1.3 - Quadratic Equations

In chapter P: We learned how to manipulate expressions.

Ex: ① Changed x^2-5x+6 to $(x-3)(x-2)$, but there was no "solving for x".

② Changed $\frac{x^5}{x^3}$ to x^2, but no "solving".

③ We multiplied $(x+2)(x^3-7x^2) = x^4-5x^3-14x^2$, but no "solving".

In 1.1 & 1.2: We introduced equations, i.e. mathematical statements with equal signs. Now, we can "solve for x". In 1.1 & 1.2, solving for x in an equation was simple, because we dealt with linear equations (highest power on x is 1).

Ex
$$
\begin{aligned}
2x-7 &= 1 \\
2x &= 8 \quad \text{(add 7)} \\
x &= 4 \quad \text{(divide by 2)}.
\end{aligned}
$$

Now in 1.3: 4 methods to "solve for x" in an eqn. where the highest power on x is 2.

Def: A quadratic equation in the variable x is an equation of the form $ax^2+bx+c=0$, where $a \neq 0$, and b, c are any real number.

Ex ① $3x^2+5x-1=0$

② $2x^2+17=0$

③ $2x^2-5x+7=x^2+3x-1$

$x^2-8x+8=0$

Finding Solutions to Quadratic Equations

I Factoring

- The zero-product property says that $A \cdot B = 0$ if and only if $A=0$ or $B=0$.

Ex ① $2x^2+5x=3$

$2x^2+5x-3=0$

$\cancel{(2x-3)(x+1)=0}$

$\cancel{\overset{F}{2x^2} \underbrace{\overset{O}{+2x} \quad \overset{I}{-3x}}_{-x} \quad \overset{L}{-3}}$

$\overset{F}{2x^2} \underbrace{\overset{O}{+6x} \quad \overset{I}{-x}} \quad \overset{L}{-3}$

$(2x-1)(x+3)=0$

$2x-1=0$ or $x+3=0$

$2x=1$

$\boxed{x=\frac{1}{2}}$ $\boxed{x=-3}$

② $3t^2=2t$

$3t^2-2t=0$

$t(3t-2)=0$

$\boxed{t=0}$ or $3t-2=0$

$3t=2$

$\boxed{t=\frac{2}{3}}$

③ $x^2+4x+4=0$

$(x+2)(x+2)=0$

$x+2=0$

$\boxed{x=-2}$

Caution! We need the right-hand-side to be 0 in order to apply the zero product property.
If $(x+3)(x-2)=100$, we can't say $x+3=0$ & $x-2=0$.

II Square Root Property

- If u is an algebraic expression and $d \geq 0$ with $u^2=d$, then $u=\pm\sqrt{d}$.

Ex: ① $x^2=25$

$x=\pm\sqrt{25}$

$x=\pm 5$

- When asked to evaluate $\sqrt{25}$, we look for a positive number whose square is 25. Only 5 matches that description, so $\sqrt{25}=5$.
- When asked to solve $x^2=25$, we look for all x-values whose square is 25. So $x=5$ & $x=-5$.

② $x^2-13=0$

$x^2=13$

$x=\pm\sqrt{13}$

③ $(x+2)^2-17=0$

$(x+2)^2=17$

$x+2=\pm\sqrt{17}$

$$x = -2 \pm \sqrt{17}$$

$\boxed{\text{III}}$ Completing the Square

- Given an expression of the form x^2+bx, we convert the expression to a perfect square by adding to x^2+bx the square of half of b.

Ex ① x^2+6x

$\rightsquigarrow$ add $\left(\frac{6}{2}\right)^2$ to x^2+6x

So $x^2+6x+9 = (x+3)^2$.

② x^2+3x

$\rightsquigarrow$ add $\left(\frac{3}{2}\right)^2$ to x^2+3x

So, $x^2+3x+\frac{9}{4} = \left(x+\frac{3}{2}\right)^2$.

Now, we use this method of "completing the square" to solve quadratic equations.

Ex: ① $x^2+10x+8=0$

$$x^2+10x = -8$$
$$x^2+10x+\left(\frac{10}{2}\right)^2 = -8+\left(\frac{10}{2}\right)^2$$
$$x^2+10x+25 = -8+25$$
$$(x+5)^2 = 17$$
$$x+5 = \pm\sqrt{17}$$
$$x = -5 \pm \sqrt{17}$$

② $x^2 - 6x + 7 = 0$

$$x^2 - 6x = -7$$

$$x^2 - 6x + \left(-\frac{6}{2}\right)^2 = -7 + \left(-\frac{6}{2}\right)^2$$

$$(x-3)^2 = 2$$

$$x - 3 = \pm\sqrt{2}$$

$$x = 3 \pm \sqrt{2}$$

③ $3x^2 - 4x - 1 = 0$

· Want left-hand-side to look like $x^2 - bx$.

$$3x^2 - 4x - 1 = 0$$

$$\frac{3x^2}{3} - \frac{4x}{3} = \frac{1}{3}$$

$$x^2 - \frac{4}{3}x = \frac{1}{3}$$

$$x^2 - \frac{4}{3}x + \left(-\frac{4}{3}\cdot\frac{1}{2}\right)^2 = \frac{1}{3} + \left(-\frac{4}{3}\cdot\frac{1}{2}\right)^2$$

$$x^2 - \frac{4}{3}x + \left(-\frac{2}{3}\right)^2 = \frac{1}{3} + \left(\frac{2}{3}\right)^2$$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = \frac{1}{3} + \frac{4}{9}$$

$(x + \frac{b}{2})^2$

$$\left(x - \frac{2}{3}\right)^2 = \frac{3}{9} + \frac{4}{9}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{7}{9}$$

$$x - \frac{2}{3} = \pm\sqrt{\frac{7}{9}}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{7}}{3}.$$

$\boxed{\text{IV}}$ Using the Quadratic Formula.

- A quadratic equation of the form

$$ax^2+bx+c=0$$

has solutions

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Ex ① $3x^2=5x+2$

$$3x^2-5x-2=0$$

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(3)(-2)}}{2(3)}$$

$$=\frac{5\pm\sqrt{25+24}}{6}$$

$$=\frac{5\pm\sqrt{49}}{6}$$

$$=\frac{5\pm7}{6}$$

$$\frac{5+7}{6}=\frac{12}{6}=2$$

$$\frac{5-7}{6}=\frac{-2}{6}=-\frac{1}{3}$$

Bonus! $3x^2-5x-2=\underbrace{(x-2)}_{\text{since}}\underbrace{(3x+1)}_{\text{since}}$

since $x=2$, $x-2=0$

since $x=-\frac{1}{3}$, $3x=-1$, $3x+1=0$