3.5 - The Complex Zeros of a Polynomial

Before: We found the real zeros of polynomials.

Now: We will find all zeros of polynomials,
including complex ones.

Factorization Theorem

- Every polynomial $P(x)$ can be factored as

$$P(x) = a(x-r_1)(x-r_2)\cdots\cdots(x-r_n)$$

where $a, r_1, r_2, \ldots, r_n$ are complex numbers.

Number of Zeros Theorem: A polynomial of degree n has exactly n zeros, counting multiplicities.

Ex ① x^2+1

$$x^2+1 = (x-A)(x-B)$$

$$x^2+1 \overset{set}{=} 0$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1}$$

$$x = \pm i$$

So $\quad x^2+1 = (x-(i))(x-(-i))$

$$= (x-i)(x+i)$$

② Find a polynomial of degree 4, with leading coeficient 2, and zeros $-1, 3, i, -i$.

$$P(x) = 2x^4 + \cdots\cdots$$

$$= 2(x-(-1))(x-3)(x-i)(x-(-i))$$

$$= 2(x+1)(x-3)\underbrace{(x-i)(x+i)}_{(a-b)(a+b)=a^2-b^2}$$

$$= 2(x^2-2x-3)(x^2-i^2)$$
$$= 2(x^2-2x-3)(x^2-(-1))$$
$$= 2(x^4-2x^3-2x^2-2x-3)$$
$$= 2x^4-4x^3-4x^2-4x-6.$$

Conjugate Pairs Theorem:

Let $P(x)$ be a polynomial with real coefficients. If $a+bi$ is a zero of P, then so is $a-bi$.

Ex ① See example 2 on Pg 379.

② Given that $2-i$ is a zero of
$$P(x) = x^4-6x^3+14x^2-14x+5,$$
find the remaining zeros.

$$* \quad P(x) = (x-(2-i))(x-(2+i))(\text{somethin' else})$$

$$(x-(2-i))(x-(2+i)) = (x-2+i)(x-2-i)$$
$$= ((x-2)+i)((x-2)-i)$$
$$= (x-2)^2 - i^2$$
$$= x^2-4x+4-(-1)$$
$$= x^2-4x+5$$

$$(a-b)(a+b) = a^2-b^2$$

$$P(x) = (x^2-4x+5)(\text{somethin' else})$$

$$\frac{P(x)}{x^2-4x+5} = (\text{somethin' else}).$$

$$\begin{array}{r|l} & x^2-2x+1 \\ \hline x^2-4x+5 & x^4-6x^3+14x^2-14x+5 \end{array}$$

So $P(x) = (x^2-4x+5)(x^2-2x+1)$

* $P(x) = (x-\underline{(2-i)})(x-\underline{(2+i)})(x-\underline{1})(x-\underline{1})$

Zeros of P: $2-i$, $2+i$, 1, 1

③ Find all the zeros of the polynomial

$$P(x) = x^4 - x^3 + 7x^2 - 9x - 18$$

- Use Rational Zeros Theorem.

→ All <u>possible</u> rational roots are

$$\frac{\text{Factors of } 18}{\text{Factors of } 1}$$

$$= \frac{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}{1}$$

$\longrightarrow$ 1, -1, 2, -2, 3, -3, 6, -6, 9, -9, 18, -18

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & 7 & -9 & -18 \\ & \downarrow & 2 & 2 & 18 & 18 \\ \hline & 1 & 1 & 9 & 9 & 0 \end{array}$$ remainder

$$\frac{x^4 - x^3 + 7x^2 - 9x - 18}{x-2} = x^3 + x^2 + 9x + 9$$

$$\frac{P(x)}{x-2} = \frac{x^4 - x^3 + 7x^2 - 9x - 18}{x - 2} = x^3 + x^2 + 9x + 9$$

$$* \; P(x) = (x-2)(x^3 + x^2 + 9x + 9)$$

$$P(x) = (x-2)\big(x^2(x+1) + 9(x+1)\big)$$

$$P(x) = (x-2)\;(x^2+9)(x+1)$$

$$\swarrow$$

$$x^2 + 9 \overset{set}{=} 0$$

$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

$$x = \pm 3i$$

So $x^2+9 = (x-(3i))(x-(-3i))$

$$P(x) = (x-2)(x-3i)(x+3i)(x+1)$$

Zeros of P: 2, 3i, -3i, -1