |z| = 1$. Next we will make a change of variabe so we can work with z instead of θ:
$$\begin{aligned} 5+4\sin(\theta) &= 5+4[\frac{1}{2i}(e^{i\theta}-e^{-i\theta})] \\ &= 5+\frac{2}{i}(z-\frac{1}{z}) \\ &= \frac{5iz}{iz}+\frac{2z^2-2}{iz} \\ &= \frac{2z^2+5iz-2}{iz}. \end{aligned}$$

Now that we know how it will substitute in we can make our subtitutions:
$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)} &= \int_\gamma \frac{iz}{2z^2+5iz-2}(\frac{1}{iz})dz \\ &= \int_\gamma \frac{dz}{(2z+i)(z+2i)}. \end{aligned}$$

From here we will say that $f(z) = \frac{1}{(2z+i)(z+2i)}$. Here note that there are poles of order $m = 1$ at both $z = \frac{-i}{2}$ and $z = -2i$. Since $f(z)$ is anaytic in $\mathbb{C}$, except for at the two poles listed above, and γ is a closed recticfiable curve in $\mathbb{C}$ that doesn't pass through either singularity and $\gamma \approx 0$ in $\mathbb{C}$ we can use the Residue Theorem (p112) to say that:
$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)} &= \int_\gamma \frac{dz}{(2z+i)(z+2i)} \\ &= 2\pi i[\eta(\gamma,\frac{-i}{2})Res(f,\frac{-i}{2})+\eta(\gamma,-2i)Res(f,-2i)] \\ &= 2\pi i[(1)Res(f,\frac{-i}{2})+(0)Res(f,-2i)] \\ &= (2\pi i)Res(f,\frac{-i}{2}). \end{aligned}$$

Note that $\eta(\gamma, -2i) = 0$ as $|-2i| > 1$. Now we need to calculate the residue around $\frac{-i}{2}$. Since this is a pole we are able to use Proposition 2.4 from page 113. First we need to find our function:

$$g(z) = (z - (\frac{-i}{2})f(z) = (z + \frac{i}{2})\frac{1}{2(z+\frac{i}{2})(z+2i)} = \frac{1}{2(z+2i)}.$$

Thus the residue will be:

$$\begin{aligned} Res(f, \frac{-i}{2}) &= \frac{1}{0!}g^{(0)}(\frac{-i}{2}) \\ &= \frac{1}{2(\frac{-i}{2}+2i)} = \frac{1}{3i}. \end{aligned}$$

After that we are finally ready to finish our calculation:

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)} &= (2\pi i)Res(f, \frac{-i}{2}) \\ &= (2\pi i)\frac{1}{3i} \\ &= \frac{2\pi}{3}. \end{aligned}$$

□

Problem 10.6 (UI Urbana-Champaign, May 2015-Q5)**.** *Let $\mathscr{F}$ be a family of all analytic functions satisfying the inequality:*

$$|f(z)| \leq \frac{1}{(1-|z|)^{2015}}$$

for all $z \in \mathbb{D}$. Prove $\mathscr{F}$ is normal.

Solution. [Andrew Pensoneault]
Attempt:
As the problem asks first for the statement of Montel's theorem, I first thought it would be best to show that the family is locally bounded, which turned out to be the correct approach. I initially drew the picture as seen below and thought we would be splitting the problem into two cases, the set of points inside of a disk in $\mathbb{D}$, and the set of points outside of that disk. This approach allows us to find a uniform bound $\left(|f(z)| \leq \frac{1}{(1-|R|)^{2015}} \text{ for } |a-z| < (R-r)\right)$ for all points inside of the inner disk, however, it turns out this is unnecessary, as local boundedness is a pointwise property.

Solution:

Let $a \in \mathbb{D}$. As $\mathbb{D}$ is open, there exists a $r > 0$ such that $B(a, r) \subset \mathbb{D}$, and therefore we have $\overline{B}(a, \frac{r}{2}) \subset \mathbb{D}$. Define $\tilde{g} : \mathbb{D} \to \mathbb{R}$ such that $\tilde{g}(z) = (1-|z|)^{2015}$. This is a continuous function which does not vanish in $\mathbb{D}$, thus the function $g(z) = \frac{1}{\tilde{g}(z)}$ is continuous. As $\overline{B}(a, \frac{r}{2})$ is a compact set, by the Extreme Value Theorem, g obtains a maximum value M on $\overline{B}(a, \frac{r}{2})$, and thus $g(z) \leq M$ on $|z-a| < \frac{r}{2}$. Now, by definition, if $f \in \mathscr{F}$, then $|f(z)| \leq |g(z)| \leq M$ on $|z-a| < \frac{r}{2}$, and as our choice of a was arbitrary, we have shown $\mathscr{F}$ is locally bounded. By Montel's Theorem, $\mathscr{F}$ is normal. □

Problem 10.7 (UI Urbana-Champaign, May 2015-Q6). *Let f be an entire function such that*

$$|f(z)| \leq 5|z|^{3/2} \quad \text{for all } z \in \mathbb{C},\ |z| \geq 1.$$

Prove that $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| + |b| \leq 5$.

Solution. [W. Tyler Reynolds]

Attempt:

My initial approach to the first part of the problem was to use Mobius transformations to create a new function which satisfied the hypotheses of Schwarz's Lemma. The Maximum Modulus Theorem was helpful in exhibiting that these conditions could in fact be satisfied. My hope was that the implications of Schwarz's Lemma would yield some useful information about the original function. However, the resulting inequalities proved to be of little use. Next, I tried the power series approach with Cauchy's Estimate, and it worked! In hindsight, this approach is more straightforward because it directly utilizes the fact the f is entire. For the second part of the problem, I tried working with Cauchy's Representation Theorem, Mobius transformations, and plain inequalities to get adequate estimates on $|a|$ and $|b|$. The best I could say at first was that $|a| \leq 5$, $|b| \leq 5$, and $|a+b| \leq 5$. If only the triangle inequality could be reversed! While this can't happen in general, the geometry of the plane does allow for the triangle inequality to become equality, namely when either $|a|$ or $|b|$ is zero, or when a and b point in the same direction (i.e., when $\arg a = \arg b$). When two vectors don't point the same direction, you can of course rotate them so that they do!

Solution:

Since f is entire, f has a power series about 0 defined on all of $\mathbb{C}$, say $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We wish to show that $a_n = 0$ for $n \geq 2$. To this end, let $n \geq 2$ and let $R \geq 1$. By assumption, $|f(z)| \leq 5R^{3/2}$ for $|z| = R$; it follows from the Maximum Modulus Theorem that $|f(z)| \leq 5R^{3/2}$ for $|z| \leq R$. Thus, we can differentiate the power series and apply Cauchy's Estimate to obtain $|a_n| = n! \cdot |f^{(n)}(0)| \leq \dfrac{n! \cdot 5R^{3/2}}{R^n}$. Since $n > \frac{3}{2}$, letting $R \to \infty$ yields $|a_n| = 0$.

Letting $a = a_0$ and $b = a_1$, we now have $f(z) = az + b$. If $a = 0$ or $b = 0$, then $|f(1)| = |a + b| = |a| + |b| \leq 5$. Otherwise, there is some $c \in \mathbb{C}$ with $|c| = 1$ such that $\arg ca = \arg b$. In this case,

$$|a| + |b| = |ca| + |b| = |ca + b| = |f(c)| \leq 5.$$

□

Problem 10.8 (UI Urbana-Champaign, January 2017-Q2)**.** *Prove that the equation*

$$z \sin z = 1, \quad z \in \mathbb{C}$$

has only real roots.

Solution. [Qing Zou]
Attempt:
I know the proof of this problem relies on Rouché's theorem, which is on page 125 of our textbook (Theorem 3.8). While I failed because I cannot prove that $|z \sin z| > 1$ on a chosen circle. But when I tried to prove that $|z \sin z| > 1$ on a chosen circle, I found that I can prove that $|z \sin z| > 1$ on a chosen square box. Then I decided to use the general form of Rouché's theore, which allows us to use the square box rather than a circle.
Solution:
Theorem (General form of Rouché's theorem, Rudin, "Real and Complex Analysis", Page 229) Let Ω be the interior of a compact set K in the plane. Suppose f and g are continuous on K and holomorphic in Ω, and $|f(z) - g(z)| < |f(z)|$ for $z \in K - \Omega$. Then f and g have the same number of zeros in Ω, where each zero is counted as many times as its muliplicity.

Consider

$$R = \{z = x + iy : -(n + \frac{1}{2})\pi < x < (n + \frac{1}{2})\pi, \;\; -(n + \frac{1}{2})\pi < y < (n + \frac{1}{2})\pi, \;\; n \in \mathbb{N}\}.$$

Then $\bar{R}$ is a compact set and R is the interior of $\bar{R}$.

Let $f(z) = z \sin z$ and $g(z) = z \sin z - 1$.

First, we show that $|f(z)| > 1$ on the square box ∂R.

Since on ∂R,

$$|z \sin z| = |z| \cdot |\sin z| > |\sin z|.$$

Then in order to show $|f(z)| > 1$ on ∂R, it suffices to show that $|\sin z| \geq 1$, which is equivalent to show that $|\sin z|^2 \geq 1$.

Note that we have

$$|\sin z|^2 = |\sin(x + iy)|^2 = \cosh^2 y - \cos^2 x.$$

(This is not something easy to obtain, one can try it as an exercise).

- If $x = (n + \frac{1}{2})\pi$ or $x = -(n + \frac{1}{2})\pi$, then $\cos^2 x = 0$ and since $\cosh y$ is always greater than or equal to 1. So we can get that $|\sin z|^2 \geq 1$ in this case.

- If $y = (n + \frac{1}{2})\pi$ or $y = -(n + \frac{1}{2})\pi$, then $\cosh^2 y > 4$ (Since as n increases, $\cosh y$ also increases. Then $\cosh^2 y > 4$ because $\cosh(\frac{\pi}{2}) \approx 2.509$). So we can get that in this case, $\cosh^2 y - \cos^2 x > 1$, i.e. $|\sin z|^2 > 1$.

To sum up, we can get that on ∂R, $|f(z)| > 1$.

Then we know that on ∂R,

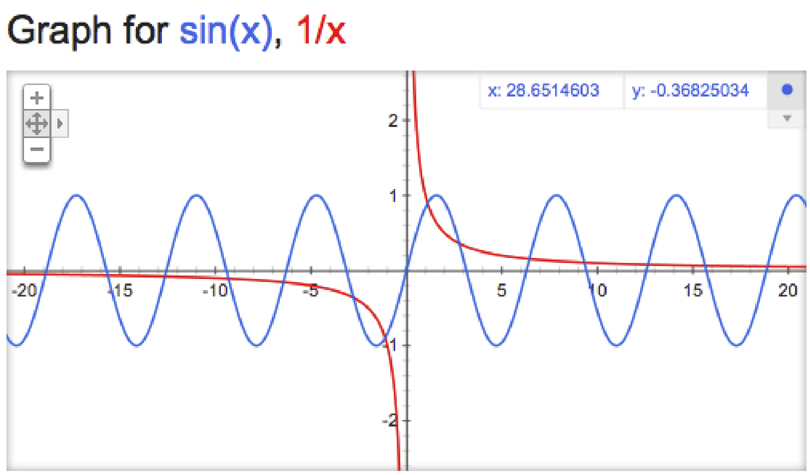
$$1 = |f(z) - g(z)| < |f(z)|.$$

So, by the general form of Rouché's theorem, we know that $f(z)$ and $g(z)$ have the same number of zeros in R.

Now, let us consider the roots of $z \sin z$ in R. If

$$z \sin z = 0 \Leftrightarrow \begin{cases} z = 0 \\ \sin z = 0 \end{cases} \Leftrightarrow \begin{cases} z = 0 \\ z = n\pi \quad n \in \mathbb{Z} \end{cases}.$$

Note that $z = 0$ is a zero of muliplicity 2 since $z = 0$ is also included in $z = n\pi$. Thus, the number of roots (count muliplicity) of $z \sin z$ is $2n + 2$. Therefore, we know that $g(z) = z \sin z - 1$ has $2n + 2$ roots in R.



From the graph, we can find $2n + 2$ real roots for $g(z)$ in R, then we know that all the roots of $g(z)$ are real in R. This conclusion holds for all $n \in \mathbb{N}$.

If we let $n \to \infty$, then R becomes the whole plane and then we can say that the equation

$$z \sin z = 1, \quad z \in \mathbb{C}$$

has only real roots.

$\square$

Problem 10.9 (UI Urbana-Champaign, January 2017-Q3). *Fix $R > 0$, show that there exists $0 < n \in \mathbb{N}$ such that for all $m \geq n$ the polynomial $f_m(z) = \sum_{k=0}^{m} \frac{z^k}{k!}$ has no roots w with $|w| < R$.*

Solution. [Jared Grove]
Attempt:
None, I nailed it the first time. **Solution:**

Notice that $\lim_{m\to\infty} f_m(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$. We know that $e^z \neq 0$ for all $z \in \mathbb{C}$, thus for any $R > 0$, $e^z \neq 0$ for all $|z| = R$. Since $\bar{B}(0,R) \subset \mathbb{C}$, $\{f_m\} \in H(G)$ and $\lim_{m\to\infty} f_m = e^z$, Hurwitz's Theorem (p152) tells us that there exists some $N \in \mathbb{N}$ such that for all $m \geq N$, $f_m(z)$ and e^z have the same number of zeros in $|z| < R$, namely 0.

□

Problem 10.10 (UI Urbana-Champaign, January 2017 Q4)**.**
Use residues to calculate:

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2},$$

where $a > 1$.

Solution. [Alex Bates]
Solution:
For any $\theta \in \mathbb{R}$, $z = e^{i\theta}$ lies on the unit circle and as such is nonzero. Now,

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}.$$

Then:

$$\frac{1}{(a+\cos\theta)^2} = \frac{1}{\left(a + \frac{z^2+1}{2z}\right)^2} = \frac{1}{\left(\frac{z^2+2az+1}{2z}\right)^2} = \frac{4z^2}{(z^2+2az+1)^2}. \tag{59}$$

By application of the quadratic formula, we find that the roots of $z^2 + 2az + 1$ are $\alpha = -a + \sqrt{a^2-1}$ and $\beta = -a - \sqrt{a^2-1}$. Hence, Equation59 becomes:

$$\frac{4z^2}{(z^2+2az+1)^2} = \frac{4z^2}{((z-\alpha)(z-\beta))^2} = \frac{4z^2}{(z-\alpha)^2(z-\beta)^2}. \tag{60}$$

By manipulating trig identities and clever calculus, one can easily establish[3] that:

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}. \tag{61}$$

Define $\gamma(\theta) = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Now actually performing the substitution $z = e^{i\theta}$ we

[3]Rewrite $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} + \frac{1}{2}\int_0^{\pi} \frac{dx}{(a+\cos x)^2}$ and show that $\int_0^{\pi} \frac{dx}{(a+\cos x)^2} = \int_{\pi}^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$ via the substitution $x = -\theta + 2\pi$.

obtain $d\theta = -i\frac{dz}{z}$. Hence,

$$\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_\gamma \frac{4z^2}{(z^2+2az+1)^2}\left(-i\frac{dz}{z}\right) \quad \text{(Equation 59)}$$

$$= -i\frac{1}{2}\int_\gamma \frac{4z^2}{(z-\alpha)^2(z-\beta)^2}\left(\frac{dz}{z}\right) \quad \text{(Equation 60)}$$

$$= \frac{1}{2i}\int_\gamma \underbrace{\frac{4z}{(z-\alpha)^2(z-\beta)^2}}_{=:f(z)} dz$$

$$= \frac{1}{2i}\int_\gamma f(z)dz. \tag{62}$$

Since $a > 0$, $|\beta| > 1$ and $|\alpha| < 1$, so α lies inside γ, which winds around α precisely once. The Residue Theorem then tells us that $\frac{1}{2\pi i}\int_\gamma f(z)dz = Res(f,\alpha)$, that is,

$$\frac{1}{2i}\int_\gamma f(z)dz = \pi Res(f,\alpha). \tag{63}$$

Define $g(z) = (z-\alpha)^2 f(z) = \frac{4z}{(z-\beta)^2}$. By application of the quotient rule, $g'(z) = \frac{-4(z+\beta)}{(z-\beta)^3}$. We then have that:

$$Res(f,\alpha) = g'(\alpha)$$

$$= \frac{-4(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{-4(-2a)}{(2\sqrt{a^2-1})^3}$$

$$= \frac{a}{(a^2-1)^{\frac{3}{2}}}. \tag{64}$$

Therefore, by combining Equations 61, 62, 63 and 64, we obtain:

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{\frac{3}{2}}}.$$

□

11 Additional Practice

11.1 Group Work I

Problem 11.1 (Group Work I Complex, Number 1)**.**
Let $\mathscr{F}$ be the collection of all holomorphic mappings from $Q = \{z : \Re(z) > 0, \Im(z) > 0\}$ to $\mathbb{D}$. What is $\sup_{f\in\mathscr{F}} |f'(1+i)|$? Find a bijective map h from Q onto the unit disc with $h(1+i) = 1/2$.

Solution. [Alex Bates, Jared Grove, Meghan Malachi]
Solution:
Let $f \in \mathscr{F}$ and define $a := f(1+i) \in \mathbb{D}$. Since $|a| < 1$ may select a modular map $\phi_a : \mathbb{D} \to \mathbb{D}$ defined by $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ so that $\phi_a(a) = \phi_a(f+i) = 0$. Now, notice that the function $z \mapsto z^2$ maps Q conformally onto the upper half plane $\mathbb{H}$. The map $z \mapsto \frac{z-i}{z+i}$ maps $\mathbb{H}$ conformally onto the unit disc $\mathbb{D}$.[4] Define their composition by $\phi(z) = \frac{z^2-i}{z^2+i}$. Since this map is conformal, there exists an analytic inverse $\phi^{-1} : \mathbb{D} \to Q$ so that, given $\phi(1+i) = \frac{1}{3}$, we have $\phi^{-1}(\frac{1}{3}) = 1+i$. Let $\phi_{-\frac{1}{3}} : \mathbb{D} \to \mathbb{D}$ be the modular map defined by $\phi_{-\frac{1}{3}}(z) = \frac{z+\frac{1}{3}}{1+\frac{1}{3}z}$ and observe that $\phi_{-\frac{1}{3}}(0) = \frac{1}{3}$.

What does all that get us? Observe that by the following diagram,

$$\mathbb{D} \xrightarrow{\phi_{-\frac{1}{3}}} \mathbb{D} \xrightarrow{\phi^{-1}} Q \xrightarrow{f} \mathbb{D} \xrightarrow{\phi_a} \mathbb{D}$$

the composition $\psi := \phi_a \circ f \circ \phi^{-1} \circ \phi_{-\frac{1}{3}} : \mathbb{D} \to \mathbb{D}$ is a well-defined analytic map with the property that $\psi(0) = 0$. Schwarz's Lemma then guarantees us that $|\psi'(0)| \leq 1$. We use this fact to obtain an upper bound for $|f'(1+i)|$. Now,

$$\begin{aligned} \psi'(0) &= (\phi_a \circ f \circ \phi^{-1} \circ \phi_{-\frac{1}{3}})'(0) \\ &= [\phi_a'(a)] \cdot [f'(1+i)] \cdot [(\phi^{-1})'(1/3)] \cdot [\phi'_{-\frac{1}{3}}(0)], \end{aligned} \tag{65}$$

so let's tackle this piece-by-piece. Since $|1/3| < 1$, we're told from Proposition VI.2.2 of Conway (pg. 131) that

$$\phi'_{-\frac{1}{3}}(0) = 1 - |1/3|^2 = 1 - 1/9 = 8/9. \tag{66}$$

While we're at it, this Proposition also tells us that

$$\phi_a'(a) = \frac{1}{1-|a|^2}. \tag{67}$$

By Corollary IV.7.6 of Conway (pg. 99), $(\phi^{-1})'(1/3) = \frac{1}{\phi'(\phi^{-1}(1/3))}$. By the chain rule, $\phi'(z) = \frac{4iz}{(z^2+i)^2}$ so that $\phi'(\phi^{-1}(1/3)) = \phi'(1+i) = \frac{4i(1+i)}{((1+i)^2+i)^2} = \frac{4-4i}{9}$, hence

$$(\phi^{-1})'(1/3) = \frac{9}{4-4i}. \tag{68}$$

[4] This map is also known as the Cayley Transformation.

Combining the result that $|\psi'(0)| \leq 1$ (from Schwarz's Lemma) along with Equations 65, 66, 67, and 68 we obtain:

$$\begin{aligned}\left|\frac{1}{1-|a|^2}\right| \cdot |f'(1+i)| \cdot \left|\frac{9}{4-4i}\right| \cdot \left|\frac{8}{9}\right| &\leq 1 \\ \frac{1}{1-|a|^2} \cdot |f'(1+i)| \cdot \left|\frac{2}{1-i}\right| &\leq 1 \\ |f'(1+i)| \cdot \frac{2}{\sqrt{2}} &\leq 1-|a|^2 \\ |f'(1+i)| &\leq (1-|f(1+i)|^2) \cdot \frac{\sqrt{2}}{2}. \qquad (69)\end{aligned}$$

Now, since $0 \leq |f(z)| < 1$ for all $z \in Q$, it follows that $1-|f(1+i)|^2 \leq 1$ so that Equation 69 becomes:

$$|f'(1+i)| \leq \frac{\sqrt{2}}{2}.$$

Since f was arbitrary and this upper bound can actually be attained (by perhaps selecting an analytic map $g : Q \to \mathbb{D}$ with $g(1+i) = 0$), it follows that $\sup_{f\in\mathscr{F}} |f'(1+i)| = \frac{\sqrt{2}}{2}$.

Lastly, we need to find a bijective map $h : Q \to \mathbb{D}$ with $h(1+i) = \frac{1}{2}$. The map $\phi : Q \to \mathbb{D}$ above is bijective and analytic and does $\phi(1+i) = \frac{1}{3}$, so it would suffice to find a bijective map from $\mathbb{D}$ to $\mathbb{D}$ with the property that $\frac{1}{3} \mapsto \frac{1}{2}$. We attempt to find a modular map that does so. Provisionally assume that there is $|a| < 1$ so that $\phi_a(\frac{1}{3}) = \frac{\frac{1}{3}-a}{1-\bar{a}\frac{1}{3}} = \frac{1}{2}$. Solving, we obtain $\bar{a} - 6a = 1$. Performing the substitution $a = x + iy$ for $x, y \in \mathbb{R}$, we see that $-5x - 7iy = 1$ so that $y = 0$, $x = -\frac{1}{5}$, i.e., $a = -\frac{1}{5}$. Certainly, $|-1/5| < 1$ so that $\phi_{-1/5} : \mathbb{D} \to \mathbb{D}$ is bijective analytic, hence the map $\phi_{-1/5} \circ \phi : Q \to \mathbb{D}$ is bijective with $(\phi_{-1/5} \circ \phi)(1+i) = \frac{1}{2}$, by construction. □

Problem 11.2 (Group Work I Complex, Number 3)**.** *Let $f : \mathscr{U} \to \mathbb{C}$ be a holomorphic function on an open connected set $\mathscr{U} \subset \mathbb{C}$. Find all possible values of f if f satisfies $f(z)^2 = \overline{f(z)}$.*

Solution. [Michael Kratochvil, Amrei Oswald, Yanqing Shen] Say $f(x+iy) = u(x,y) + iv(x,y)$. Then we have

$$f(z)^2 = \overline{f(z)} \implies (u+iv)^2 = u - iv \implies u^2 - v^2 + 2iuv = u - iv.$$

Since the imaginary and real parts above must be equal, we have that

$$u^2 - v^2 = u \text{ and } 2uv = v.$$

Say that $v \neq 0$. Then the above gives us $u = 1\frac{1}{2}$ and therefore $v^2 = \frac{3}{4} \implies v = \pm\frac{\sqrt{3}}{2}$.

If $v = 0$, then $u^2 = u \implies u = 1$. If $u = 0$, then $v = 0$.

Note that since f is continous on a connected set $\mathscr{U}$ and $u \in \left\{-\frac{1}{2}, 0, 1\right\}$ and $v \in \left\{\pm\frac{\sqrt{3}}{2}, 0\right\}$, f must be constant. Therefore, $f(z) \in \left\{-\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, 1, 0\right\}$. □

Problem 11.3 (Group Work I Complex, Number 5). *Let $\mathscr{F}$ be the family of all holomorphic functions on $\mathbb{D}$ such that $f(0) = 0$, $f^{'}(0) = 2017$ and $|f(z)| \leq 2017$. Prove that there exists a constant $c > 0$ such that $c\mathbb{D} \subset f(\mathbb{D})$ for every $f \in \mathscr{F}$.*

Solution. [Kaitlin Healy, Sara Reed, W. Tyler Reynolds, Adam Wood]
Attempt:
Solving this problem came down to seeing that Schwarz's Lemma is applicable after just a tiny bit of work. We have included a direct proof using Schwarz's Lemma, as well as a proof where we actually prove Schwarz's Lemma along the way, since the latter would presumable be more satisfactory for the purposes of the qualifying exam.
Solution:
Easy Proof. Given $f \in \mathbb{F}$, let $g(z) = \dfrac{f(z)}{2017}$ on $\mathbb{D}$. Then g is analytic on $\mathbb{D}$ with $g(0) = 0$ and $g^{'}(0) = 1$. By Schwarz's Lemma, there is a constant a with $|a| = 1$ such that $g(z) = az$. Thus $f(z) = 2017az$. Notice that $z \mapsto 2017z$ is a map of $\mathbb{D}$ onto $2017\mathbb{D}$, and the map $z \mapsto az$ is a rotation of $2017\mathbb{D}$. Since f is the composition of these maps, we have $2017\mathbb{D} = f(\mathbb{D})$. Since f is arbitrary, we obtain our result by setting $c = 2017$.

Alternate Proof. Let $f \in \mathscr{F}$, and define $g = \dfrac{f(z)}{2017z}$ on $\mathbb{D} \setminus \{0\}$. Note that g is analytic. Since $f(0) = 0$, we have

$$\lim_{z\to 0} zg(z) = \lim_{z \to 0} \frac{f(z)}{2017} = 0.$$

Thus, g has a removable singularity at $z = 0$. We may therefore assume that g is defined and analytic on all of $\mathbb{D}$. By continuity,

$$g(0) = \lim_{z\to 0} \frac{f(z)}{2017z} = \frac{1}{2017} \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \frac{f^{'}(0)}{2017} = 1.$$

By the Maximum Modulus Theorem, $|g(z)| \leq \dfrac{1}{r}$ for $|z| \leq r$ and $0 < r < 1$. Letting $r \to 1$, we obtain $|g(z)| \leq 1$ for $z \in \mathbb{D}$. Since $g(0) = 1$, it follows from the Maximum Modulus Theorem that $g(z) = az$ for some a with $|a| = 1$. Thus $f(z) = 2017az$. Notice that $z \mapsto 2017z$ is a map of $\mathbb{D}$ onto $2017\mathbb{D}$, and the map $z \mapsto az$ is a rotation of $2017\mathbb{D}$. Since f is the composition of these maps, we have $2017\mathbb{D} = f(\mathbb{D})$. Since f is arbitrary, we obtain our result by setting $c = 2017$.

□

11.2 Group Work II

11.3 Practice Exam I

11.4 Practice Exam II