3.1 - Quadratic Functions

- A function f given by the rule

$$f(x) = ax^2+bx+c \quad , \quad a \neq 0$$

is called a quadratic function

- Standard Form

$$f(x) = a(x-h)^2+k,$$ where (h,k) is the vertex of the parabola.

Ex: ① Find standard form of a function f whose graph has vertex $(2,-1)$ & passes through $(1,4)$.

- $f(x) = a(x-2)^2+(-1)$
  $= a(x-2)^2-1$
- $f(1)=4$
  $4 = f(1) = a(1-2)^2-1$
  $= a(-1)^2-1$
  $= a-1$

So $a=5$.

$f(x) = 5(x-2)^2-1$

② Vertex: $(-3,2)$, passes through: $(1,4)$

$f(x) = a(x+3)^2+2$

$f(1)=4$

$4 = f(1) = a(4)^2+2$

$= 16a+2$

$4 = 16a+2$

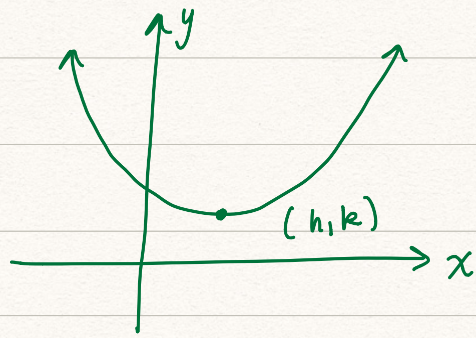
$2 = 16a$

$a = \frac{2}{16} = \frac{1}{8}$

$f(x) = \frac{1}{8}(x+3)^2 + 2$

Facts: ① If $a > 0$, the parabola opens up.
If $a < 0$, " ——————— " down.

② If $a > 0$, the function f has a minimum value of k at $x = h$.



If $a < 0$, " ——————— " maximum value of k at $x = h$.

Graphing a Quadratic Function

Steps: ① Complete the square to get into standard form.
② Opens Up/Down?
③ Find Vertex
④ Find x-intercepts & y intercepts (if any).

Ex: ① $f(x) = 2x^2 + 8x - 10$

$$= 2(x^2 + 4x - 5)$$

$$= 2(\underbrace{x^2 + 4x + 4}\underbrace{4 - 4}_{\text{adding } 0} - 5)$$

$$= 2((x+2)^2 - 4 - 5)$$

$$= 2((x+2)^2 - 9)$$

$$= 2(x+2)^2 - 18$$

vertex: $(-2, -18)$

Opens Up/Down? opens up, since $2 > 0$.

x-intercepts: $0 = 2x^2 + 8x - 10$

$$x = \frac{-8 \pm \sqrt{64 - 4(2)(-10)}}{2(2)}$$

$$= \frac{-8 \pm \sqrt{144}}{4}$$

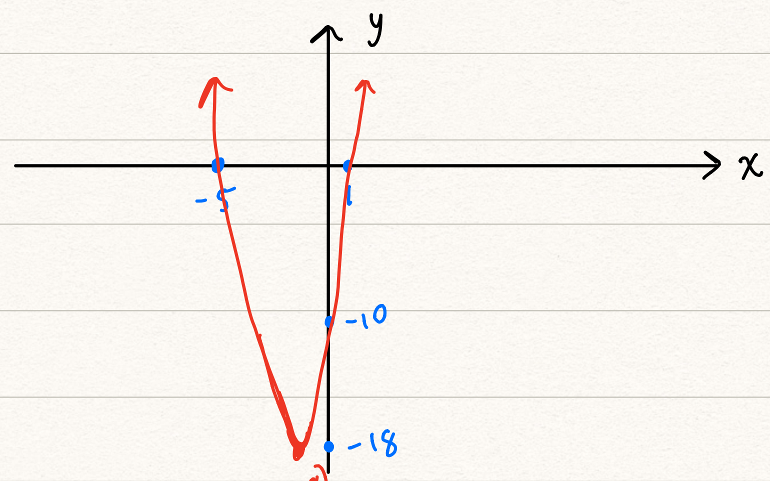
$$= \frac{-8 \pm 12}{4}$$

$$\frac{-8+12}{4} = 1$$

$$\frac{-8-12}{4} = -5$$

y-intercept: $f(0) = 2(0)^2 + 8(0) - 10$

$$= -10$$



$(-2,-18)$

Note: Given $f(x) = ax^2 + bx + c$, the x-value of the vertex is $h = \frac{-b}{2a}$, and the y-value of the vertex is $k = f(h)$.

Ex: Find vertex of parabola: $f(x) = 2x^2 + 8x - 10$.

$$h = \frac{-b}{2a} = \frac{-8}{2(2)} = \frac{-8}{4} = -2$$

$$\begin{aligned} k = f(h) = f(-2) &= 2(-2)^2 + 8(-2) - 10 \\ &= 2(4) - 16 - 10 \\ &= 8 - 16 - 10 \\ &= -18 \end{aligned}$$

Applications:

① Total cost of a product is $C(x) = 200 - 50x + x^2$ where x = number of units produced. Find the x-value for which the total cost is minimum.

- 1 = coeff. on x^2 is > 0. So parabola opens up. Hence the minimum value of C is $y = k$ at $x = h$, where (h, k) is the vertex.

$$h = \frac{-b}{2a} = \frac{-(-50)}{2(1)} = \frac{50}{2} = 25.$$

② We want to build a rectangular pen that borders a river on one side. We have 100 ft. of fence to work with. What is the maximum area that can be enclosed?

- Area is length times width.
  Let x = width, y = length.
- The perimeter of our pen is
  $$x + y + x = 2x + y.$$
  We knew we only have 100 ft. of fence to work with, so our perimeter is $100 = 2x + y$.

- We want to maximize area $= xy$. But we need a quadratic expression in order to find a maximum, and xy is not a quadratic expression.
  However, since $2x + y = 100$ then $y = 100 - 2x$.
  So xy becomes
  $$xy = x(100 - 2x) = 100x - 2x^2,$$
  which <u>is</u> a quadratic expression.

Since $-2 < 0$, the parabola opens <u>down</u>, and from Fact ②, the maximum value is $y = k$, the y value of the vertex. Lets first find the x-value of the vertex:

$$h = \frac{-b}{2a} = \frac{-100}{2(-2)} = \frac{-100}{-4} = 25.$$

Remember, 25 is <u>not</u> the maximum. It is the x-value that gives the maximum value k when we plug 25 into the expression. So: $100(25) - 2(25)^2 = 2500 - 1050$

$= 1450$