3.3 - Dividing Polynomials

- We say 5 is a <u>factor</u> of 10, since $10 = 5 \cdot 2$.
- Similarly, a polynomial $d(x)$ is a <u>factor</u> of a polynomial $p(x)$ if

$$p(x) = d(x) \cdot q(x) \quad \text{for some polynomial } q(x).$$

<u>Ex</u> $x-1$ is a factor of x^3-1, since

$$x^3 - 1 = (x-1)(x^2+x+1)$$

- 5 is not a factor of 13, but we can still write $13 = 2 \cdot 5 + 3$.

$$\frac{13}{5} = 2 + \frac{3}{5}$$

- Recall long division of numbers:

$$\begin{array}{r} 41 \\ 6 \overline{)\,246} \\ -(24) \\ \hline 06 \\ -\ 6 \\ \hline 0 \end{array} \qquad\qquad \begin{array}{r} 41\frac{2}{6} \\ 6 \overline{)\,248} \\ -24 \\ \hline 08 \\ -\ 6 \\ \hline 2 \end{array}$$

$$\frac{248}{6} = 41 + \frac{1}{3}$$

Long Division of Polynomials

① Compute $\dfrac{2x^2+x^5+7+4x^3}{x^2+1-x}$.

$$x^3+x^2+4x+5$$

$$x^2-x+1 \,\big)\, x^5+0x^4+4x^3+2x^2+0x+7$$

$$-(x^5-x^4+x^3)$$

$$0+x^4+3x^3+2x^2+0x+7$$

$$-(x^4-x^3+x^2)$$

$$0+4x^3+x^2+0x+7$$

$$-(4x^3-4x^2+4x)$$

$$0+5x^2-4x+7$$

$$-(5x^2-5x+5)$$

$$0 \quad x+2 \longrightarrow \text{Remainder}$$

So: $x^3+x^2+4x+5+\dfrac{x+2}{x^2-x+1} = \dfrac{x^5+4x^3+2x^2+7}{x^2-x+1}$

Synthetic Division:

- Use when dividing by a polynomial of the form $x-a$

Ex: ① Divide $2x^3-5x^2+3x-14$ by $x-3$

$$\begin{array}{c|cccc} 3 & 2 & -5 & 3 & -14 \\ & & \overset{\text{add}}{6} & \overset{\text{add}}{3} & \overset{\text{add}}{18} \\ \hline & 2 & 1 & 6 & 4 \end{array}$$

mult. mult. mult.

$$\frac{2x^3-5x^2+3x-14}{x-3} = 2x^2+1x+6 + \frac{4}{x-3}$$

② $$\frac{2x^4-3x^2+5x-63}{x+3}$$ $\qquad x+3 = x-(-3)$

$$\begin{array}{r|rrrrr} -3 & 2 & 0 & -3 & 5 & -63 \\ & \downarrow & -6 & 18 & -45 & 120 \\ \hline & 2 & -6 & 15 & -40 & 57 \end{array}$$

$$\frac{2x^4-3x^2+5x-63}{x+3} = 2x^3-6x^2+15x-40+\frac{57}{x+3}$$

Remainder Theorem

- If a polynomial $F(x)$ is divided by $x-a$, then the remainder R is given by

$$R = F(a).$$

Ex ① Find the remainder when $F(x) = 2x^5-4x^3+5x^2-7x+2$ is divided by $x-1$.

$$\begin{aligned} R = F(1) &= 2(1)^5-4(1)^3+5(1)^2-7(1)+2 \\ &= 2-4+5-7+2 \\ &= -2+5-5 \\ &= -2 \end{aligned}$$

So: $$\frac{2x^5-4x^3+5x^2-7x+2}{x-1} = \sim\sim\sim + \frac{-2}{x-1}$$

② Let $f(x) = x^4 + 3x^3 - 5x^2 + 8x + 75$. Find $f(-3)$.

· By Remainder Theorem $f(-3)$ is the remainder of $\dfrac{f(x)}{x-(-3)}$.

$$
\begin{array}{r|rrrrr}
-3 & 1 & 3 & -5 & 8 & 75 \\
 & \downarrow & -3 & 0 & 15 & -69 \\
\hline
 & 1 & 0 & -5 & 23 & 6
\end{array}
$$

So $\dfrac{f(x)}{x-(-3)} = x^3 + 0x^2 - 5x + 23 + \dfrac{6}{x-(-3)}$

So: $f(-3) = 6$.

③ What is the remainder of:

ⓐ $\dfrac{2x^3 + 3x^2 - 1}{x}$

ⓑ $\dfrac{3x^4 - 7x}{x}$

ⓒ $\dfrac{14x^2 - 3x + 7}{x+1}$

$x+1 = x-(-1)$

ⓐ $R = -1$

ⓑ $R = 0$

ⓒ $R = 14(-1)^2 - 3(-1) + 7 = 14 + 3 + 7 = 24$

Factor Theorem

- A polynomial $f(x)$ has $x-a$ as a factor if and only if $f(a)=0$.
  (When can I write $f(x)=(x-a)(\cdots)$?)

Ex ① Is $x^4+3x^3-5x^2+8x+75$ divisible by $x+3$?

Is $x+3$ a factor of $x^4+3x^3-5x^2+8x+75$?

-3	1	3	-5	8	75
	↓	-3	0	15	-69
	1	0	-5	23	6

$$\frac{x^4+3x^3-5x^2+8x+75}{x+3} = x^3-5x+23+\frac{6}{x+3}$$

② Given that 2 is a zero of

$$f(x)=3x^3+2x^2-19x+6,$$

find all the zeros of f.

- By Factor Theorem, since $f(2)=0$, we know that $x-2$ is a factor of $3x^3+2x^2-19x+6$.

$f(x)=(x-2)(\text{-----})$

2	3	2	-19	6
	↓	6	16	-6
	3	8	-3	0

→ remainder 0.

$$\frac{f(x)}{x-2} = 3x^2+8x-3$$

$$f(x) = (x-2)(3x^2+8x-3)$$

$$f(x) = (x-2)(3x-1)(x+3)$$

When is $f(x)=0$?

$x-2=0$ $\qquad$ $3x-1=0$ $\qquad$ $x+3=0$

$x=2$ $\qquad$ $x=\frac{1}{3}$ $\qquad$ $x=-3$

Rational Zeros Theorem

Ex: ① Find all rational zeros of $F(x)=2x^3+5x^2-4x-3$

Rational zeros theorem says:

If there are any zeros of F that are rational, they must be of the form

$$\frac{\text{Factor of } -3}{\text{Factor of } 2}$$

Factors of -3: $\pm 1, \pm 3$

Factors of 2: $\pm 1, \pm 2$

A rational zero of F must be

$$\frac{\pm 1}{\pm 1}, \quad \frac{\pm 1}{\pm 2}, \quad \frac{\pm 3}{\pm 1}, \quad \frac{\pm 3}{\pm 2}.$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$1, -1$ $\qquad$ $\frac{1}{2}, -\frac{1}{2}$ $\qquad$ $3, -3$ $\qquad$ $\frac{3}{2}, -\frac{3}{2}$