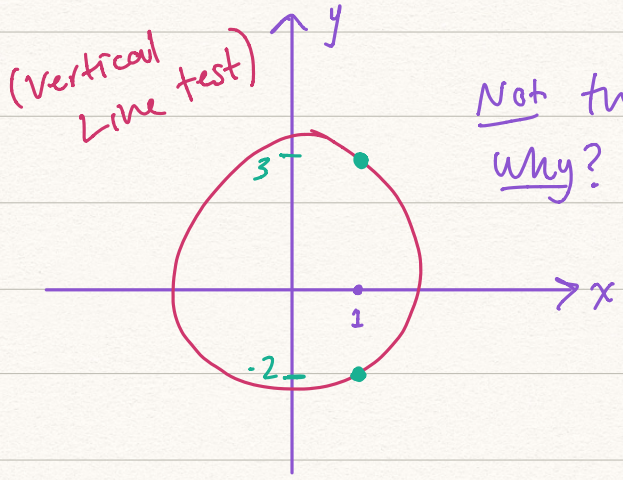
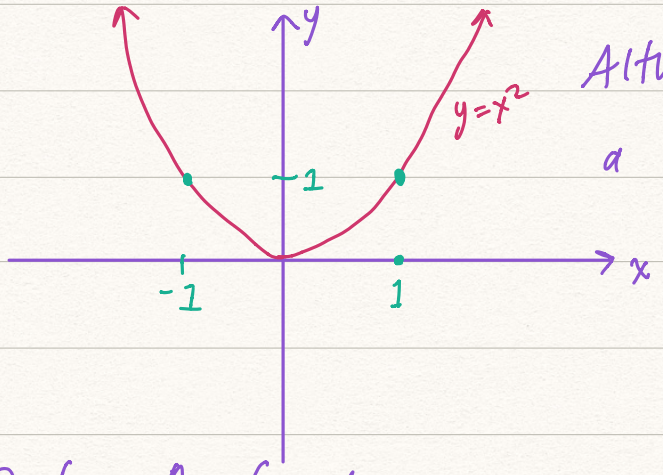
2.9 - Inverse Function



(vertical line test)

Not the graph of a function!

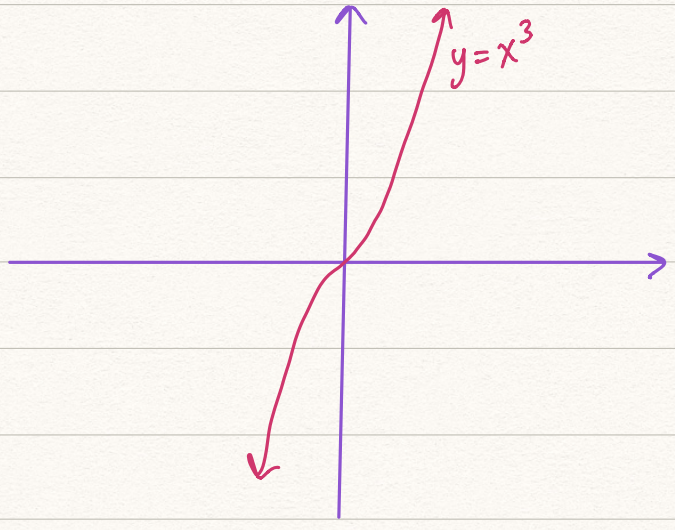
Why? Only allow one output for each input.



Although this is the graph of a function, this graph does not pass the horizontal line test.

So $y=x^2$ is not one-to-one.

Def: A function is one-to-one if no two inputs share an output.



Def: If $f(x)$ is a one-to-one function, then it has an inverse function, f^{-1}, which is defined by the following rule:

If $f(x) = y$, then $f^{-1}(y) = x$.

* Domain of f = Range of f^{-1} *
* Domain of f^{-1} = Range of f *

Ex: Suppose f is one-to-one. Find f^{-1} of the given number.

(1) $f(3) = 5$. What is $f^{-1}(5)$?

$f^{-1}(5) = 3$.

(2) $f(\pi) = 17\pi^2$. What is $f^{-1}(17\pi^2)$?

$f^{-1}(17\pi^2) = \pi$.

* The inverse of f^{-1} is f *
* $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ * (1)

$x \longrightarrow \boxed{f^{-1}} \longrightarrow f^{-1}(x) \longrightarrow \boxed{f} \longrightarrow f(f^{-1}(x)) = x$

* $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ * (2)

Ex: Check if the given functions are inverse to each other:

(1) $f(x) = x + 2$ Check if (1) & (2) hold.

$g(x) = x - 2$

$(g \circ f)(x) \stackrel{?}{=} x$

$(g \circ f)(x) = g(f(x)) = g(x+2) = (x+2) - 2 = x$ ✓

$(f \cdot g)(x) \stackrel{?}{=} x$

$(f \cdot g)(x) = f(g(x)) = f(x-2) = (x-2)+2 = x$ ✓

So, f and g are inverse to each other.

In other words: $f^{-1} = g$, and $g^{-1} = f$.

Finding Inverse Functions

Ex: (1) $f(x) = \dfrac{x+1}{x-2}$. Find f^{-1}.

Step 1: Set $y = \dfrac{x+1}{x-2}$

Step 2: Interchange x and y:

$$x = \frac{y+1}{y-2}.$$

Step 3: Solve for y.

$$\left.\begin{aligned} x(y-2) &= y+1 \\ xy - 2x &= y+1 \end{aligned}\right\}$$

$$xy - y = 2x+1 \quad *$$

$$y(x-1) = 2x+1 \quad \longleftarrow$$

$$y = \frac{2x+1}{x-1}.$$

So $f^{-1}(x) = \dfrac{2x+1}{x-1}$.

(2) Find f^{-1}, where $f(x) = \frac{x}{x+3}$. Find domain & range of f.

Step 1: $y = \frac{x}{x+3}$

Step 2: $x = \frac{y}{y+3}$

Step 3: $x(y+3) = y$

$xy + 3x = y$

$xy - y = -3x \rightarrow y - xy = 3x$

$y(x-1) = -3x \qquad y(1-x) = 3x$

$y = \frac{-3x}{x-1} \qquad y = \frac{3x}{1-x}$

So, $f^{-1}(x) = \frac{-3x}{x-1}$

Domain of f:

$(-\infty, -3) \cup (-3, \infty)$

Range of f:

= Domain of f^{-1}

$= (-\infty, 1) \cup (1, \infty)$