4.1 - Exponential Functions

Before: $f(x) = 3x^2 + 2x + 1$

$$f(2) = 3(2)^2 + 2(2) + 1 = 3\cdot 4 + 4 + 1 = 17$$

Now:

- **Def:** An exponential function is a function of the form

$$f(x) = a^x, \quad \text{where } a > 0,\ a \neq 1.$$

The base of f is a, and the exponent is x.

Ex ① $f(x) = 3^{x-2}$

$$f(4) = 3^{4-2} = 3^2 = 9$$

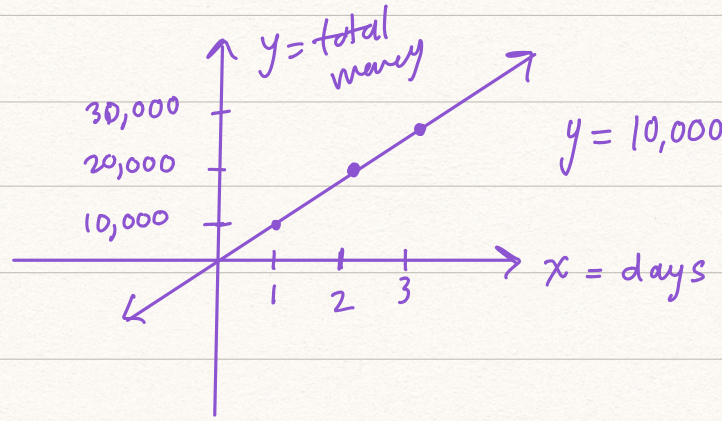
$$f(2) = 3^{2-2} = 3^0 = 1$$

② $f(x) = \left(\frac{1}{4}\right)^x$

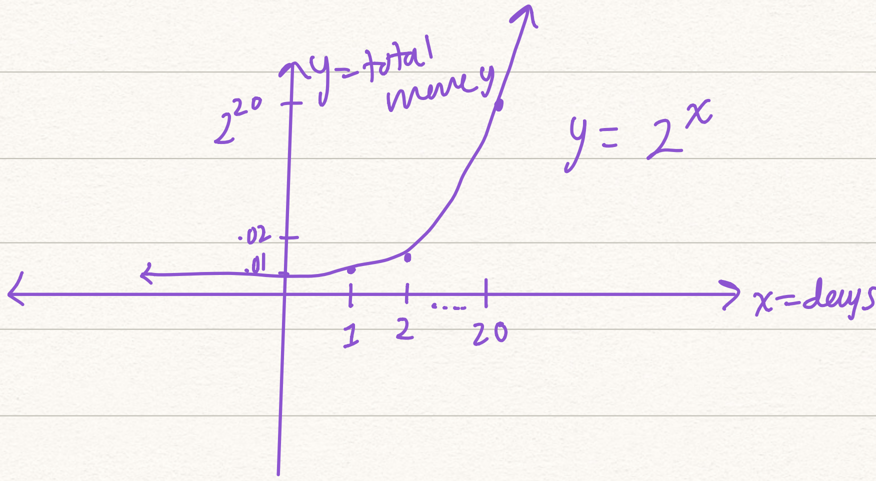
$$f(2) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$f\left(-\frac{1}{2}\right) = \left(\frac{1}{4}\right)^{-1/2} = \frac{1}{\left(\frac{1}{4}\right)^{1/2}} = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{\sqrt{1}}{\sqrt{4}}}$$

$$= \frac{1}{\frac{1}{2}} = 2.$$



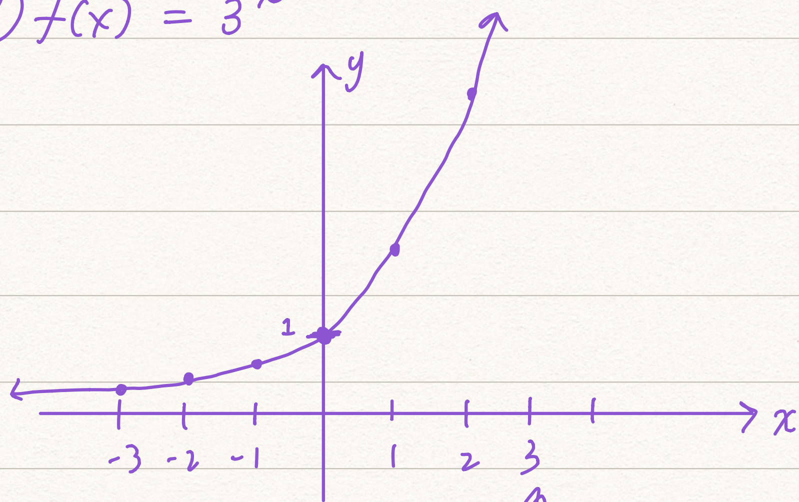
$y = 10{,}000x$



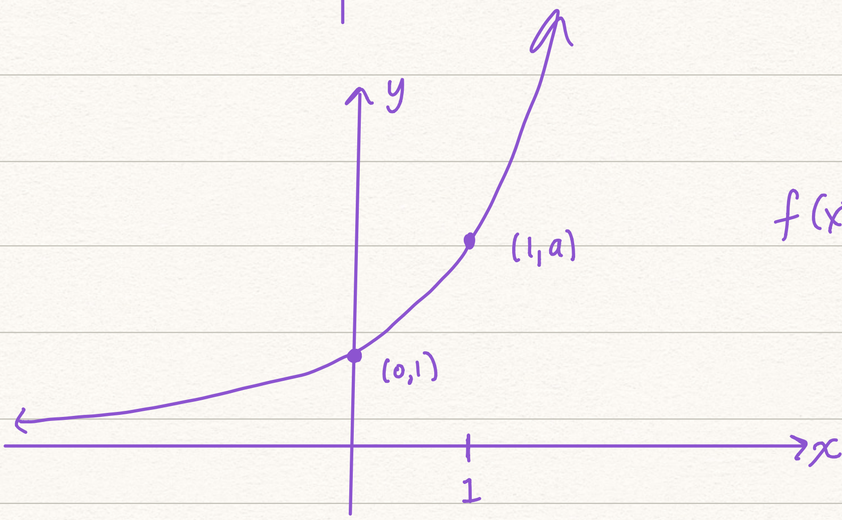
$y = 2^x$

Graphing Exponential Functions

① $f(x) = 3^x$



x	y
-3	$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$
-2	$3^{-2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$
3	$3^3 = 27$



$f(x) = a^x$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to \infty} f(x) = \infty$

② $f(x) = \left(\frac{1}{3}\right)^x$

Transformations of Exponential Functions

- Shift $f(x)=3^x$ left 3 & reflected about the y-axis.

$$y = 3^x$$
$$y = 3^{x+3} \qquad \text{(shift left 3)}$$
$$y = 3^{-x+3} \qquad \text{(reflect about y-axis)}$$

Finding Exponential Functions:

① Find the exponential function $f(x) = ca^x$, whose graph contains the points $(-1, 18)$ & $(4, \frac{2}{27})$.

$$18 = f(-1) = ca^{-1}$$
$$18 = ca^{-1}$$
$$18 = \frac{c}{a}$$
$$18a = c$$

$$\frac{2}{27} = f(4) = ca^4$$
$$\frac{2}{27} = ca^4$$

$$\frac{2}{27} = (18a)a^4$$

$$\frac{2}{27} = 18a^5$$

$$\frac{1}{243} = a^5$$

$$\sqrt[5]{\frac{1}{243}} = a$$

$$a = \frac{1}{3}$$

So $c = 18(a) = 18\left(\frac{1}{3}\right) = 6.$

Hence $f(x) = c \cdot a^x = 6\left(\frac{1}{3}\right)^x$

(2) Same problem for $(-2, 16)$ and $\left(3, \frac{1}{2}\right)$.

$$f(x) = c \cdot a^x$$

$16 = f(-2) = c\,a^{-2}$

$16 = c\,a^{-2}$

$16 = \frac{c}{a^2}$

$16a^2 = c$

$\frac{1}{2} = f(3) = c\,a^3$

$\frac{1}{2} = c\,a^3$

$\frac{1}{2} = (16a^2)\,a^3$

$\frac{1}{2} = 16a^5$

$\frac{1}{32} = a^5$

$a = \sqrt[5]{\frac{1}{32}} = \frac{\sqrt[5]{1}}{\sqrt[5]{32}} = \frac{1}{2}$

$a = \frac{1}{2}$

$c = 16\left(\frac{1}{2}\right)^2 = 16\left(\frac{1}{4}\right) = 4$

$$f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$$

Interest Formulas

① Simple Interest

$$I = Prt$$

$I =$ interest earned
$P =$ principal amount
$r =$ annual interest rate (in decimal form)
$t =$ years

Ex: We deposit \$8,000 in a bank for 5 years at a simple interest rate of 6%.

a) How much interest will we earn?

b) How much money is in the account after 5 years?

Solution: a) $I = (8000)(.06)(5)$

$$= (8000)\left(\frac{6}{100}\right)(5)$$

$$= \frac{(8000)(30)}{100}$$

$$= (80)(30)$$

$$= 2{,}400$$

b) $8{,}000 + 2{,}400 = \$10{,}400$

② Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$A =$ amount after t years
$P =$ principal amount
$r =$ annual rate
* $n =$ # of times interest is compounded <u>per year</u>.
$t =$ years

Ex: \$100 is deposited in a bank that pays 5% annual interest. How much money is in the bank after 3 years if the interest is compounded quarterly?

$$A = (100)\left(1 + \frac{.05}{4}\right)^{(4)(3)} \approx \$116.08$$

Daily? $n = 365$

Semiannually? $n = 2$

Annually? $n = 1$

③ Continuous Compound Interest

$$A = Pe^{rt}$$

A = amount after t years

P = principal

r = annual rate

t = years

$e \approx 2.71828\ldots$

Exponential Growth & Decay

$$A(t) = A_0 e^{kt}$$

$A_0 = A(0)$ = initial amount (at $t=0$)

k = growth rate/decay rate

$k > 0$ (growth)

$k < 0$ (decay)

t = time

Ex: In year 2000, there were 6.08 billion people in the world. Assume a growth rate after 1990 of 1.5%. Estimate population in the years

a) 2030

b) 1990.

$$A(t) = A_0 e^{kt}$$

- $A_0 = A(0) = 6.08$ billion
- $k = .015$

So $A(t) = (6.08)e^{(.015)t}$

a) $A(30) = (6.08)e^{(.015)(30)}$

b) $A(-10) = (6.08)e^{(.015)(-10)}$

2 Fishing boat is purchased for \$20,000

Boat value depreciates @ a rate of 15% / year.

Find boat value after 5 years.

$$A(t) = A_0 e^{kt}$$

- $A_0 = A(0) = 20{,}000$
- $k = -.15$

$$A(t) = (20{,}000)e^{(-.15)(t)}$$

$$A(5) = (20{,}000)e^{(-.15)(5)} \approx \$10{,}392.06$$